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Rank and eigenvalues of a supersymmetric tensor, the multivariate homogeneous polynomial and the algebraic hypersurface it defines

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Abstract

A real n-dimensional homogeneous polynomial $f(x)$ of degree m and a real constant c define an algebraic hypersurface S whose points satisfy $f(x) = c$. The polynomial f can be represented by Ax^m where A is a real mth order n-dimensional supersymmetric tensor. In this paper, we define rank, base index and eigenvalues for the polynomial f, the hypersurface S and the tensor A. The rank is a nonnegative integer r less than or equal to n. When r is less than n, A is singular, f can be converted into a homogeneous polynomial with r variables by an orthogonal transformation, and S is a cylinder hypersurface whose base is r-dimensional. The eigenvalues of f, A and S always exist. The eigenvectors associated with the zero eigenvalue are either recession vectors or degeneracy vectors of positive degree, or their sums. When $c \neq 0$, the eigenvalues with the same sign as c and their eigenvectors correspond to the characterization points of S, while a degeneracy vector generates an asymptotic ray for the base of S or its conjugate hypersurface. The base index is a nonnegative integer d less than m. If $d = k$, then there are nonzero degeneracy vectors of degree $k-1$, but no nonzero degeneracy vectors of degree k. A linear combination of a degeneracy vector of degree k and a degeneracy vector of degree j is a degeneracy vector of degree $k+j-m$ if $k+j \geq m$. Based upon these properties, we classify such algebraic hypersurfaces in the nonsingular case into ten classes.

Keywords: Algebraic hypersurface; Rank; Eigenvalue; Homogeneous polynomial; Supersymmetric tensor; Base index

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1. Introduction

The study of algebraic hypersurfaces is an important topic in algebraic geometry and topology (Cox et al., 1998; Sturmfels, 2002). A general algebraic hypersurface can be represented by

$$S = \{x \in \mathbf{R}^n : f(x) = 0\}, \tag{1}$$

where $f(x) \in \mathbf{R}[x_1, \ldots, x_n]$, $x = (x_1, \ldots, x_n)$, is a real n-dimensional polynomial of degree m. For general m and n, the classification of such algebraic hypersurfaces is not trivial. Newton's classification of cubic curves in the late 1600s was the first great success of analytic geometry apart from its role in calculus (Bix, 1998). Newton divided cubic curves into 72 species. Sterling identified four more species, and Jean-Paul de Gua de Malves found another two in 1740, giving a total of 78 species (Bix, 1998). The case of quartic curves is more complicated (Lawrence, 1972). In 2002, Korchagin and Weinberg in their 93 page paper (Korchagin and Weinberg, 2002) classified quartic curves to 516 classes. Instead of studying the general algebraic hypersurface defined by (1), in this paper, we study the algebraic hypersurface defined by

$$S = \{x \in \mathbf{R}^n : f(x) = c\}, \tag{2}$$

where $f(x) \in \mathbf{R}[x_1, \ldots, x_n]$, $x = (x_1, \ldots, x_n)$, is a real n-dimensional homogeneous polynomial of degree m and c is a real constant. This will be the first step for studying the general case (1).

A real n-dimensional homogeneous polynomial $f(x)$ of degree m can be represented by tensor operations of a real mth order n-dimensional supersymmetric tensor and the vector x. A real mth order n-dimensional supersymmetric tensor A consists of n^m real entries:

$$A_{i_1,\ldots,i_m} \in \mathbf{R},$$

which are invariant under any permutation of their indices, for $i_j = 1, \ldots, n$, $j = 1, \ldots, m$ (Kofidis and Regalia, 2002). Then the homogeneous polynomial $f(x)$ can be represented as

$$f(x) \equiv Ax^m := \sum_{i_1,\ldots,i_m=1}^{n} A_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m}, \tag{3}$$

where x^m can be regarded as an mth order n-dimensional rank-one tensor with entries $x_{i_1} \cdots x_{i_m}$ (De Lathauwer et al., 2000; Kofidis and Regalia, 2002; Zhang and Golub, 2001), and Ax^m is the inner product of A and x^m.

When $m = 2$, A is a real symmetric matrix. It has n real eigenvalues with corresponding eigenvectors. The rank r of A is a nonnegative integer less than or equal to n. When r is less than n, A is singular, f can be converted into a homogeneous quadratic polynomial with r variables by an orthogonal transformation, and S is a cylinder hypersurface whose base is r-dimensional. When $c \neq 0$, the quadratic hypersurface S surrounds a bounded region if and only if all the eigenvalues have the same sign as c. When $c \neq 0$, the vertices of S in the nonsingular case and the vertices of the base of S in the singular case are uniquely determined by c, m, the eigenvalues with the same sign as c, and the eigenvectors associated with these eigenvalues. We may use these properties to analyze and to classify such quadratic hypersurfaces.

Can these properties be extended to the general case such that m is an arbitrary positive integer? We explore this answer in this paper, and the answer is "yes".

Throughout this paper, we assume that m and n are two positive integers, and assume that the supersymmetric tensor A, the homogeneous polynomial $f(x)$ and the algebraic hypersurface S

are defined by (2) and (3). We use $\|\cdot\|$ to denote the Euclidean norm in $\mathbf{R}^n$. When we talk about S defined by (2), we assume that $c > 0$. If $c < 0$, we may discuss the conjugate hypersurface $\bar{S}$ of S, defined by

$$\bar{S} = \{x \in \mathbf{R}^n : f(x) = -c\}.$$

We do not miss the case that $c = 0$. In Section 5, we will discuss the structure of

$$S_0 = \{x \in \mathbf{R}^n : Ax^m = 0\}.$$

In Section 2, we define eigenvalues and eigenvectors for A, $f(x)$ and S. We show that they always exist and are invariant under orthogonal transformations. When $c > 0$, the distance from S and the closest point on S to the origin are given by c, the largest eigenvalue and its eigenvector. We show that when $c > 0$, the algebraic hypersurface S surrounds a bounded region if and only if all the eigenvalues are positive, i.e., A is positive definite. In this case, the largest distance from S and the farthest point on S to the origin are determined by c, the smallest eigenvalue and its eigenvector.

We define the rank of A, $f(x)$ and S in Section 3. We first define recession vectors of A, $f(x)$ and S. A vector $y \in \mathbf{R}^n$ is a recession vector of A, $f(x)$ and S if and only if $Ay = 0$. A linear combination of recession vectors is still a recession vector, i.e., the set of all the recession vectors forms a linear subspace of $\mathbf{R}^n$. We then define the rank of A, $f(x)$ and S. The rank is a nonnegative integer r less than or equal to n. When $r < n$, A is called singular, f can be converted into a homogeneous polynomial with r variables by an orthogonal transformation, and S is a cylinder hypersurface whose base is r-dimensional. A unit recession vector is an eigenvector associated with the zero eigenvalue but not vice versa. Furthermore, we show that a recession vector is perpendicular to any eigenvector associated with a nonzero eigenvalue.

In Section 4, we discuss asymptotic rays of S in the case that $c > 0$. We show that any unit vector $y \in S_0$ which is not a recession vector of S generates an asymptotic ray of S, or its conjugate hypersurface $\bar{S}$, or both. We also define the degree of an asymptotic ray there.

In Section 5, we discuss the structure of S_0. This case is important as the general algebraic hypersurface defined in (1) can be regarded as the intersection of the algebraic hypersurface

$$\{x \in \mathbf{R}^{n+1} : g(x) = 0\}$$

and the hyperplane

$$\{x \in \mathbf{R}^{n+1} : x_0 = 1\},$$

where $x = (x_0, x_1, \ldots, x_n)$, $g(x) \in \mathbf{R}[x_0, x_1, \ldots, x_n]$ is a real $(n+1)$-dimensional homogeneous polynomial of degree m.

In Section 5, we call a vector in the base of S_0 a degeneracy vector of A, $f(x)$ and S, and define its degree k for $0 \le k \le m-2$. A unit degeneracy vector of positive degree is also an eigenvector associated with the zero eigenvalue. We show that a linear combination of a degeneracy vector of degree k and a degeneracy vector of degree j is a degeneracy vector of degree $k+j-m$ if $k+j \ge m$. We also define the base index of A, $f(x)$ and S in Section 5. The base index is a nonnegative integer d less than m. If $d = k$, then there are degeneracy vectors of degree $k-1$ but no degeneracy vectors of degree k. If $d = 0$, then there are no asymptotic rays.

Based upon these properties, we classify the algebraic hypersurfaces defined by (2) in the nonsingular case into ten classes in Section 6.

When $c > 0$, a positive eigenvalue and its eigenvector correspond to a stationary point of the distance function from S to the origin. Such a point may be regarded as a characterization point of S. In Section 7, we present conditions under which such a point is a local minimizer or maximizer of the distance function.

In Section 8, we discuss another property of eigenvalues and eigenvectors. Recently, researchers studied rank-one tensors and best rank-one approximation to tensors (De Lathauwer et al., 2000; Kofidis and Regalia, 2002; Zhang and Golub, 2001). Rank-one decomposition of tensors is important in signal processing (Comon, 2000; De Lathauwer et al., 2000; Kofidis and Regalia, 2002; Zhang and Golub, 2001). We show that the largest absolute value of the eigenvalues and its corresponding eigenvector of A form the best rank-one supersymmetric approximation to A.

Some final remarks are given in Section 9.

2. Eigenvalues and eigenvectors

We call a real number λ an **eigenvalue** of A, $f(x)$ and S, and a real vector $x \in \mathbf{R}^n$ an *eigenvector* of A, $f(x)$ and S, associated with the eigenvalue λ, respectively, if they are solutions of the following system:

$$\begin{cases} Ax^{m-1} = \lambda x \\ x^{\mathrm{T}} x = 1. \end{cases} \tag{4}$$

We say that A is *positive semidefinite* if $Ax^m \geq 0$ for all $x \in \mathbf{R}^n$. We say that A is *positive definite* if $Ax^m > 0$ for all $x \in \mathbf{R}^n$ and $x \neq 0$. If $-A$ is positive semidefinite (positive definite), then we say that A is negative semidefinite (negative definite). Clearly, only when m is even, A can be positive or negative definite.

Theorem 1. *We have the following conclusions on eigenvalues and eigenvectors of A, $f(x)$ and S:*

(a) *Eigenvalues and eigenvectors of A, $f(x)$ and S exist.*
(b) *When m is even, the supersymmetric tensor A is positive definite (positive semidefinite) if and only if all the eigenvalues are positive (nonnegative).*
(c) *Assume that $c > 0$. Let the largest eigenvalue be denoted as $\lambda_{\max}$. If $\lambda_{\max} \leq 0$, then S is empty. Otherwise, S is not empty and the distance from S to the origin is*

$$\sigma_{\min} = \left(\frac{c}{\lambda_{\max}} \right)^{\frac{1}{m}},$$

and this distance occurs at the point $y = \sigma_{\min} x$, where x is an eigenvector associated with $\lambda_{\max}$.
(d) *Assume that $c > 0$. Denote the smallest eigenvalue as $\lambda_{\min}$. If A is positive definite, then the largest distance from S to the origin is*

$$\sigma_{\max} = \left(\frac{c}{\lambda_{\min}} \right)^{\frac{1}{m}},$$

and this distance occurs at the point $y = \sigma_{\max} x$, where x is an eigenvector associated with $\lambda_{\min}$.
(e) *Assume that $c > 0$. The algebraic hypersurface S surrounds a bounded region if and only if m is even and all the eigenvalues are positive, i.e., A is positive definite.*

Proof. (a) We see that (4) is the optimality condition of

$$\max \left\{ Ax^m : \sum_{i=1}^{n} x_i^2 = 1, x \in \mathbf{R}^n \right\} \tag{5}$$

and

$$\min \left\{ Ax^m : \sum_{i=1}^{n} x_i^2 = 1, x \in \mathbf{R}^n \right\}. \tag{6}$$

As the feasible set is compact and the function Ax^m is continuous, the global maximizer and minimizer always exist. This shows that (4) always has real solutions, i.e., A, $f(x)$ and S always have eigenvalues.

(b) Since A is positive definite (positive semidefinite) if and only if the optimal value of (6) is positive (nonnegative), we have this conclusion.

(c) By (4), we have

$$Ax^m = \lambda. \tag{7}$$

If $\lambda_{\max} \le 0$, then $Ax^m \le 0$ for all $x \in \mathbf{R}^n$. This implies that S is empty as $c > 0$. We now assume that $\lambda_{\max} > 0$.

The distance from S to the origin can be found by

$$\min \left\{ \sum_{i=1}^{n} y_i^2 : Ay^m = c, y \in \mathbf{R}^n \right\}. \tag{8}$$

The optimality condition of (8) is

$$\begin{cases} \mu A y^{m-1} = y \\ Ay^m = c. \end{cases} \tag{9}$$

We now show that a solution of (9) corresponds to a solution (λ, x) of (4), where $\lambda \neq 0$. Let (μ, y) be a solution of (9). Since $c > 0$, we have $y \neq 0$, and hence $\mu \neq 0$. Let

$$x = \frac{y}{\|y\|}.$$

Then

$$Ax^{m-1} = \frac{Ay^{m-1}}{\|y\|^{m-1}} = \frac{y}{\mu \|y\|^{m-1}} = \frac{x}{\mu \|y\|^{m-2}}.$$

Let

$$\lambda = \frac{1}{\mu \|y\|^{m-2}}. \tag{10}$$

Then (λ, x) is a solution of (4), where $\lambda \neq 0$.

Similarly, we may show that a solution (λ, x) of (4) corresponds to a solution of (9) if $\lambda \neq 0$.

Comparing (9) with (7), we have

$$\|y\| = \left(\frac{c}{\lambda}\right)^{\frac{1}{m}}, \tag{11}$$

and now the conclusion follows.

(d) This conclusion also follows from the relation of the solutions of (4), (5) and (9), as described above, and (11).

(e) If A is positive definite, then by (d), S surrounds a region which is contained in the compact set

$$\{y \in \mathbf{R}^n : \|y\| \leq \sigma_{\max}\}.$$

Since that f is homogeneous, S intersects with the ray

$$\{y \in \mathbf{R}^n : y = \sigma x, \sigma \geq 0\} \tag{12}$$

at most once, where x is a nonzero vector in $\mathbf{R}^n$. Suppose that S surrounds a bounded region. Then the origin has to be an interior point of that bounded region. Otherwise, let x be an interior point of that bounded region. Then S would intersect with the ray (12) at least twice, a contradiction. Thus, if S surrounds a bounded region, S would intersect with the ray (12) exactly once for any nonzero vector x in $\mathbf{R}^n$.

If A is not positive definite, i.e., one eigenvalue $\lambda \leq 0$, then S cannot intersect with the ray (12), where x is an eigenvector associated with the eigenvalue $\lambda \leq 0$. This implies that S does not surround a bounded region. ■

Motivated by the study of positive definiteness (Anderson et al., 1975; Bose and Kamt, 1974; Bose and Modaress, 1976; Bose and Newcomb, 1974; Hasan and Hasan, 1996; Jury and Mansour, 1981; Ku, 1965; Wang and Qi, 2005) of $f(x)$ defined in (3), We introduced in Qi (2005) the concepts of H-eigenvalues and Z-eigenvalues of an even-order real supersymmetric tensor A. When m is even, the eigenvalues introduced here are the same as the Z-eigenvalues introduced in Qi (2005). H-eigenvalues and their complex extensions have some good algebraic properties and are associated with the concept of hyperdeterminants (Cayley, 1845; Gelfand et al., 1994). But here we find that Z-eigenvalues and Z-eigenvectors have important geometric properties for such algebraic hypersurfaces, while H-eigenvalues and H-eigenvectors have no such properties. Also we find that the concepts and geometric properties of Z-eigenvalues and Z-eigenvectors can be extended to the case that m is odd. This also does not work for H-eigenvalues and H-eigenvectors. Hence, in this paper, we extend Z-eigenvalues and Z-eigenvectors to the case that m is odd and simply call them eigenvalues and eigenvectors, as we do not need to distinguish two different kinds of eigenvalues and eigenvectors here. For more discussion on Z-eigenvalues, see Qi (in press) and Ni et al. (in press). Independently, with a variational approach, Lim also defines eigenvalues of tensors in Lim (2005).

Theorems 1(a), (b) and 2 below have been given in Qi (2005) when m is even. For completeness, we give the proof of Theorem 1(a) and (b) here. Otherwise, the properties of eigenvalues and eigenvectors studied in this paper are all new.

Note that the concepts of eigenvalues and eigenvectors work for $m = 1$ or $n = 1$ too.

Let $P = (p_{ij})$ be an $n \times n$ real matrix. Define $B = P^m A$ as an mth order n-dimensional tensor with its entries as

$$B_{i_1,\ldots,i_m} = \sum_{j_1,\ldots,j_m=1}^{n} p_{i_1 j_1} \cdots p_{i_m j_m} A_{j_1,\ldots,j_m}.$$

It is easy to see that B is also a supersymmetric tensor. Furthermore, if P is nonsingular and $Q = P^{-1}$, then $A = Q^m B$.

If P is a real orthogonal matrix and $B = P^m A$, then $A = (P^{\mathrm{T}})^m B$. In this case, we say that A and B are *orthogonally similar*.

The following theorem was proved in Qi (2005). In Qi (2005), m is restricted to be even. But this theorem and its proof also hold when m is odd. Since the proof follows directly from the definition and is given in Qi (2005), we omit the proof here.

Theorem 2. *If supersymmetric tensors A and B are orthogonally similar, then they have the same eigenvalues. In particular, if $B = P^m A$, λ is an eigenvalue of A and x is an eigenvector of A associated with λ, where P is an $n \times n$ real orthogonal matrix, then λ is also an eigenvalue of B and $y = Px$ is an eigenvector of B associated with λ.*

3. Recession vectors and rank

For two subsets S and T of $\mathbf{R}^n$, we denote $W = S + T$ if

$$W = \{x + y : x \in S, y \in T\}.$$

If furthermore, we have $x^T y = 0$ for any $x \in S$ and $y \in T$, we denote that

$$W = S \dot{+} T.$$

We call a vector $y \in \mathbf{R}^n$ a *recession vector* of A, $f(x)$ and S if for any $x \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$,

$$A(x + \alpha y)^m = Ax^m. \tag{13}$$

For any $x, y \in \mathbf{R}^n$ and an integer k satisfying $0 \le k \le m$, we define

$$Ax^{m-k}y^k = \sum_{i_1,\ldots,i_m}^{n} A_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_{m-k}} y_{i_{m-k+1}} \cdots y_{i_m}.$$

Then we have

$$Ax^{m-k}y^k = Ay^k x^{m-k}, \tag{14}$$

$$Ax^{m-1}y = \left(Ax^{m-1}\right)^T y, \tag{15}$$

and

$$A(x + \alpha y)^m = \sum_{k=0}^{m} \binom{m}{k} \alpha^k Ax^{m-k}y^k, \tag{16}$$

for any $x, y \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$.

Also, for $1 \le k \le m - 1$, we may regard Ay^k as an $(m - k)$th order n-dimensional supersymmetric tensor, whose entries are:

$$(Ay^k)_{i_1,\ldots,i_{m-k}} = \sum_{i_{m-k+1},\ldots,i_m}^{n} A_{i_1,\ldots,i_m} y_{i_{m-k+1}} \cdots y_{i_m}.$$

It is not difficult to prove that

$$Ay^k = 0 \tag{17}$$

if $Ax^{m-k}y^k = 0$ for all $x \in \mathbf{R}^n$. If (17) holds, we also have $Ay^j = 0$ for $j = k + 1, \ldots, m$.

Theorem 3. *We have the following conclusions on recession vectors of A, $f(x)$ and S:*

(a) *A vector* $y \in \mathbf{R}^n$ *is a recession vector if and only if*

$$Ay = 0. \tag{18}$$

(b) *A linear combination of recession vectors is still a recession vector, i.e., the set of all the recession vectors forms a linear subspace* V_R *of* $\mathbf{R}^n$.

(c) *A unit recession vector is an eigenvector associated with the zero eigenvalue of* A, $f(x)$ *and* S.

(d) *A recession vector is perpendicular to any eigenvector associated with a nonzero eigenvalue.*

Proof. (a) Suppose that y is a recession vector of A, $f(x)$ and S, $x \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$ and $\alpha \neq 0$. By (16), we have

$$\sum_{k=1}^{m} \binom{m}{k} \alpha^k A x^{m-k} y^k = 0.$$

Dividing it by α, we have

$$\sum_{k=1}^{m} \binom{m}{k} \alpha^{k-1} A x^{m-k} y^k = 0.$$

Let $\alpha \to 0$. Then

$$Ax^{m-1}y = 0$$

for all $x \in \mathbf{R}^n$. This implies that (18) holds.

On the other hand, suppose that (18) holds. Then

$$Ay^j = 0 \tag{19}$$

for $j = 1, \ldots, m$. By (16), (13) holds for any $x \in \mathbf{R}^n$ and any $\alpha \in \mathbf{R}$. This implies that y is a recession vector. The proof of (a) is completed.

(b) This follows from (18) directly.

(c) This follows from (19) and (4).

(d) Suppose that y is a recession vector of A, $f(x)$ and S, x is an eigenvector associated with a nonzero eigenvalue λ of A, $f(x)$ and S, $\alpha \in \mathbf{R}$ and $\alpha \neq 0$. By (18), we have

$$Ax^{m-1}y = 0.$$

Since x is an eigenvector associated with a nonzero eigenvalue λ, by this, (14), (15) and (4), we have

$$\lambda x^{\mathrm{T}} y = \left(Ax^{m-1}\right)^{\mathrm{T}} y = Ax^{m-1}y = 0.$$

This shows that y is perpendicular to x. ■

We call V_R the *recession space* of A, $f(x)$ and S. Then there is another subspace V_B of $\mathbf{R}^n$ such that

$$\mathbf{R}^n = V_R \dot{+} V_B.$$

We call V_B the *base space* of A, $f(x)$ and S, and call the dimension r of V_B the rank of A, $f(x)$ and S. Let

$$S_B = S \cap V_B.$$

We call S_B the *base* of S. Then

$$S = V_R \dot{+} S_B.$$

Clearly, $0 \le r \le n$. When $r < n$, we say that A and $f(x)$ are *singular*.

Theorem 4. *When the rank r of A, $f(x)$ and S is less than n, the polynomial f can be converted into a homogeneous polynomial with r variables by an orthogonal transformation, and S is a cylinder hypersurface whose base is r-dimensional.*

Proof. Let $x^{(1)}, \ldots, x^{(n-r)}$ be an orthonormal basis of V_R. Let P be an $n \times n$ real orthogonal matrix whose last $n - r$ rows are $x^{(1)}, \ldots, x^{(n-r)}$. Then

$$f(x) = Ax^m = g(y) = By^m,$$

where $B = P^m A$ and $y = Px$. Then the last $n - r$ unit vectors $e^{(r+1)}, \ldots, e^{(n)}$ are recession vectors of B and $g(y)$. By (13), we see that $g(y)$ is a homogeneous polynomial with r variables $y_1, \ldots, y_r$. We now see that S is a cylinder hypersurface whose base S_B is r-dimensional. ■

Hence, if A and $f(x)$ are singular, we may convert $f(x)$ into a homogeneous polynomial with r variables by an orthogonal transformation, and study the r-dimensional base S_B of S.

4. Asymptotic rays

Suppose that $y \in \mathbf{R}^n$ and $y^T y = 1$. Assume that S is defined by (2) where $c > 0$. We say that the ray

$$L = \{\alpha y : \alpha \ge 0\} \tag{20}$$

is an *asymptotic ray* for S if $\alpha y \notin S$ for all $\alpha \ge 0$ and there is a sequence

$$\{u^{(j)} \in S : j = 1, 2, \ldots\}$$

such that $\|u^{(j)}\| \to \infty$ and the distance from $u^{(j)}$ to L tends to zero as $j \to \infty$, i.e.,

$$\lim_{j \to \infty} \left\| u^{(j)} - \left[\left(u^{(j)}\right)^T y \right] y \right\| = 0.$$

If L is an asymptotic ray for S or $\bar{S}$, then L is called an asymptotic ray for A and $f(x)$. Using the notions in projective geometry, an asymptotic ray is a tangent ray of S, which touches S at infinity (Edwards, 2003).

Clearly, if L is an asymptotic ray for A and $f(x)$, then

$$Ay^m = 0. \tag{21}$$

The questions are: (a) In which case does a unit vector y satisfying (21) give an asymptotic ray for A and $f(x)$? (b) Is there a measure on the asymptotic degree?

We first define the asymptotic degree. We say that the asymptotic ray L for S is of degree d if there is a sequence

$$\{u^{(j)} \in S : j = 1, 2, \ldots\}$$

such that $\|u^{(j)}\| \to \infty$, the distance from $u^{(j)}$ to L tends to zero as $j \to \infty$, and

$$\left\| u^{(j)} - \left[\left(u^{(j)} \right)^{\mathrm{T}} y \right] y \right\| = O\left(\left\| \frac{u^{(j)}}{\|u^{(j)}\|} - y \right\|^{d} \right),$$

where d is a positive number.

Theorem 5. *Suppose that $y \in \mathbf{R}^n$, $y^{\mathrm{T}} y = 1$ and y satisfies (21). Then*

(a) *The ray L, defined by (20), is an asymptotic ray for A and $f(x)$, if and only if y is not a recession vector of A, $f(x)$ and S.*
(b) *The ray L, defined by (20), is an asymptotic ray of degree $1 - \frac{1}{m}$, if and only if y is not an eigenvector of A, $f(x)$ and S, associated with the zero eigenvalue.*
(c) *The ray L, defined by (20), is an asymptotic ray of degree $1 - \frac{k}{m}$, where k is an integer, $2 \le k \le m - 1$, if and only if y is an eigenvector of A, $f(x)$ and S, associated with the zero eigenvalue,*

$$Ay^{m-k+1} = 0 \tag{22}$$

and

$$Ay^{m-k} \neq 0.$$

Proof. It is easy to see that if y is a recession vector of A, $f(x)$ and S, then the ray L, defined by (20), is not an asymptotic ray for A and $f(x)$. Suppose now that $y \in \mathbf{R}^n$, $y^{\mathrm{T}} y = 1$, $Ay^m = 0$ and y is not a recession vector of A, $f(x)$ and S. Then by Theorem 3, there is an integer k and $u \in \mathbf{R}^n$ such that $1 \le k \le m - 1$, $u^{\mathrm{T}} u = 1$, (22) holds and

$$Ay^{m-k} u^k \neq 0. \tag{23}$$

Then u and y are linearly independent since $Ay^m = 0$. This implies that

$$u - (u^{\mathrm{T}} y) y \neq 0. \tag{24}$$

Without loss of generality, we may assume that

$$Ay^{m-k} u^k > 0. \tag{25}$$

If $Ay^{m-k} u^k < 0$, we may use $\bar{S}$ instead of S in the following discussion.

Assume that $0 < \beta < 1$. Let

$$u^{(j)} = \alpha_j (y + \beta^j u) \in S$$

for $j = 1, 2, \ldots$. Then by (16) and the above assumptions,

$$c = A\left(u^{(j)}\right)^m = \alpha_j^m A(y + \beta^j u)^m = \alpha_j^m \left[\sum_{i=k}^{m} \binom{m}{i} \beta^{ij} Ay^{m-i} u^i \right],$$

i.e.,

$$\alpha_j = \left[\frac{c}{\sum_{i=k}^{m} \binom{m}{i} \beta^{ij} Ay^{m-i} u^i} \right]^{\frac{1}{m}}. \tag{26}$$

Then

$$\left\|u^{(j)} - \left[\left(u^{(j)}\right)^{\mathrm{T}} y\right] y\right\| = \left\|\alpha_j(y + \beta^j u) - \left[\alpha_j(1 + \beta^j u^{\mathrm{T}} y)\right] y\right\|$$
$$= \left\|\alpha_j \beta^j \left[u - (u^{\mathrm{T}} y) y\right]\right\|. \tag{27}$$

Combining (23), (25), (26) and (27), we have

$$\left\|u^{(j)} - \left[\left(u^{(j)}\right)^{\mathrm{T}} y\right] y\right\| = O\left(\beta^{\left(1-\frac{k}{m}\right)j}\right). \tag{28}$$

Note that

$$\|y + \beta^j u\| = \sqrt{1 + 2\beta^j y^{\mathrm{T}} u + \beta^{2j}} = 1 + \beta^j u^{\mathrm{T}} y + o(\beta^j).$$

Then

$$\left\|\frac{u^{(j)}}{\|u^{(j)}\|} - y\right\| = \frac{\left\|y\left[1 - \|y + \beta^j u\|\right] + \beta^j u\right\|}{\|y + \beta^j u\|} = \frac{\beta^j \left\|u - (u^{\mathrm{T}} y) y\right\| + o(\beta^j)}{1 + \beta^j u^{\mathrm{T}} y + o(\beta^j)}.$$

Combining this with (24), we have

$$\beta^j = O\left(\left\|\frac{u^{(j)}}{\|u^{(j)}\|} - y\right\|\right). \tag{29}$$

By (28) and (29), we have

$$\left\|u^{(j)} - \left[\left(u^{(j)}\right)^{\mathrm{T}} y\right] y\right\| = O\left(\left\|\frac{u^{(j)}}{\|u^{(j)}\|} - y\right\|^{1-\frac{k}{m}}\right). \tag{30}$$

This proves (a). When y is not an eigenvector of A, $f(x)$ and S, associated with the zero eigenvalue, $k = 1$. When y is an eigenvector of A, $f(x)$ and S, associated with the zero eigenvalue, $2 \le k \le m - 1$. By (30), we have (b) and (c). The proof is completed. ■

5. Degeneracy vectors and base index

In this section, we discuss the structure of S_0. The importance of S_0 has been explained in the introduction. Denote the base of S_0 as S_{0B}. Then $S_{0B} = S_0 \cap V_B$ and

$$S_0 = V_R \dot{+} S_{0B}.$$

For $0 \le k \le m - 2$, we call a vector $y \in V_B$ a degeneracy vector of degree k of A, $f(x)$ and S if

$$Ay^{m-k} = 0.$$

Thus, S_{0B} consists of degeneracy vectors of degree k for $0 \le k \le m - 2$.

We now define the base index d of A, $f(x)$ and S. If there are nonzero degeneracy vectors of degree $k - 1$ but no nonzero degeneracy vectors of degree k, then we define $d = k$. If there are no nonzero degeneracy vectors, let $d = 0$. Then the base index d is a nonnegative integer less than m. If $d = 0$, then there are no asymptotic rays.

The following theorem is about properties of degeneracy vectors.

Theorem 6. *We have the following conclusions on degeneracy vectors.*

(a) *If x is a degeneracy vector of degree k and $j < k$, then x is also a degeneracy vector of degree j.*
(b) *If x is a degeneracy vector of degree k and α is a real number, then αx is also a degeneracy vector of degree k.*
(c) *If A is positive or negative semidefinite, then any degeneracy vector is at least of degree* 1.
(d) *A linear combination of a degeneracy vector of degree k and a degeneracy vector of degree j is a degeneracy vector of degree $k + j - m$ if $k + j \geq m$.*

Proof. Conclusions (a) and (b) follow from the definition directly.

(c) If A is positive or negative definite, then the only degeneracy vector is zero. The conclusion holds naturally. Suppose now that A is positive semidefinite but not positive definite. If $f(x) = Ax^m = 0$ at x, then (6) attains its minimum at x. Thus $\nabla f(x) = mAx^{m-1} = 0$, i.e., if x is a degeneracy vector, its degree is at least 1. The case that A is negative semidefinite is similar.

(d) Suppose that $k + j \geq m$, x is a degeneracy vector of degree k and y is a degeneracy vector of degree j, α and β are two real numbers. Then we have

$$\begin{aligned}
A(\alpha x + \beta y)^{m-(k+j-m)} &= A(\alpha x + \beta y)^{2m-k-j} \\
&= \sum_{i=0}^{2m-k-j} \binom{2m-k-j}{i} \alpha^{2m-k-j-i} \beta^i A x^{2m-k-j-i} y^i \\
&= \sum_{i=0}^{m-j} \binom{2m-k-j}{i} \alpha^{2m-k-j-i} \beta^i (Ax^{m-k}) x^{m-j-i} y^i \\
&\quad + \sum_{i=1}^{m-k} \binom{2m-k-j}{m-j+i} \alpha^{m-k-i} \beta^{m-j+i} (Ay^{m-j}) x^{m-k-i} y^i \\
&= 0.
\end{aligned}$$

This completes the proof. ■

This theorem reveals that S_{0B} has a very special structure.

6. Classification of algebraic hypersurfaces

If A and $f(x)$ are singular, as discussed in Section 3, we may convert $f(x)$ into a homogeneous polynomial with r variables by an orthogonal transformation, and study the r-dimensional base of S. In this section, we assume that A and $f(x)$ are nonsingular, and $c \geq 0$. Note that here we allow $c = 0$ in general.

Let

$$S_{00} = \{x \in \mathbf{R}^n : Ax^{m-1} = 0\},$$
$$S_{01} = S_0 \setminus S_{00},$$
$$S_+ = \{x \in \mathbf{R}^n : Ax^m > 0\}$$

and

$$S_- = \{x \in \mathbf{R}^n : Ax^m \leq 0\}.$$

Since A and $f(x)$ are nonsingular, by Section 5, S_0 consists of the degeneracy vectors, the degeneracy vectors in S_{01} are of degree 0, while the degeneracy vectors in S_{00} are of degree k, for $1 \leq k \leq d - 1$, where d is the base index of S.

Based upon the properties of eigenvalues and asymptotic rays, discussed in Sections 2 and 4, we may classify the algebraic hypersurfaces defined by (2) in the nonsingular case into ten classes:

(A) m is even and $\lambda_{\max} \le 0$.
(B) m is even and $\lambda_{\min} > 0$.
(C) m is even, $\lambda_{\min} = 0$, $n \ge 3$, and S has only one branch.
(D) m is even, $\lambda_{\min} = 0$, and S has several branches.
(E) m is even, $\lambda_{\min} < 0 < \lambda_{\max}$, there is no zero eigenvalue, $n \ge 3$, and S has only one branch.
(F) m is even, $\lambda_{\min} < 0 < \lambda_{\max}$, there is no zero eigenvalue, and S has several branches.
(G) m is even, $\lambda_{\min} < 0 < \lambda_{\max}$, there is a zero eigenvalue, $n \ge 3$, and S has only one branch.
(H) m is even, $\lambda_{\min} < 0 < \lambda_{\max}$, there is a zero eigenvalue, and S has several branches.
(I) m is odd and there is no zero eigenvalue.
(J) m is odd and there is a zero eigenvalue.

In Case (A), if $c > 0$, then S is empty by Theorem 1(c). In Case (B), S surrounds a bounded region by Theorem 1(e). Note that this bounded region may be not convex. For example, let $n = 2, m = 4, c = 1$ and

$$f(x) = x_1^4 + \frac{5}{3}x_2^4 + \frac{4}{3}x_1^3 x_2.$$

Then (4) is

$$\begin{cases} x_1^3 + x_1^2 x_2 = \lambda x_1, \\ \frac{1}{3}x_1^3 + \frac{5}{3}x_2^3 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

It has four eigenvalues

$$\lambda_1 = \frac{5}{3}, \qquad \lambda_2 = \sqrt{2}, \qquad \lambda_3 = \frac{8 + 3\sqrt{6}}{\sqrt{8 + 2\sqrt{6}}}, \qquad \lambda_4 = \frac{8 - 3\sqrt{6}}{\sqrt{8 - 2\sqrt{6}}}.$$

Since $\lambda_i > 0$ for $i = 1, 2, 3, 4$, A is positive definite and S surrounds a bounded region. On the other hand,

$$\nabla^2 f(1, 0) = \begin{pmatrix} 12 & 4 \\ 4 & 0 \end{pmatrix},$$

which is indefinite. Thus, f is not convex and the bounded region surrounded by S is not convex.

In Cases (B)–(H), m is even. Then S is symmetric to the origin. In Cases (I) and (J), m is odd. Then S is not symmetric to the origin.

Suppose that $c > 0$ in this paragraph. We cannot discuss asymptotic rays for $c = 0$. In Case (B), $d = 0$ and there are no asymptotic rays. In Cases (C) and (D), all the asymptotic rays are of degree $1 - \frac{k}{m}$, where $2 \le k \le m - 1$. In Cases (E), (F) and (I), $d = 1$, all the asymptotic rays are of degree $1 - \frac{1}{m}$. In Cases (G), (H) and (J), there may be both kinds of such asymptotic rays.

In Cases (C), (E) and (G), S has only one branch. It surrounds a spider-shape region. This spider has a bounded body and several unbounded feet. There are two kinds of feet of the spider. One kind of feet is of thorn-type. They surround and approximate the asymptotic rays which are of degree $1 - \frac{k}{m}$, where $2 \le k \le m - 1$. They become thinner and thinner as they stretch to infinity. Another kind of feet is of trumpet-type. They surround some parts of S_-, and become larger and larger as they stretch to infinity. In Case (C), there are only thorn-type feet. In Case (E), there are only trumpet-type feet. In Case (G), there may be both kinds of such feet.

For example, let $n = 3, m = 4$ and

$$f(x) = Ax^4 = 2x_1^2x_2^2 + 2x_1^2x_3^2 + x_2^4 + 2x_2^2x_3^2 + x_3^4.$$

Then A is positive semidefinite, A, $f(x)$ and S have only two asymptotic rays

$$L_1 = \{(\alpha, 0, 0) : \alpha \geq 0\}$$

and

$$L_2 = \{(-\alpha, 0, 0) : \alpha \geq 0\}.$$

The hypersurface S surrounds a region with a bounded body and two infinite feet which surround and approximate L_1 and L_2. By calculus, we may find that though this region has two infinite feet, it has a finite volume:

$$V = \int_0^\infty \frac{2\pi c}{\sqrt{t^4 + c} + t^2} \mathrm{d}t.$$

We conjecture that in Case (C) the spider-shape region surrounded by S always has a finite volume.

In Cases (D), (F), (H), (I) and (J), S_+ is separated by S_0 to several parts. In each part of S_+, there is a branch of S.

Suppose $c > 0$ again in this paragraph. In Cases (B)–(J), the points on S which are the closest to the origin, and their distances from the origin are given by Theorem 1(c). In Case (B), the points on S which are the farthest from the origin, and their distances from the origin are given by Theorem 1(d).

We may subdivide each of the above classes to subclasses, according to the order m, the dimension n and the base index d. We may further subdivide such subclasses to species according to the number of branches and vertices. In the next section, we will discuss vertices of S in the nonsingular case for $c > 0$.

7. Characterization points

We assume that $c > 0$ in this section. Similar to the proof of Theorem 1(c) and (d), we may show that a positive eigenvalue λ and its eigenvector x correspond to a stationary point y of the distance function from S to the origin such that $y = \sigma x$, where σ is the distance from y to the origin: $\sigma = \left(\frac{c}{\lambda}\right)^{\frac{1}{m}}$. We may regard such a point as a characterization point of S.

The question is: when does (λ, x) give a local minimizer or maximizer of the distance function? Such a point may be more important for describing S.

In Cases (D), (F), (H), (I) and (J) in the last section, S has several branches. Then there is at least one point in each branch which is the closest point from that branch to the origin. Such a point is a local minimizer of the distance function.

Note that some other stationary points of the distance function may be also important. First, a local maximizer of the distance function from S_B to the origin is also important. In the singular case, such a point is not a local maximizer of the distance function from S to the origin. Hence we need to consider the distance function from S_B to the origin in the singular case. We do not go into such detail. Second, even in the nonsingular case, some other stationary points of the distance function may be also important. For example, when $n = 3$, $m = 2$, $c = 1$ and

$$f(x) = x_1^2 + \left(\frac{x_2}{2}\right)^2 + \left(\frac{x_3}{3}\right)^2,$$

S is an ellipsoid surface. The point (0, 1, 0) is a stationary point but not a local minimizer or maximizer of the distance function from S to the origin, but it is still important for studying this ellipsoid and it is still given by an eigenvalue and its eigenvector of A.

The following discussion follows from the theory of nonlinear programming. We only give an outline here.

As studied in Section 2, the distance from S to the origin can be found by

$$\min \left\{ \sum_{i=1}^{n} y_i^2 : Ay^m = c, y \in \mathbf{R}^n \right\}. \tag{31}$$

The optimality condition of (31) is

$$\begin{cases} \mu A y^{m-1} = y \\ Ay^m = c. \end{cases} \tag{32}$$

We may write the Lagrange function of (31) as

$$L(y, \mu) = \frac{1}{2} y^{\mathrm{T}} y - \frac{\mu}{m}(Ay^m - c).$$

Then the first equation of (32) is $\nabla_y L(y, \mu) = 0$, and

$$\nabla^2_{yy} L(y, \mu) = I - (m-1)\mu A y^{m-2},$$

where I is the n-dimensional unit matrix. Let λ be a positive eigenvalue of A, $f(x)$ and S. Let x be an eigenvector associated with λ. Assume that $y = \sigma x$. Then we have

$$\mu \sigma^{m-2} = \frac{1}{\lambda}$$

and

$$\nabla^2_{yy} L(y, \mu) = I - \frac{m-1}{\lambda} A x^{m-2}.$$

The tangent plane of S at y is

$$T = \{u \in \mathbf{R}^n : Ay^{m-1}u = 0\}.$$

But

$$Ay^{m-1}u = \left(Ay^{m-1}\right)^{\mathrm{T}} u = \sigma^{m-1} \left(Ax^{m-1}\right)^{\mathrm{T}} u = \lambda \sigma^{m-1} x^{\mathrm{T}} u.$$

Since $\lambda > 0$ and $\sigma > 0$, we have

$$T = \{u \in \mathbf{R}^n : x^{\mathrm{T}} u = 0\}.$$

The second-order sufficient (necessary) optimality condition for y to be a local minimizer/maximizer of (31) is that the symmetric matrix

$$\nabla^2_{yy} L(y, \mu) = I - \frac{m-1}{\lambda} A x^{m-2}$$

is positive definite (positive semidefinite)/negative definite (negative semidefinite) on T. Note that $2-m$ is an eigenvalue of $I - \frac{m-1}{\lambda}Ax^{m-2}$ with the eigenvector x. Thus, we have the following theorem.

Theorem 7. *Assume that $m \geq 2$. Suppose that $\lambda > 0$ is an eigenvalue of A, $f(x)$ and S, with an eigenvector x. Let $y = \sigma x$ and $\sigma = (\frac{c}{\lambda})^{\frac{1}{m}}$.*

(a) *A sufficient (necessary) condition for y to be a local minimizer of* (31) *is that except for $2-m$, all the other $n-1$ eigenvalues of $I - \frac{m-1}{\lambda}Ax^{m-2}$ are positive (nonnegative).*
(b) *A sufficient (necessary) condition for y to be a local maximizer of* (31) *is that except for $2-m$, all the other $n-1$ eigenvalues of $I - \frac{m-1}{\lambda}Ax^{m-2}$ are negative (nonpositive).*
(c) *When $m = 2$, the two necessary conditions in* (a) *and* (b) *are also sufficient.*

Note that

$$\nabla^3_{yyy}L(y, \mu) = -\frac{(m-1)(m-2)}{\lambda\sigma}Ax^{m-3}$$

and

$$\nabla^4_{yyyy}L(y, \mu) = -\frac{(m-1)(m-2)(m-3)}{\lambda\sigma^2}Ax^{m-4}.$$

We may establish the following third order and fourth order sufficient (necessary) conditions for a local minimizer of (31).

Theorem 8. *Assume that $m \geq 3$. Suppose that $\lambda > 0$ is an eigenvalue of A, $f(x)$ and S, with an eigenvector x. Let $y = \sigma x$ and $\sigma = (\frac{c}{\lambda})^{\frac{1}{m}}$. Suppose that the necessary condition in Theorem 7(a) holds. Let*

$$T_1 = \left\{ u \in \mathbf{R}^n : x^{\mathrm{T}}u = 0, u = \frac{m-1}{\lambda}Ax^{m-2}u \right\}.$$

(a) *If $m = 3$, a sufficient and necessary condition for y to be a local minimizer of* (31) *is that for any $u \in T_1$,*

$$Ax^{m-3}u^3 = 0.$$

(b) *If $m \geq 4$, a necessary condition for y to be a local minimizer of* (31) *is that for any $u \in T_1$,*

$$Ax^{m-3}u^3 = 0$$

and

$$Ax^{m-4}u^4 \leq 0.$$

(c) *If $m \geq 4$, a sufficient condition for y to be a local minimizer of* (31) *is that for any $u \in T_1$ with $u \neq 0$,*

$$Ax^{m-3}u^3 = 0$$

and

$$Ax^{m-4}u^4 < 0.$$

Similarly, we may establish the third order and fourth order sufficient (necessary) conditions for a local maximizer of (31). Continuing this discussion, we may establish higher order conditions. We do not go into the details.

8. Best rank-one approximation

Let A be a supersymmetric n-dimensional tensor of order m. We now define the Frobenius norm of A as

$$\|A\|_F := \sqrt{\sum_{i_1,\ldots,i_m=1}^{n} A_{i_1,\ldots,i_m}^2}.$$

Then λx^m, where $\lambda \in \mathbf{R}$, $x \in \mathbf{R}^n$, $\|x\|_2 = 1$ and $\|x\|_2$ is the Euclidean norm of x in $\mathbf{R}^n$, is called the best rank-one supersymmetric approximation of A if it is a global minimizer of

$$\{\|A - \lambda x^m\|_F : \lambda \in \mathbf{R}, x \in \mathbf{R}^n, \|x\|_2 = 1\}.$$

Theorem 9. *Suppose that λ is an eigenvalue of A with x as its associated eigenvector. Then*

$$\lambda = Ax^m \tag{33}$$

and

$$\|A - \lambda x^m\|_F^2 = \|A\|_F^2 - \lambda^2 \geq 0. \tag{34}$$

Furthermore, the largest absolute value of the eigenvalues and the corresponding eigenvector of A form its best rank-one supersymmetric approximation.

Proof. From (4), we have (33). Since $\|x\| = 1$, we have

$$\|A - \lambda x^m\|_F^2 = \|A\|_F^2 - 2\lambda Ax^m + \lambda^2\|x^m\|_F^2 = \|A\|_F^2 - 2\lambda Ax^m + \lambda^2.$$

By (33), we have (34).

On the other hand, the best rank-one supersymmetric approximation of A is the solution of

$$\begin{array}{lll} \min\limits_{\lambda,x} & \|A - \lambda x^m\|_F^2 & \\ \text{s.t.} & \|x\| & = \; 1. \end{array} \tag{35}$$

By (34) and (35) is equivalent to

$$\begin{array}{lll} \max & (Ax^m)^2 - \|A\|_F^2 & \\ \text{s.t.} & x^{\mathrm{T}}x & = \; 1. \end{array}$$

The conclusion now follows from (4). ■

9. Final remarks

In this paper, we define eigenvalues, rank and base index for a supersymmetric tensor A, a homogeneous polynomial $f(x)$ and an algebraic hypersurface S, defined by (2). We see that eigenvalues and eigenvectors have strong geometric meanings. The eigenvectors of the zero eigenvalues correspond to either recession vectors or asymptotic rays of S. The nonzero eigenvalues and their eigenvectors correspond to characterization points of S. The discussion on recession vectors, asymptotic rays and degeneracy vectors reveals that S and S_0 have very special structure. The base index d is a new concept. It is useful in discussing the case that $m > 2$.

We have the five final comments here.

(a) According to Newton there are 5 irreducible cubic cones required for obtaining his 59 irreducible cubic sections (Korchagin and Weinberg, 2005). Recently, Korchagin and Weinberg

(2005) further showed that 1037 quartic cones are required for obtaining a similar classification of irreducible quartic sections. Note that an mth order cone corresponds to an mth order three-dimensional supersymmetric tensor. Can we classify third order and fourth order three-dimensional supersymmetric tensors by their eigenvalues? If such a classification exists, is there a relation with the classification of irreducible cubic cones by Newton and irreducible quartic cones by Korchagin and Weinberg?

(b) We conjecture that there are n linearly independent eigenvectors of A. Let the rank of A be r. Then there are $n - r$ linearly independent unit recession vectors, which are eigenvectors associated with the zero eigenvalue, perpendicular to V_B, which is in an r-dimensional subspace. It is not reasonable that all the characterization points of S_B are in a subspace whose dimension is less than r.

(c) If we know an upper bound of the number of eigenvalues, then we have an upper bound of the numbers of branches and characterization points of S. In Qi (2005), when m is even, in the nonsingular case, such an upper bound is given as $n(m-1)^{n-1} - 1$. This upper bound has been further improved in Ni et al. (in press) as

$$d = \sum_{k=0}^{n-1}(m-1)^k = \begin{cases} n, & \text{if } m = 2, \\ \frac{(m-1)^n - 1}{m-2}, & \text{otherwise.} \end{cases}$$

It is interesting to know the situation when m is odd.

(d) The classification of quadratic hypersurfaces is related with the classification of second order partial differential equations. Is the classification of cubic and quartic hypersurfaces related with the classification of third and fourth order partial differential equations?

(e) When the rank r is less than n, A is singular, f can be converted into a homogeneous polynomial with r variables by an orthogonal transformation. As pointed out by a referee, in a very recent reprint (Carlini, 2005) by Carlini, by using catalecticant matrices instead of supersymmetric tensors to represent a polynomial, similar results are achieved.

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