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FIRST ORDER CONDITIONS FOR NONSMOOTH DISCRETIZED CONSTRAINED OPTIMAL CONTROL PROBLEMS*

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Abstract. This paper studies first order conditions (Karush–Kuhn–Tucker conditions) for discretized optimal control problems with nonsmooth constraints. We present a simple condition which can be used to verify that a local optimal point satisfies the first order conditions and that a point satisfying the first order conditions is a global or local optimal solution of the optimal control problem.

Key words. optimal control, first order condition, nonsmooth, discretization

AMS subject classifications. 49K20, 35J25

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1. Introduction.

We consider the distributed optimal control problem

$$\begin{aligned}
&\text{minimize} \quad \frac{1}{2}\int_\Omega (y-y_d)^2 d\omega + \frac{\alpha}{2}\int_\Omega (u-u_d)^2 d\omega \\
&\text{subject to} \ -\triangle y + \lambda \max(0,y) = u \quad \text{in} \quad \Omega, \quad y = g \quad \text{on}\, \Gamma, \\
&\qquad\qquad u \in U,
\end{aligned} \tag{1.1}$$

where $y_d, u_d \in L^2(\Omega)$, $g \in C(\Gamma)$, $\alpha > 0$, and $\lambda > 0$ are constants, Ω is an open bounded convex subset of $R^N, N \leq 3$, with smooth boundary Γ, and

$$U = \{u \in L^2(\Omega) \,|\, u(x) \leq q(x) \text{ a.e in}\, \Omega\},$$

$q \in L^\infty(\Omega)$.

This problem is a special case of semilinear elliptic control problems whose constraints involve a semilinear elliptic equation [3]

$$\begin{aligned}
\triangle y &= f(x,y,u) \qquad \text{in} \quad \Omega, \\
y &= g \qquad\qquad\quad\ \ \text{on} \quad \Gamma,
\end{aligned}$$

where $f : \Omega \times R^2 \to R$ is a continuous function. Optimality conditions for semilinear elliptic control problems have been studied extensively. However, most of papers assume that f is continuously differentiable with respect to the second and third variables [3]. These results are not applicable to (1.1) because the elliptic equation in (1.1) has a nonsmooth term $\lambda \max(0,y)$. Such nonsmooth equations can be found in equilibrium analysis of confined magnetohydrodynamics (MHD) plasmas [4, 5, 11], thin stretched membranes partially covered with water [9], or reaction-diffusion problems [1].

In this paper, we study first order conditions for the discretized nonsmooth constrained optimal control problems derived from a finite difference approximation or a finite element approximation of (1.1), which has the form

$$\text{minimize}\, \frac{1}{2}(y-y_d)^T H(y-y_d) + \frac{\alpha}{2}(u-u_d)^T M(u-u_d)$$

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$$\text{subject to } Ay + \lambda D \max(0, y) = Nu, \quad u \le b. \tag{1.2}$$

Here $y_d \in R^n$, $u_d, b \in R^m$, $H \in R^{n\times n}$, $M \in R^{m\times m}$, $A \in R^{n\times n}$,$D \in R^{n\times n}$, $N \in R^{n\times m}$, and $\max(\cdot)$ is understood coordinatewise. Moreover, H, M, A, D are symmetric positive definite matrices. We assume that D is a diagonal matrix. This assumption holds for finite difference discretization and finite element discretization with mass lumping.

For every $u \in R^m$, there is a unique vector y satisfying the equality constraints in (1.2), since $Ay + \lambda D \max(0, y)$ is strongly monotone. Therefore, (1.2) is equivalent to

$$\begin{aligned} &\text{minimize } \frac{1}{2}(y(u) - y_d)^T H(y(u) - y_d) + \frac{\alpha}{2}(u - u_d)^T M(u - u_d) \\ &\text{subject to } \quad u \le b, \end{aligned} \tag{1.3}$$

where $y(u)$ is the solution function defined by the equations in the constraints of (1.2). We show that $y(\cdot)$ is a piecewise linear function in the next section.

Let $E(y)$ be an $n \times n$ diagonal matrix whose diagonal elements satisfy

$$E_{ii}(y) = \begin{cases} 1 & \text{if } y_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $E(y)$ is the Jacobian of the function $\max(0, \cdot)$ at y if y has no zero component.

Since A is a symmetric positive definite matrix and $\lambda DE(y)$ is a nonnegative diagonal matrix, the matrix $A + \lambda DE(y)$ is nonsingular and its inverse is symmetric positive definite.

We say (y, u) satisfies first order conditions for (1.2) or (y, u) is a Karush–Kuhn–Tucker (KKT) point of (1.2) if it together with some $(s, t) \in R^n \times R^m$ satisfies

$$\begin{pmatrix} H(y - y_d) + As + \lambda DE(y)s \\ \alpha M(u - u_d) - N^T s + t \\ Ay + \lambda D \max(0, y) - Nu \\ \min(t, b - u) \end{pmatrix} = 0. \tag{1.4}$$

The vectors $s \in R^n$ and $t \in R^m$ are referred to as Lagrange multipliers.

We say u satisfies first order conditions for (1.3) or u is a KKT point of (1.3) if it together with some $t \in R^m$ satisfies

$$\begin{pmatrix} ((A + \lambda DE(y(u)))^{-1}N)^T H(y(u) - y_d) + \alpha M(u - u_d) + t \\ \min(t, b - u) \end{pmatrix} = 0. \tag{1.5}$$

For $\lambda = 0$, the constraints in (1.1) involve only linear Dirichlet problems. In this case, problem (1.2) is a convex programming problem with linear constraints, and the function $y(\cdot)$ in problem (1.3) can be expressed explicitly as $y(u) = A^{-1}Nu$. Moreover, (1.4) and (1.5) are equivalent in the sense that if (y, u) is a KKT point of (1.2), then u is a KKT point of (1.3); conversely, if u is a KKT point of (1.3), then $(A^{-1}Nu, u)$ is a KKT point of (1.2). Furthermore, the convexity implies that (y, u) is a KKT point of (1.2) if and only if (y, u) is a global solution of (1.2). Therefore, problems (1.2), (1.3), (1.4), and (1.5) are equivalent in the case $\lambda = 0$. Many algorithms for solving

(1.1) based on the equivalent relation have been developed; we refer the reader to a comprehensive paper written by Bergounioux, Ito, and Kunisch [2].

For $\lambda > 0$, the constraints in (1.1) involve nonsmooth partial differential equations. In this case, problem (1.2) is a nonconvex programming problem with nondifferentiable constraints. It fails to satisfy the constraint qualification in mathematical programming [7] in the sense that the set of feasible directions and the set of feasible directions for the linearized constraint set are not same at points where the constraint is not differentiable. There are examples in [7] which show that a KKT point is not necessarily a minimizer for a nonconvex programming problem, and a minimizer is not necessarily a KKT point if the constraint qualification fails.

Recently many numerical methods for solving nonsmooth equations have been developed [4, 5, 8, 12]. We can find a solution of nonsmooth equation (1.4) or (1.5) by a fast (superlinearly convergent) algorithm. However, we do not know if the solution of (1.4) or (1.5) is a minimizer of (1.2) or (1.3). There are open questions in the relations between the four problems:

$$\begin{array}{ccc|ccc}
(1.2) & \Leftrightarrow & (1.3) & (1.2) & \Leftrightarrow & (1.3) \\
\Updownarrow & & \Updownarrow & ? & & ? \\
(1.4) & \Leftrightarrow & (1.5) & (1.4) & ? & (1.5) \\
 & & & & & \\
 & \lambda = 0 & & & \lambda > 0 &
\end{array}$$

In this paper, we provide a necessary and sufficient condition for the solution function $y(\cdot)$ to be differentiable at a point u. By using the differentiability results, we show that problems (1.4) and (1.5) are equivalent. Moreover, we present a simple condition which can be used to verify that a local optimal solution of (1.3) is a solution of (1.5) and that a solution of (1.4) is a global or local optimal solution of (1.2).

We introduce our notation. For any matrix $B \in R^{m\times n}$, let $B_{\mathcal{K}\mathcal{J}}$ be the submatrix of B whose entries lie in the rows of B indexed by $\mathcal{K}$ and the columns indexed by $\mathcal{J}$. If $\mathcal{J} = \{1, 2, \dots, n\}$, we simply denote $B_{\mathcal{K}\mathcal{J}}$ by $B_{\mathcal{K}}$. Let $e_i \in R^n$ be the ith column of the identity matrix $I \in R^{n\times n}$.

2. Differentibility. In this section, we study the function

$$F(y, u) = Ay + \lambda D \max(0, y) - Nu$$

and the solution function $y(u)$ defined by

$$F(y, u) = 0.$$

For a given $y \in R^n$, we define the index sets

$$\begin{aligned}
\mathcal{J}(y) &:= \{\, i \,|\, y_i > 0\},\\
\mathcal{K}(y) &:= \{\, i \,|\, y_i = 0\},\\
\mathcal{L}(y) &:= \{\, i \,|\, y_i < 0\}.
\end{aligned}$$

Note that $\mathcal{J}(y)$, $\mathcal{K}(y)$, and $\mathcal{L}(y)$ are mutually disjoint, and $\mathcal{J}(y) \cup \mathcal{K}(y) \cup \mathcal{L}(y) = \{1, 2, \dots, n\}$. Using the function E, we can write the functions F and $y(\cdot)$ as follows:

$$F(y, u) = (A + \lambda DE(y))y - Nu$$

and

$$y(u) = (A + \lambda DE(y(u)))^{-1} Nu.$$

Moreover, for any $u, v \in R^m$, we have

$$\max(0, y(u)) - \max(0, y(v)) = V(y(u) - y(v)),$$

where V is an $n \times n$ diagonal matrix whose diagonal elements are defined by

$$V_{ii} = \begin{cases} 1, & y_i(u) > 0, \quad y_i(v) > 0, \\ \frac{y_i(u)}{y_i(u) - y_i(v)}, & y_i(u) > 0, \quad y_i(v) \le 0, \\ \frac{-y_i(v)}{y_i(u) - y_i(v)}, & y_i(u) \le 0, \quad y_i(v) > 0, \\ 0, & y_i(u) \le 0, \quad y_i(v) \le 0. \end{cases}$$

Since $V_{ii} \in [0, 1]$, $A + \lambda DV$ is symmetric positive definite and it holds that

$$\begin{aligned} \|y(u) - y(v)\|_2 &= \|(A + \lambda DV)^{-1} N(u - v)\|_2 \\ &\le \|A^{-1}\|_2 \|N\|_2 \|u - v\|_2. \end{aligned}$$

Hence y is a Lipschitz continuous function.

THEOREM 2.1. (i) *The function $F : R^n \times R^m \to R^n$ is differentiable at (y, u) if and only if $\mathcal{K}(y) = \emptyset$; in this case the derivative of F at (y, u) is given by*

$$F'(y, u) = (A + \lambda DE(y), -N).$$

(ii) *The function $y(\cdot) : R^m \to R^n$ is differentiable at u if and only if either $\mathcal{K}(y(u)) = \emptyset$ or*

$$((A + \lambda DE(y(u)))^{-1} N)_{\mathcal{K}(y(u))} = 0; \tag{2.1}$$

in this case the derivative of $y(\cdot)$ at u is given by

$$y'(u) = (A + \lambda DE(y(u)))^{-1} N.$$

Proof. (i) If $\mathcal{K}(y) = \emptyset$, then there is an open neighborhood $\mathcal{N}_y$ of y such that for all $z \in \mathcal{N}_y$, $\mathcal{J}(z) = \mathcal{J}(y), \mathcal{L}(z) = \mathcal{L}(y)$, and

$$F(z, u) \equiv (A + \lambda DE(y))z - Nu.$$

Hence F is differentiable at (y, u) and $F'(y, u) = (A + \lambda DE(y), -N)$.

Suppose that there is an $i \in \mathcal{K}(y)$. Then for any $\epsilon > 0$, we have

$$E(y + \epsilon e_i)(y + \epsilon e_i) = E(y)y + \epsilon e_i$$

and

$$E(y - \epsilon e_i)(y - \epsilon e_i) = E(y)y.$$

It follows that

$$\begin{aligned} &F(y + \epsilon e_i, u) - F(y, u) \\ &= \epsilon A e_i + \lambda D(E(y + \epsilon e_i)(y + \epsilon e_i) - E(y)y) \\ &= (A + \lambda D)(\epsilon e_i) \end{aligned}$$

and

$$\begin{aligned} &F(y - \epsilon e_i, u) - F(y, u) \\ &= -\epsilon A e_i + \lambda D(E(y - \epsilon e_i)(y - \epsilon e_i) - E(y)y) \\ &= -\epsilon A e_i. \end{aligned}$$

This shows that

$$\lim_{\epsilon\downarrow 0}\frac{F(y+\epsilon e_i,u)-F(y,u)}{\epsilon}\neq -\lim_{\epsilon\downarrow 0}\frac{F(y-\epsilon e_i,u)-F(y,u)}{\epsilon}.$$

Hence F is not differentiable with respect to y at (y,u). Consequently, if F is differentiable at (y,u), then $\mathcal{K}(y)=\emptyset$.

(ii) If $\mathcal{K}(y(u))=\emptyset$, the differentiability of $y(\cdot)$ follows directly from the first part of this theorem and the implicit function theorem, Theorem 5.2.4 in [10].

Now we consider the case that $\mathcal{K}(y(u))\neq\emptyset$ and (2.1) holds.

In order to simplify the notation, for a given $\bar{u}\in R^n$, we denote the unique vector $y(\bar{u})\in R^n$ by $\bar{y}$ and the associated index sets by

$$\mathcal{J}=\mathcal{J}(\bar{y}),\ \mathcal{K}=\mathcal{K}(\bar{y}),\ \mathcal{L}=\mathcal{L}(\bar{y}).$$

By the continuity of $y(\cdot)$, there is a neighborhood $\mathcal{N}$ of $\bar{u}$ such that for all $w\in\mathcal{N}$ we have

$$y_{\mathcal{J}}(w)>0\quad\text{and}\quad y_{\mathcal{L}}(w)<0,$$

which implies that $\mathcal{J}\subseteq\mathcal{J}(w)$ and $\mathcal{L}\subseteq\mathcal{L}(w)$. Assume that for a vector $w\in\mathcal{N}$ there is a nonempty subset $\mathcal{K}_1$ of $\mathcal{K}$ such that $\mathcal{J}\cup\mathcal{K}_1=\mathcal{J}(y(w))$. From (2.1) and the equality

$$N(w-\bar{u})=(A+\lambda DE(\bar{y}))(y(w)-\bar{y})+\lambda D(E(y(w))-E(\bar{y}))y(w),$$

we obtain

$$\begin{aligned}
0&=((A+\lambda DE(\bar{y}))^{-1}N(w-\bar{u}))_{\mathcal{K}_1}\\
&=(y(w)-\bar{y}+\lambda(A+\lambda DE(\bar{y}))^{-1}D(E(y(w))-E(\bar{y}))y(w))_{\mathcal{K}_1}\\
&=(I_{\mathcal{K}_1\mathcal{K}_1}+\lambda(A+\lambda DE(\bar{y}))^{-1}_{\mathcal{K}_1\mathcal{K}_1}(DE(y(w)))_{\mathcal{K}_1\mathcal{K}_1})y(w)_{\mathcal{K}_1},
\end{aligned}$$

where the last equality uses that $E_{ii}(\bar{y})=0$ for $i\in\mathcal{K}_1$ and

$$E_{ii}(y(w))-E_{ii}(\bar{y})=0\quad\text{for}\quad i\notin\mathcal{K}_1.$$

This implies that $y(w)_{\mathcal{K}_1}=0$, since $\lambda(A+\lambda DE(\bar{y}))^{-1}_{\mathcal{K}_1\mathcal{K}_1}$ and $(DE(y(w)))_{\mathcal{K}_1\mathcal{K}_1}$ are symmetric positive definite. This is a contradiction to $\mathcal{K}_1\subset\mathcal{J}(y(w))$. Hence we have $\mathcal{J}(y(w))=\mathcal{J}$, which gives $E(y(w))=E(\bar{y})$. The results ensure that the solution function $y(\cdot)$ in the neighborhood $\mathcal{N}$ can be expressed by

$$y(w)=(A+\lambda DE(\bar{y}))^{-1}Nw.$$

Hence $y(\cdot)$ is differentiable at $\bar{u}$ and

$$y'(\bar{u})=(A+\lambda DE(\bar{y}))^{-1}N.$$

Conversely, we assume that $y(\cdot)$ is differentiable at $\bar{u}$. According to the positive definite property of $A+\lambda DE(\bar{y})$, for any $h\in R^m$ the system

$$(A+\lambda DE(\bar{y}))z+\phi(z)=Nh \tag{2.2}$$

has a unique solution where

$$\phi_i(z)=\begin{cases}0, & i\in\mathcal{J}\cup\mathcal{L},\\ \lambda D_{ii}\max(0,z_i), & i\in\mathcal{K}.\end{cases}$$

Letting $t > 0$ be sufficiently small, we can therefore be assured that

$$(A + \lambda DE(\bar{y}))y(\bar{u} + th) + \phi(y(\bar{u} + th)) = N(\bar{u} + th).$$

Moreover, from $\bar{y}_{\mathcal{K}} = 0$, we have

$$\phi(y(\bar{u} + th)) - \phi(\bar{y}) = \phi(y(\bar{u} + th)) = \phi(y(\bar{u} + th) - \bar{y}).$$

Hence we find

$$(A + \lambda DE(\bar{y}))(y(\bar{u} + th) - \bar{y}) + \phi(y(\bar{u} + th) - \bar{y}) = N(\bar{u} + th) - N\bar{u} = tNh.$$

Note that $(A + \lambda DE(\bar{y}))(tz) + \phi(tz) = tNh$ for $t > 0$. This establishes that the unique solution of system (2.2) is the directional derivative $y'(\bar{y}; h)$ of $y(\cdot)$ along a direction $h \in R^m$ at $\bar{u}$, since

$$y(\bar{u} + th) - y(\bar{u}) = ty'(\bar{u}; h)$$

for sufficiently small $t > 0$, which gives

$$\lim_{t \downarrow 0} \frac{y(\bar{u} + th) - y(\bar{u})}{t} = y'(\bar{u}; h).$$

Moreover, the differentiability of $y(\cdot)$ at $\bar{u}$ implies

$$y'(\bar{u})h = y'(\bar{u}; h) = -y'(\bar{u}; -h),$$

from which we obtain that

$$(A + \lambda DE(\bar{y}))y'(\bar{u}; h) + \phi(y'(\bar{u}; h)) = Nh$$

and

$$-(A + \lambda DE(\bar{y}))y'(\bar{u}; h) + \phi(-y'(\bar{u}; h)) = -Nh.$$

It follows that

$$\phi(y'(\bar{u}; h)) + \phi(-y'(\bar{u}; h)) = 0.$$

Since ϕ is nonnegative, we have $\phi(y'(\bar{u}; h)) = 0$. Consequently, we obtain

$$y'(\bar{u}; h) = (A + \lambda DE(\bar{y}))^{-1}Nh = y'(\bar{u})h \quad \text{for all } h \in R^m,$$

which implies

$$y'(\bar{u}) = (A + \lambda DE(\bar{y}))^{-1}N.$$

Now we show $(y'(\bar{u}))_{\mathcal{K}} = 0$.

We have found that in a neighborhood $\mathcal{N}$ of $\bar{u}$, the function $y(\cdot)$ is linear and can be expressed by

$$y(w) = \bar{y} + (A + \lambda DE(\bar{y}))^{-1}N(w - \bar{u}) \quad \text{for all} \quad w \in \mathcal{N}. \tag{2.3}$$

From (2.3) and the equalities

$$N(w - \bar{u}) = (A + \lambda DE(\bar{y}))(y(w) - \bar{y}) + \lambda D(E(y(w)) - E(\bar{y}))y(w)$$

we obtain

$$\lambda(A+\lambda DE(\bar{y}))^{-1}D(E(y(w))-E(\bar{y}))y(w)=0.$$

Since $\lambda(A+\lambda DE(\bar{y}))^{-1}D$ is nonsingular, this implies that $(E(y(w))-E(\bar{y}))y(w)=0$, that is, $y_i(w)\le 0$ for all $i\in\mathcal{K}$.

If there is an $i\in\mathcal{K}$ such that $y_i(w)<0$, we can choose $t>0$ sufficiently small such that

$$\tilde{w}=\bar{u}-t(w-\bar{u})\in\mathcal{N}.$$

Then from the linearity of $y(\cdot)$ in the neighborhood $\mathcal{N}$, we get

$$y(\tilde{w})=\bar{y}-t(y(w)-\bar{y})$$

and

$$\lambda(A+\lambda DE(\bar{y}))^{-1}D(E(y(\tilde{w}))-E(\bar{y}))y(\tilde{w})=0.$$

However, $((E(y(\tilde{w}))-E(\bar{y}))y(\tilde{w}))_i=y_i(\tilde{w})>0$. This is a contradiction. Therefore, we have that $y_{\mathcal{K}}(\cdot)\equiv 0$ in the neighborhood $\mathcal{N}$, and thus $(y'(\bar{u}))_{\mathcal{K}}=0$.

Since $\bar{u}$ is arbitrarily chosen, we obtain the deserved result. $\square$

A function $f:R^m\to R^n$ is called *piecewise linear* if there exists a finite number of linear functions $f^{(i)}:R^m\to R^n$, $i\in\{1,\dots,\ell\}$, such that the active index set $\{i|f(u)=f^{(i)}(u)\}$ is nonempty for every $u\in R^m$.

Theorem 2.1 states that there exists a finite number of linear functions

$$y^{(1)}u=A^{-1}Nu,\quad y^{(i)}u=\left(A+\lambda D\sum_{j\in\mathcal{I}_i}e_je_j^T\right)^{-1}Nu,$$

$\mathcal{I}_i\subseteq\{1,2,\dots,n\}$, such that the index set $\{i\mid y(u)=y^{(i)}(u)\}$ is nonempty for every $u\in R^m$. Hence $y(\cdot)$ is a piecewise linear function and the number of pieces is not more than 2^n.

The following example illustrates the differentiability of $y(\cdot)$ in the two cases of Theorem 2.1.

Example 2.1. Let $n=2,m=1,\lambda=1,b=1,D=I$,

$$A=\begin{pmatrix}2&-1\\-1&2\end{pmatrix},\qquad\text{and}\qquad N=\begin{pmatrix}3\\-1\end{pmatrix}.$$

The solution function $y(\cdot)$ can be given explicitly as

$$y(u)=\begin{cases}\begin{pmatrix}u\\0\end{pmatrix}, & u\ge 0,\\[2ex] \begin{pmatrix}5u/3\\u/3\end{pmatrix}, & u<0.\end{cases}$$

The solution function $y(\cdot)$ is differentiable at every point in R except $u=0$. Moreover, $2\in\mathcal{K}(y(u))$ for $u>0$. Let

$$y^{(1)}(u)=A^{-1}Nu=\begin{pmatrix}5/3\\1/3\end{pmatrix}u,\quad y^{(2)}(u)=(A+e_1e_1^T)^{-1}Nu=\begin{pmatrix}1\\0\end{pmatrix}u,$$

$$y^{(3)}(u) = (A + e_2 e_2^T)^{-1} N u = \begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix} u, \quad y^{(4)}(u) = (A + I)^{-1} N u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

We have

$$\begin{array}{ll} y(u) = y^{(2)}(u) = y^{(4)}(u), & u > 0, \\ y(u) = y^{(1)}(u), & u < 0, \\ y(u) = y^{(1)}(u) = y^{(2)}(u) = y^{(3)}(u) = y^{(4)}(u), & u = 0. \end{array}$$

Hence y is a piecewise linear function and the number of pieces is less than 2^n. It is interesting to notice that $(A+I)^{-1}N = (A+E(y(u)))^{-1}N$ for $u > 0$. We can explain it theoretically.

The generalized Jacobian [6] of the function $\max(0, \cdot)$ at a point y is the convex hull of the set defined by a finite number of matrices:

$$\partial \max(0, y) = \mathrm{co} \left\{ \{E(y)\} \cup \left\{ E(y) + \sum_{i \in \mathcal{I}} e_i e_i^T \quad | \quad \mathcal{I} \subseteq \mathcal{K}(y) \right\} \right\}.$$

LEMMA 2.2. *If $\mathcal{K}(y) \neq \emptyset$ and $((A + \lambda D E(y))^{-1} N)_{\mathcal{K}(y)} = 0$, then for any $W \in \partial \max(0, y)$, we have*

$$(A + \lambda D W)^{-1} N = (A + \lambda D E(y))^{-1} N.$$

Proof. For a fixed point $y \in R^n$ which has r zero components, any element $W \in \partial \max(0, y)$ can be expressed by

$$W = E(y) + \sum_{j=1}^{r} \alpha_j \sum_{i \in \mathcal{I}_j} e_i e_i^T,$$

where $1 \geq \alpha_j \geq 0$ and $\mathcal{I}_j$ are subsets of $\mathcal{K}(y)$. Let

$$V = (A + \lambda D E(y))^{-1} N \quad \text{and} \quad U = (A + \lambda D W)^{-1} N. \tag{2.4}$$

We set $B = A + \lambda D E(y)$, $C = \lambda D (W - E(y))$, $\mathcal{K} = \mathcal{K}(y)$, $\mathcal{M} = \mathcal{J}(y) \cup \mathcal{L}(y)$. From (2.4), we get

$$\begin{pmatrix} B_{\mathcal{M}\mathcal{M}} & B_{\mathcal{M}\mathcal{K}} \\ B_{\mathcal{K}\mathcal{M}} & B_{\mathcal{K}\mathcal{K}} \end{pmatrix} \begin{pmatrix} V_{\mathcal{M}} \\ V_{\mathcal{K}} \end{pmatrix} = \begin{pmatrix} B_{\mathcal{M}\mathcal{M}} & B_{\mathcal{M}\mathcal{K}} \\ B_{\mathcal{K}\mathcal{M}} & (B + C)_{\mathcal{K}\mathcal{K}} \end{pmatrix} \begin{pmatrix} U_{\mathcal{M}} \\ U_{\mathcal{K}} \end{pmatrix}.$$

From $V_{\mathcal{K}} = 0$, we have

$$B_{\mathcal{M}\mathcal{M}} (V_{\mathcal{M}} - U_{\mathcal{M}}) - B_{\mathcal{M}\mathcal{K}} U_{\mathcal{K}} = 0 \tag{2.5}$$

and

$$B_{\mathcal{K}\mathcal{M}} (V_{\mathcal{M}} - U_{\mathcal{M}}) - (B + C)_{\mathcal{K}\mathcal{K}} U_{\mathcal{K}} = 0. \tag{2.6}$$

Since B is symmetric positive definite, $B_{\mathcal{M}\mathcal{M}}$ is nonsingular. Thus (2.5) and (2.6) yield

$$(B_{\mathcal{K}\mathcal{M}} B_{\mathcal{M}\mathcal{M}}^{-1} B_{\mathcal{M}\mathcal{K}} - (B + C)_{\mathcal{K}\mathcal{K}}) U_{\mathcal{K}} = 0. \tag{2.7}$$

The submatrix $B_{\mathcal{KM}}B^{-1}_{\mathcal{MM}}B_{\mathcal{MK}} - (B+C)_{\mathcal{KK}}$ is the Schur complement of the nonsingular matrix

$$\begin{pmatrix} B_{\mathcal{MM}} & B_{\mathcal{MK}} \\ B_{\mathcal{KM}} & (B+C)_{\mathcal{KK}} \end{pmatrix},$$

and thus it is nonsingular. It follows from (2.7) that $U_{\mathcal{K}} = 0$. Moreover, (2.5) and $U_{\mathcal{K}} = 0$ imply $V_{\mathcal{M}} = U_{\mathcal{M}}$. We complete the proof. □

LEMMA 2.3. *For any $W \in \partial \max(0, y)$, the following two systems are equivalent:*

$$\begin{pmatrix} H(y - y_d) + As + \lambda DWs \\ \alpha M(u - u_d) - N^T s + t \\ Ay + \lambda D \max(0, y) - Nu \\ \min(t, b - u) \end{pmatrix} = 0 \tag{2.8}$$

and

$$\begin{pmatrix} ((A + \lambda DW)^{-1} N)^T H(y(u) - y_d) + \alpha M(u - u_d) + t \\ \min(t, b - u) \end{pmatrix} = 0. \tag{2.9}$$

Proof. Note that $e_i^T \bar{y} = 0$ for $i \in \mathcal{K}(\bar{y})$, and any element $W \in \partial \max(0, \bar{y})$ can be expressed by

$$W = E(\bar{y}) + \sum_{j=1}^{r} \alpha_j \sum_{i \in \mathcal{I}_j} e_i e_i^T,$$

where $1 \le r \le n, 0 \le \alpha_j \le 1, \mathcal{I}_j \subseteq \mathcal{K}(\bar{y})$. Hence we have

$$\max(0, \bar{y}) = E(\bar{y})\bar{y} = W\bar{y} \quad \text{for any} W \in \partial \max(0, \bar{y}). \tag{2.10}$$

Suppose that $(\bar{y}, \bar{u}, s, t)$ satisfies (2.8). Then we have $y(\bar{u}) = \bar{y}$ and

$$\begin{aligned} &((A + \lambda DW)^{-1} N)^T H(y(\bar{u}) - y_d) + \alpha M(\bar{u} - u_d) + t \\ &= -((A + \lambda DW)^{-1} N)^T (A + \lambda DW)s + \alpha M(\bar{u} - u_d) + t \\ &= -N^T s + \alpha M(\bar{u} - u_d) + t \\ &= 0. \end{aligned}$$

Suppose that $(\bar{u}, t)$ satisfies (2.9). By the definition of $y(\cdot)$ and (2.10), we have $A\bar{y} + \lambda D \max(0, \bar{y}) - N\bar{u} = 0$ with $\bar{y} = y(\bar{u})$. Let $s = -(A + \lambda DW)^{-1} H(\bar{y} - y_d)$. We get

$$H(\bar{y} - y_d) + (A + \lambda DW)s = 0$$

and

$$\begin{aligned} \alpha M(\bar{u} - u_d) - N^T s + t &= \alpha M(\bar{u} - u_d) + t \\ &+ N^T (A + \lambda DW)^{-1} H(\bar{y} - y_d) = 0. \end{aligned} \qquad \square$$

3. First order conditions. In this section we show the relations between (1.2), (1.3), (1.4), and (1.5). To simplify the notation, we denote, respectively, the objective functions of (1.2) and (1.3) by

$$f(y,u) = \frac{1}{2}(y-y_d)^T H(y-y_d) + \frac{\alpha}{2}(u-u_d)^T M(u-u_d)$$

and

$$\theta(u) = \frac{1}{2}(y(u)-y_d)^T H(y(u)-y_d) + \frac{\alpha}{2}(u-u_d)^T M(u-u_d).$$

From Lemma 2.3 and $E(y) \in \partial \max(0,y)$, we can see that (1.4) and (1.5) are equivalent.

THEOREM 3.1. *If* $(\bar{y},\bar{u},\bar{s},\bar{t})$ *is a solution of* (1.4), *then* $(\bar{u},\bar{t})$ *is a solution of* (1.5). *Conversely, if* $(\bar{u},\bar{t})$ *is a solution of* (1.5), *then* $(y(\bar{u}),\bar{u},-(A+\lambda DE(y(\bar{u})))^{-1}H(y(\bar{u})-y_d),\bar{t})$ *is a solution of* (1.4).

Now we show that (1.4) implies (1.2) and that (1.3) implies (1.5) under certain conditions.

THEOREM 3.2. *If* (y^*,u^*,s^*,t^*) *is a solution of* (1.4), *then all feasible points* (y,u) *of* (1.2) *satisfy*

$$f(y,u) \geq f(y^*,u^*) + \lambda(D(E(y)-E(y^*))y)^T s^*.$$

Moreover, if either $\mathcal{K}(y^*) = \emptyset$ *or* $((A+\lambda DE(y^*))^{-1}N)_{\mathcal{K}(y^*)} = 0$, *then* (y^*,u^*) *is a strict local optimal solution of* (1.2).

Proof. The objective function in (1.2) is quadratic. If (y^*,u^*,s^*,t^*) is a solution of (1.4), then we have

$$\begin{aligned}
&f(y,u) - f(y^*,u^*)\\
&= (y-y^*)^T H(y^*-y_d) + \alpha(u-u^*)^T M(u^*-u_d)\\
&\qquad + \frac{1}{2}(y-y^*)^T H(y-y^*) + \frac{\alpha}{2}(u-u^*)^T M(u-u^*)\\
&\geq (y-y^*)^T H(y^*-y_d) + \alpha(u-u^*)^T M(u^*-u_d)\\
&\geq (y-y^*)^T H(y^*-y_d) + \alpha(u-u^*)^T M(u^*-u_d) + (u-b)^T t^*\\
&= -(y-y^*)^T (A+\lambda DE(y^*))s^* + (u-u^*)^T(N^T s^* - t^*) + (u-u^*+u^*-b)^T t^*\\
&= -(y-y^*)^T (A+\lambda DE(y^*))s^* + (u-u^*)^T N^T s^*\\
&= -((A+\lambda DE(y^*))(y-y^*) - N(u-u^*))^T s^*\\
&= \lambda(D(E(y)-E(y^*))y)^T s^*,
\end{aligned}$$

where the second inequality uses $u \leq b$ and $t^* \geq 0$, the third equality uses $(u^*-b)^T t^* = 0$, and the fifth equality uses $(A+\lambda DE(y))y = Nu$.

If $\mathcal{K}(y^*) = \emptyset$ or $((A+\lambda DE(y^*))^{-1}N)_{\mathcal{K}(y^*)} = 0$, by Theorem 2.1, there is a neighborhood $\mathcal{N}_y$ of y^* such that for all *feasible* points $y \in \mathcal{N}_y$, we have $y = (A+\lambda DE(y^*))^{-1}Nu$ and

$$(E(y)-E(y^*))y = (E(y)-E(y^*))(A+\lambda DE(y^*))^{-1}Nu = 0.$$

Hence (y^*,u^*) is a local optimal solution of (1.2). Moreover, the first inequality above is strict if $y \neq y^*$, which implies that (y^*,u^*) is a strict local optimal solution. ∎

THEOREM 3.3. *Suppose that u^* is a local optimal solution of* (1.3). *If either $\mathcal{K}(y(u^*)) = \emptyset$ or $((A + \lambda DE(y(u^*)))^{-1}N)_{\mathcal{K}(y(u^*))} = 0$, then there is t^* such that (u^*, t^*) is a solution of* (1.5).

Proof. By Theorem 2.1, the function $y(\cdot)$ is differentiable at u^* and can be expressed by

$$y(u) = (A + \lambda DE(y^*))^{-1}Nu$$

in a neighborhood $\mathcal{N}_u$ of u^*. The gradient of θ at u^* is

$$\theta'(u^*) = ((A + \lambda DE(y^*))^{-1}N)^T H(y(u^*) - y_d) + \alpha M(u^* - u_d).$$

Let $W = (A + \lambda DE(y^*))^{-1}N$. For all $u \in \mathcal{N}_u$ we have

$$\theta(u) = \theta(u^*) + \theta'(u^*)^T(u - u^*) + (u - u^*)(W^T HW + \alpha M)(u - u^*) \geq \theta(u^*),$$

which implies that there is a sufficiently small $\epsilon > 0$ such that if $\|u - u^*\| \leq \epsilon$, then

$$\theta'(u^*)^T(u - u^*) \geq 0. \tag{3.1}$$

By choosing m feasible points

$$u_j^{(i)} = \begin{cases} u_i^* - \epsilon, & j = i, \\ u_j^*, & j \neq i, \end{cases}$$

for $i = 1, 2, \ldots m$, we find $\theta'(u^*) \leq 0$. From $u^* \leq b$, this gives

$$\theta'(u^*)^T(b - u^*) \leq 0.$$

If $\theta'(u^*)^T(b - u^*) < 0$, then there is an $i(1 \leq i \leq m)$ such that $\theta'_i(u^*) < 0$ and $b_i - u_i^* > 0$. We define a feasible point $\tilde{u}$ by

$$\tilde{u}_i = u_i^* + \min(\epsilon, b_i - u_i^*), \quad \tilde{u}_j = u_j^* \ (j \neq i).$$

Then we have

$$\theta'(u^*)^T(\tilde{u} - u^*) < 0.$$

This contradicts (3.1). Hence we obtain $\theta'(u^*)^T(b - u^*) = 0$. Let $t^* = -\theta'(u^*)$. Then (u^*, t^*) is a solution of (1.5). We complete the proof. ∎

COROLLARY 3.4.

1. *Let (y^*, u^*, s^*, t^*) be a solution of* (1.4). *If $y^* = y_d$, then (y^*, u^*) is a global optimal solution of* (1.2).
2. *Let u^* be a local optimal solution of* (1.3). *If $y(u^*) = y_d$, then $(u^*, -\alpha M(u^* - u_d))$ is a solution of* (1.5). *Moreover, u^* is a global optimal solution of* (1.3).

Proof. 1. From the first equality of (1.4), we find that $s^* = 0$. Then the result is derived from Theorem 3.2.

2. By the relation between (1.2) and (1.3), (y_d, u^*) is a local optimal solution of (1.2) and

$$f'(y_d, u^*)^T(u - u^*) = \alpha(u - u^*)^T M(u^* - u_d) \geq 0$$

for u being sufficiently close to u^*. Let $t^* = \alpha M(u^* - u_d)$. We can show that $\max(t^*, b - u^*) = 0$ by using the same technique in the proof for Theorem 3.3. Moreover, by Theorem 3.1 and part 1 of this corollary, u^* is a global optimal solution of (1.3). ∎

4. Final remarks. This paper presents new theoretical results on the first order conditions for the discretization problem arising from nonsmooth constrained optimal control problems. The results generalize existing results for smooth discretized constrained optimal control problems and answer some open questions on first order conditions for two kinds of discretization problems. Theorem 3.1 states that the first order conditions for problem (1.2) are equivalent to the first order conditions for problem (1.3). Theorems 3.2 and 3.3 show that the condition $\mathcal{K}(y) = \emptyset$ and $((A + \lambda DE(y))^{-1}N)_{\mathcal{K}(y)} = 0$ can be used to verify a local optimal point satisfies the first order conditions and that a point satisfying the first order conditions is a local optimal solution. Notice that $((A + \lambda DE(y))^{-1}N)_{\mathcal{K}(y)} = 0$ does not imply that $F(y, u) = Ay + \lambda D \max(0, y) - Nu$ is differentiable at (y, u).

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