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J. Math. Anal. Appl. 329 (2007) 1218-1229

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

# The degree of the E-characteristic polynomial of an even order tensor

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Received 15 March 2006

Available online 17 August 2006

Submitted by Jerzy Filar

#### Abstract

The E-characteristic polynomial of an even order supersymmetric tensor is a useful tool in determining the positive definiteness of an even degree multivariate form. In this paper, for an even order tensor, we first establish the formula of its E-characteristic polynomial by using the classical Macaulay formula of resultants, then give an upper bound for the degree of that E-characteristic polynomial. Examples illustrate that this bound is attainable in some low order and dimensional cases.

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Keywords: Tensor; E-Characteristic polynomial; E-Eigenvalue; Resultant; Upper bound

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<sup>1</sup> The work of this author was supported by the Hong Kong Research Grant Council and the National Natural Science Foundation of China (No. 60572135).

 $^{2}$  The work of this author was supported by the Research Grant Council of Hong Kong.

<sup>3</sup> The work of this author was supported by a Hong Kong Polytechnic University Postdoctoral Fellowship.

0022-247X/\$ – see front matter © 2006 Elsevier Inc. All rights reserved. doi:10.1016/j.jmaa.2006.07.064

### 1. Introduction

An *m*th order tensor is an *m*-way array whose entries accessed via *m* indices. It arises in diverse fields such as signal and image processing, data analysis, nonlinear continuum mechanics, higher-order statistics, as well as independent component analysis [5,7,8,12,15,16,19,22,29,31]. It is well known that supersymmetric tensors and homogeneous polynomials are bijectively associated [8,13], and when *m* is even, the positive definiteness of a homogeneous polynomial plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [1-4,9,11,14,30]. Motivated by this issue, Qi [24] introduced the concepts of eigenvalues and E-eigenvalues of a supersymmetric tensor, and established their close relationship with the theory of resultants [6,10,28].

An *m*th degree homogeneous polynomial form of *n* variables f(x) can be represented as the product of two tensors

$$f(x) \equiv Ax^{m} = \sum_{i_{1},\dots,i_{m}=1}^{n} a_{i_{1}\dots i_{m}} x_{i_{1}} \cdots x_{i_{m}},$$
(1.1)

where tensor A is a supersymmetric tensor, i.e., its entries  $a_{i_1\cdots i_m}$  are invariant under any permutation of their indices  $i_1, \ldots, i_m = 1, \ldots, n$ , and  $x^m$  is a supersymmetric tensor with entries  $x_{i_1}x_{i_2}\cdots x_{i_m}$ .

A supersymmetric tensor A is called positive definite if it satisfies

 $Ax^m > 0, \quad \forall x \in \mathbb{R}^n, \ x \neq 0.$ 

For a vector  $x \in C^n$ , we denote its *i*th component by  $x_i$ . By the tensor product [27],  $Ax^{m-1}$  is a vector in  $C^n$  whose *i*th component is

$$\sum_{i_2,\ldots,i_m=1}^n a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}.$$

In [24], Qi introduced eigenvalues, eigenvectors, E-eigenvalues, E-eigenvectors, characteristic polynomials and E-characteristic polynomials for supersymmetric tensors. When  $m \ge 3$ , eigenvalues and E-eigenvalues may not be real. An eigenvalue (E-eigenvalue) with a real eigenvector (E-eigenvector) is called an H-eigenvalue (Z-eigenvalue). An even order supersymmetric tensor always has H-eigenvalues and Z-eigenvalues. It is positive (semi)definite if and only if all of its H-eigenvalues or all of its Z-eigenvalues are positive (nonnegative). A complex number is an eigenvalue of a supersymmetric tensor if and only if it is a root of the characteristic polynomial of that tensor. Based upon these, an H-eigenvalue method for the positive definiteness identification problem was developed in [21].

By [24], the degree of the characteristic polynomial of an mth order n-dimensional supersymmetric tensor is

$$d = n(m-1)^{n-1}.$$

For example, if m = 4 and n = 3, then d = 27.

On the other hand, the degree of the E-characteristic polynomial is lower than this. In [26], E-eigenvalues and E-characteristic polynomials were further discussed. The definitions of eigenvalues, eigenvectors, E-eigenvalues, E-eigenvectors, E-characteristic polynomials were generalized to nonsymmetric tensors. It was shown in [26] that the degree of the E-characteristic polynomial of an *m*th order *n*-dimensional tensor varies for different tensors. Sometimes there

may be zero E-characteristic polynomials. The maximum of degrees of E-characteristic polynomial of *m*th order *n*-dimensional tensors is denoted as d(m, n) when *m* is even. When *m* is odd, the E-characteristic polynomial of an *m*th order *n*-dimensional tensor only contains even degree terms. Thus, the maximum of degrees of E-characteristic polynomial of *m*th order *n*-dimensional tensors is denoted as 2d(m, n) when *m* is odd. It was shown in [26], that  $d(1, n) \equiv 1$ , d(2, n) = n, d(m, 2) = m for  $m \ge 3$  and

$$d(m,n) \leqslant m^{n-1} + m^{n-2} + \dots + m \tag{1.2}$$

for  $m, n \ge 3$ . When m = 4 and n = 3, (1.2) gives an upper bound 20 for d(m, n). This shows that the degree of the E-characteristic polynomial is much lower than the degree of the characteristic polynomial, and a Z-eigenvalue method for the positive definiteness identification problem may be better than the H-eigenvalue method.

The upper bound for d(m, n) given in (1.2) can be improved. In this paper, we do this when *m* is even. In particular, we show that d(4, 3) = 13, which is much smaller than 20, the upper bound given in (1.2) and 27, the degree of the characteristic polynomial when m = 4 and n = 3. In [20], using the result d(4, 3) = 13 in this paper, a Z-eigenvalue method for the positive definiteness identification problem for a quartic form of three variables is developed. Numerical results show that this method is better than the existing global polynomial optimization methods [23], applied to this problem.

In the following sections, for an even order tensor, we first establish the formula of its Echaracteristic polynomial by using the classical Macaulay formula of resultants, then give an upper bound for the degree of that E-characteristic polynomial. Examples illustrate that this bound is tough in some low order and dimensional cases.

In [25], geometric meanings of Z-eigenvalues are discussed. In [26], it was also shown that E-eigenvalues are invariant under co-ordinate changes in the sense of tensor analysis used in nonlinear mechanics [12,29]. This shows an additional merit of E-eigenvalues. Independently, with a variational approach, Lim also defines eigenvalues of tensors in [17] in the real field. The  $l^2$  eigenvalues of tensors defined in [17] are Z-eigenvalues in [24], while the  $l^k$  eigenvalues of tensors defined in [17] are H-eigenvalues in [24]. Notably, Lim [17] proposed a multilinear generalization of the Perron–Frobenius theorem based upon the notion of  $l^k$  eigenvalues (H-eigenvalues) of tensors.

#### 2. A formula of the E-characteristic polynomial

In this section, we will first review the definition of E-eigenvalues, E-characteristic polynomials, and their properties. Then we will review the classical Macaulay formula of the resultant for a polynomial system, stated in [6]. Finally, we will use the Macaulay formula to establish a formula of the E-characteristic polynomial of an even order tensor *A*.

**Definition 2.1.** For a real tensor *A*, a number  $\lambda \in C$  is called an *E-eigenvalue* of *A* and a nonzero vector  $x \in C^n$  is called an *E-eigenvector* of *A* associated with the E-eigenvalue  $\lambda$ , if they are solutions of the following polynomial equation system:

$$\begin{cases} Ax^{m-1} = \lambda x, \\ x^T x = 1. \end{cases}$$
(2.3)

If x is real, then  $\lambda$  is also real. In this case,  $\lambda$  and x are called a Z-eigenvalue of A and a Z-eigenvector of A associated with the Z-eigenvalue  $\lambda$ , respectively.

Assume that m is even. Let A be an mth order tensor and

$$F_{\lambda}(x) = Ax^{m-1} - \lambda I(x)x = 0,$$
(2.4)

where  $I(x) = (x^T x)^{\frac{m-2}{2}}$ . Then the resultant of  $F_{\lambda}(x)$ , denoted by  $\operatorname{Res}(F_{\lambda}(x))$ , is the *E*-characteristic polynomial  $\phi(\lambda)$  of *A*, i.e.,

$$\phi(\lambda) = \operatorname{Res}(F_{\lambda}(x)).$$

indicator of the positive definiteness of A.

The tensor A is called *regular* if there is no vector  $x \neq 0$  such that

$$\begin{cases} Ax^{m-1} = 0\\ x^T x = 0. \end{cases}$$

The following theorem was shown in [26].

**Theorem 2.1.** Assume that  $m, n \ge 2$ . Let d(m, n) be the maximum of degrees of *E*-characteristic polynomials of mth order *n*-dimensional tensors. Then the following statements hold:

- (a) An *E*-eigenvalue of A is a root of the *E*-characteristic polynomial  $\phi$ . If A is regular, then a complex number is an *E*-eigenvalue of A if and only if it is a root of  $\phi$ .
- (b) d(2, n) = n. For  $m \ge 3$ , d(m, 2) = m. For  $m, n \ge 3$ ,

$$d(m,n) \leqslant m^{n-1} + \dots + m.$$

This theorem holds for all  $m, n \ge 2$ . But in this paper, we only discuss the case that m is even.

We denote by  $k[x_1, ..., x_n]$  the collection of all polynomials in  $x_1, ..., x_n$  with coefficients in k, where k is a field. For homogeneous polynomials  $F_1, F_2, ..., F_n \in C[x_1, x_2, ..., x_n]$  of total degrees  $d_1, d_2, ..., d_n$ , set

$$\bar{d} = \sum_{i=1}^{n} (d_i - 1) + 1 = \sum_{i=1}^{n} d_i - n + 1.$$

Let S be the set of the monomials  $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  of total degree  $\overline{d}$  and divide it into the following n sets:

$$S_{1} = \{x^{\alpha}: |\alpha| = \bar{d}, x_{1}^{d_{1}} \text{ divides } x^{\alpha}\},$$

$$S_{2} = \{x^{\alpha}: |\alpha| = \bar{d}, x_{1}^{d_{1}} \text{ does not divide } x^{\alpha} \text{ but } x_{2}^{d_{2}} \text{ does}\},$$

$$\vdots$$

$$S_{n} = \{x^{\alpha}: |\alpha| = \bar{d}, x_{1}^{d_{1}}, \dots, x_{n-1}^{d_{n-1}} \text{ do not divide } x^{\alpha} \text{ but } x_{n}^{d_{n}} \text{ does}\}.$$

$$(2.5)$$

Consider the system of homogeneous equations of degree d:

$$\begin{cases} x^{\alpha}/x_1^{d_1} \cdot F_1(x) = 0 & \text{for all } x^{\alpha} \in S_1, \\ \dots \\ x^{\alpha}/x_n^{d_n} \cdot F_n(x) = 0 & \text{for all } x^{\alpha} \in S_n. \end{cases}$$

$$(2.6)$$

Since  $F_i$  has degree  $d_i$ , it follows that  $x^{\alpha}/x_i^{d_i} \cdot F_i$  has total degree  $\bar{d}$ . Thus each polynomial on the left side of (2.6) can be written as a linear combination of monomials of total degree  $\bar{d}$ . Suppose that there are N such monomials, where  $N = \begin{pmatrix} \bar{d}+n-1\\n-1 \end{pmatrix}$ . Then observe that the total number of equations is the number of elements in  $S_1 \cup \cdots \cup S_n$ , which is also N. Thus, regarding the monomials of total degree  $\bar{d}$  as unknowns, we get a system of N linear equations in N unknowns.

Denote the coefficient matrix of the  $N \times N$  system of equations by M. A monomial  $x^{\alpha}$  of total degree  $\bar{d}$  is called *reduced* if  $x_i^{d_i}$  divides  $x^{\alpha}$  for exactly one i, where i = 1, ..., n. Denote M' the submatrix of the coefficient matrix of (2.6) obtained by deleting all rows and columns corresponding to reduced monomials  $x^{\alpha}$ .

Macaulay [18] gave the following formula for the resultant as a quotient of two determinants.

**Theorem 2.2.** When  $F_1, F_2, \ldots, F_n$  are universal polynomials, the resultant of  $\{F_1, F_2, \ldots, F_n\}$  is given by

$$\operatorname{Res} = \pm \frac{\det(M)}{\det(M')}.$$
(2.7)

Furthermore, if k is a field and  $F_1, F_2, \ldots, F_n \in k[x_1, x_2, \ldots, x_n]$ , then the formula for Res holds whenever det $(M') \neq 0$ .

Now we discuss the resultant of  $F_{\lambda}(x)$  based on the above discussion. For convenience, we denote (2.4) by

$$F_{\lambda}(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_n(x) \end{pmatrix} = 0 \quad \text{and} \quad \bar{F}(x) = Ax^{m-1}.$$

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Obviously,

$$F_i(x) = \overline{F}_i(x) - \lambda I(x)x_i \quad \text{for } i = 1, \dots, n.$$
(2.8)

Let  $d_1, \ldots, d_n = m - 1$ . Then we have

$$\bar{d} = \sum_{i=1}^{n} (m-1-1) + 1 = n(m-2) + 1.$$

Let *S* be the set of the monomials  $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  of total degree  $\bar{d}$  and divide it into *n* sets as (2.5), where  $d_1 = d_2 = \cdots = d_n = m - 1$ .

Consider the system of homogeneous equations of degree d:

$$x^{\alpha}/x_i^{m-1} \cdot F_i(x) = 0 \quad \text{for all } x^{\alpha} \in S_i,$$
(2.9)

for i = 1, ..., n.

Regarding the monomials of total degree  $\overline{d}$  as unknowns. Then we get a system of N linear equations in N unknowns, where  $N = \binom{n(m-1)}{n-1}$ .

Denote by  $M_{\lambda}$  the coefficient matrix of the  $N \times N$  system of equations and  $M'_{\lambda}$  the submatrix of  $M_{\lambda}$  obtained by deleting all rows and columns corresponding to reduced monomials  $x^{\alpha}$ .

**Theorem 2.3.** Assume that A is a universal mth order n-dimensional supersymmetric tensor and m is even. Then the E-characteristic polynomial of A is given by

$$\phi(\lambda) = \pm \frac{\det(M_{\lambda})}{\det(M_{\lambda}')}.$$
(2.10)

Furthermore, if A is a real tensor, then the above formula holds whenever  $det(M'_{\lambda}) \neq 0$ .

**Proof.** The conclusion follows from Theorems 2.1 and 2.2.  $\Box$ 

#### **3.** An upper bound of the degree of $\phi(\lambda)$

In this section, we consider the degree of the E-characteristic polynomial of A. Assume that A is a universal even order tensor. We assume that  $m, n \ge 2$ . We rewrite  $\phi(\lambda)$  as follows:

$$\phi(\lambda) = \sum_{i=0}^d c_i \lambda^i,$$

where d is the degree of  $\phi(\lambda)$  with respect to  $\lambda$ ,  $c_i$ 's are homogeneous polynomials in elements of A.

Denote the degrees of det $(M_{\lambda})$  and det $(M'_{\lambda})$  with respect to  $\lambda$  by  $d_M$  and  $d_{M'}$ , respectively. Obviously, we have

$$d(m,n) \leqslant d = d_M - d_{M'}. \tag{3.11}$$

Theorem 3.1.

$$d_M = \left( \begin{array}{c} (n-1)(m-1)+1\\ n-1 \end{array} \right).$$

**Proof.** Consider the elements of  $M_{\lambda}$  that contains  $\lambda$ . We rewrite (2.9) as follows:

$$(x^{\alpha}/x_i^{m-1})\bar{F}_i - \lambda(x^{\alpha}/x_i^{m-2})I(x) = 0 \quad \text{for all } x^{\alpha} \in S_i, \ i = 1, \dots, n.$$
 (3.12)

If there exist  $\alpha^{(i)} \in S_i$  and  $\alpha^{(j)} \in S_j$  for some  $1 \leq i \neq j \leq n$  such that

$$x^{\alpha^{(i)}}/x_i^{m-2} = x^{\alpha^{(j)}}/x_j^{m-2},$$

then the equation

$$(x^{\alpha^{(j)}}/x_j^{m-1})\bar{F}_j - \lambda(x^{\alpha^{(j)}}/x_j^{m-2})I(x) = 0$$

can be replaced by

$$(x^{\alpha^{(j)}}/x_j^{m-1})\bar{F}_j - (x^{\alpha^{(i)}}/x_i^{m-1})\bar{F}_i = 0.$$

This procedure is equivalent to perform the elementary row operation on the corresponding row of the matrix  $M_{\lambda}$ .

Repeat this process till it cannot be executed. Then we get a new system of "linear" equations with a coefficient matrix, denoted by  $\hat{M}$ .

Denote by *r* the number of elements in  $\frac{S_1}{x_1^{m-2}} \cup \cdots \cup \frac{S_n}{x_n^{m-2}}$ . It is easy to observe that the number of rows containing  $\lambda$  in  $\hat{M}$  is *r* and  $d_M = r$ .

We now compute r. Let  $S' = \bigcup_{i=1}^{n} S'_i$ , where

$$S'_i = \{x^{\alpha}: x^{\alpha} x_i^{m-2} \in S_i\}, \quad i = 1, 2, \dots, n,$$

and

$$S'' = \{x^{\alpha} \colon |\alpha| = (n-1)(m-2) + 1\}.$$

It is clear that  $\frac{S_i}{x_i^{m-2}} = S'_i$  for all i = 1, ..., n. Moreover, we claim that S' = S''. In fact, obviously,  $S' \subseteq S''$ . On the other hand, since  $S_1, ..., S_n$  constitute a partition of the set S, combining this with the definition of these sets, we conclude that for any  $x^{\alpha} \in S''$ , there exists at least one index i such that  $1 \le i \le n$  and  $x^{\alpha} x_i^{m-2} \in S_i$ , so  $x^{\alpha} \in S'$ . Hence, S' = S''. From the combinatory theory, we can compute the cardinality of the set S'' equals

$$\binom{(n-1)+(n-1)(m-2)+1}{n-1} = \binom{(n-1)(m-1)+1}{n-1}.$$

This completes the proof.  $\Box$ 

Now, we consider the degree of  $\lambda$  in det $(M'_{\lambda})$ . Denote by |B| the number of entries of the set *B*.

## **Theorem 3.2.** $d_{M'} = |\hat{S}''|$ , where $\hat{S}''$ is defined by (3.15).

**Proof.** By the definition of reduced monomials, we know that a monomial  $x^{\alpha}$  of total degree  $\bar{d}$  is not reduced if and only if there exist at least two distinct indices  $i, j \ge 1$  such that  $\alpha_i \ge m - 1$  and  $\alpha_j \ge m - 1$ . So all entries of  $S_n$  are reduced, and all nonreduced monomials can be divided into n - 1 sets according to (2.5) as follows:

$$\hat{S}_{1} = \{x^{\alpha} \colon x^{\alpha} \in S_{1}, \ \alpha_{i} \ge m - 1 \text{ for some } 2 \le i \le n\}, 
\hat{S}_{2} = \{x^{\alpha} \colon x^{\alpha} \in S_{2}, \ \alpha_{i} \ge m - 1 \text{ for some } 3 \le i \le n\}, 
\vdots 
\hat{S}_{n-1} = \{x^{\alpha} \colon x^{\alpha} \in S_{n-1}, \ \alpha_{n} \ge m - 1\}.$$
(3.13)

Recall that the matrix  $M'_{\lambda}$  is the submatrix of the coefficient matrix  $M_{\lambda}$  obtained by deleting all rows and columns corresponding to reduced monomials  $x^{\alpha}$ , so the rows of  $M'_{\lambda}$  correspond to the following polynomial system:

$$\begin{cases} \left(x^{\alpha}/x_{1}^{m-1}\right)\hat{F}_{1}-\lambda\left(x^{\alpha}/x_{1}^{m-2}\right)I(x)=0 & \text{ for all } x^{\alpha}\in\hat{S}_{1}, \\ \dots \\ \left(x^{\alpha}/x_{n-1}^{m-1}\right)\hat{F}_{n-1}-\lambda\left(x^{\alpha}/x_{n-1}^{m-2}\right)I(x)=0 & \text{ for all } x^{\alpha}\in\hat{S}_{n-1}. \end{cases}$$
(3.14)

For each polynomial of (3.14), we consider its term whose monomial is

$$\frac{x^{\alpha}}{x_i^{m-2}} \cdot x_t^{m-2} \quad \text{for some } x^{\alpha} \in \hat{S}_i.$$

Denote it by  $x(i, t, \alpha)$ , where t is the first index of  $\alpha$  such that  $\alpha_t \ge 1$ . It is clear that  $1 \le t \le i$ , and

$$x(i,t,\alpha)\in \hat{S}_t,$$

which implies that its coefficient is an element of  $M'_{\lambda}$  and contains a linear term of  $\lambda$ . Hence, there exists  $\lambda$  in each row of  $M'_{\lambda}$ .

Perform the same row operation on the matrix  $M'_{\lambda}$  as that on  $M_{\lambda}$  in the proof of Theorem 3.1, we obtain a new matrix, denoted by  $\overline{M}'$ . Let  $\hat{S}' = \bigcup_{i=1}^{n-1} \hat{S}'_i$ , where

$$\hat{S}'_i = \{x^{\alpha}: x^{\alpha} x_i^{m-2} \in \hat{S}_i\}, \quad i = 1, 2, \dots, n-1\}$$

and

$$\hat{S}'' = \left\{ x^{\alpha} \colon |\alpha| = (n-1)(m-2) + 1, \ \alpha_i \ge 1, \ \alpha_j \ge m-1 \text{ for some } 1 \le i < j \le n \right\}.$$

It is clear that

$$\frac{\hat{S}_i}{x_i^{m-2}} = \hat{S}'_i \quad \text{for all } i = 1, \dots, n-1.$$

Moreover, similarly to the proof of Theorem 3.1, we have that  $\hat{S}' = \hat{S}''$ . Denote by r' the number of entries of the set  $\hat{S}''$ . Then, it is easy to observe that there are r' rows containing  $\lambda$  in  $\bar{M}'$  and  $r' = d_{M'}$ . This completes the proof.  $\Box$ 

We now compute  $|\hat{S}''|$ .

#### Theorem 3.3.

$$|\hat{S}''| = \sum_{k=1}^{n-2} (-1)^{k-1} \sum_{i=1}^{n-k} \binom{n-i}{k} \cdot \binom{(n-1-k)(m-1)+1-i}{n-i}.$$

Proof. Denote

$$P_i = \left\{ x^{\alpha} \colon |\alpha| = (n-1)(m-2) + 1, \ \alpha_1 = \dots = \alpha_{i-1} = 0, \\ \alpha_i \ge 1, \ \alpha_j \ge m-1 \text{ for some } i < j \le n \right\}$$

for all i = 1, 2, ..., n - 1. Then

$$\hat{S}'' = \bigcup_{i=1}^{n-1} P_i$$
 and  $P_{i_1} \cap P_{i_2} = \emptyset$  if  $i_1 \neq i_2$ .

Hence,

$$|\hat{S}''| = \sum_{i=1}^{n-1} |P_i|.$$
(3.16)

Denote

$$P_i^j = \left\{ x^{\alpha} \colon |\alpha| = (n-1)(m-2) + 1, \ \alpha_1 = \dots = \alpha_{i-1} = 0, \ \alpha_i \ge 1, \ \alpha_j \ge m-1 \right\}$$

for all i = 1, 2, ..., n - 1, j = i + 1, ..., n. Then

(3.15)

$$|P_{i}| = \sum_{j=i+1}^{n} |P_{i}^{j}| - \sum_{i+1 \leq j_{1} < j_{2} \leq n} |P_{i}^{j_{1}} \cap P_{i}^{j_{2}}| + \sum_{i+1 \leq j_{1} < j_{2} < j_{3} \leq n} |P_{i}^{j_{1}} \cap P_{i}^{j_{2}} \cap P_{i}^{j_{3}}| - \cdots$$
$$= \sum_{k=1}^{n-i} (-1)^{k-1} \sum_{i+1 \leq j_{1} < j_{2} < \cdots < j_{k} \leq n} |P_{i}^{j_{1}} \cap P_{i}^{j_{2}} \cap \cdots \cap P_{i}^{j_{k}}|.$$
(3.17)

From the combinatory theory, it is clear that

$$|P_i^j| = \binom{((n-1)(m-2)+1) - m + n - i}{n-i} = \binom{(n-2)(m-1) + 1 - i}{n-i}$$

for j = i + 1, ..., n, and

$$|P_i^{j_1} \cap \dots \cap P_i^{j_k}| = \begin{pmatrix} ((n-1)(m-2)+1) - (k(m-1)+1) + n - i \\ n - i \end{pmatrix}$$
$$= \begin{pmatrix} (n-1-k)(m-1) + 1 - i \\ n - i \end{pmatrix},$$
(3.18)

where k = 1, ..., n - i and  $i + 1 \le j_1 < \dots < j_k \le n$ . By (3.16)–(3.18), we have that

$$\begin{aligned} |\hat{S}''| &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} (-1)^{k-1} \sum_{i+1 \leq j_1 < \dots < j_k \leq n} \left| P_i^{j_1} \cap \dots \cap P_i^{j_k} \right| \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} (-1)^{k-1} \sum_{i+1 \leq j_1 < \dots < j_k \leq n} \left( \frac{(n-1-k)(m-1)+1-i}{n-i} \right) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} (-1)^{k-1} \binom{n-i}{k} \cdot \binom{(n-1-k)(m-1)+1-i}{n-i} \\ &= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{i=1}^{n-k} \binom{n-i}{k} \cdot \binom{(n-1-k)(m-1)+1-i}{n-i}. \end{aligned}$$
(3.19)

Note that, if k = n - 1, then

$$\binom{(n-1-k)(m-1)+1-i}{n-i} = 0.$$
(3.20)

By (3.19) and (3.20), we have the desired result.  $\Box$ 

**Theorem 3.4.** Assume that A is a universal even order tensor. Then the degree of  $\phi(\lambda)$  is given by

$$d = \sum_{k=0}^{n-1} (m-1)^k = \begin{cases} n, & \text{if } m = 2, \\ \frac{(m-1)^n - 1}{m-2}, & \text{otherwise.} \end{cases}$$
(3.21)

Furthermore, if A is a real even order tensor, then the above number d is an upper bound of the degree of  $\phi(\lambda)$ .

**Proof.** By (3.11) and Theorems 3.1–3.3, the degree of  $\phi(\lambda)$  is given by

$$d = \binom{(n-1)(m-1)+1}{n-1} - \sum_{k=1}^{n-2} (-1)^{k-1} \sum_{i=1}^{n-k} \binom{n-i}{k} \cdot \binom{(n-1-k)(m-1)+1-i}{n-i}$$

By induction, we have (3.21). But for a real even order tensor *A*, the leading coefficient  $c_d$  of the E-characteristic polynomial  $\phi(\lambda)$  may be zero. In this case,  $\deg(\phi) < d$ , which follows the second statement.  $\Box$ 

It is clear that the upper bound given by Theorem 3.4 is much smaller than that given by Theorem 2.1 when  $m, n \ge 3$  and m is even.

#### **Corollary 3.1.**

$$d(2,n) = n.$$

**Proof.** This follows from (3.21) directly.  $\Box$ 

**Corollary 3.2.** *Assume that m is even and m*  $\ge$  2. *Then* 

 $d(m, 2) \leq m$ .

**Proof.** This also follows from (3.21) directly.

The above two corollaries are the same as the corresponding contents of Theorem 2.1. In fact, we have d(m, 2) = m for all  $m \ge 2$ . The following corollary is sharper than the corresponding content of Theorem 2.1.

**Corollary 3.3.** *Assume that m is even and m \ge 2. Then* 

 $d(m,3) \leqslant m^2 - m + 1.$ 

In particular, we have d(4, 3) = 13.

**Proof.** The first statement also follows from (3.21) directly.

Let A be a 4th order 3-dimensional unit tensor, i.e.,  $a_{1111} = a_{2222} = a_{3333} = 1$  and other entries are zero. We have

$$\det(M_{\lambda}) = (1 - 3\lambda)^4 (-1 + \lambda)^{10} (-1 + 2\lambda)^7,$$
  
$$\det(M'_{\lambda}) = (-1 + \lambda)^7 (-1 + 2\lambda).$$

Hence, its E-characteristic polynomial is given by

$$\phi(\lambda) = \pm \frac{\det(M_4)}{\det(M'_4)} = \pm (1 - 3\lambda)^4 (-1 + \lambda)^3 (-1 + 2\lambda)^6.$$

Its E-eigenvalues are all Z-eigenvalues. They are  $\lambda = 1$  (three multiple), 1/2 (six multiple) and 1/3 (four multiple). Totally, it has 13 Z-eigenvalues. Hence, when m = 4, the upper bound  $m^2 - m + 1 = 13$  is attainable, i.e., d(4, 3) = 13.  $\Box$ 

We conjecture that the upper bound given in Theorem 3.4 is attainable and thus gives the exact value of d(m, n). We also conjecture that Theorem 3.4 also holds when *m* is odd.

The following is an example that the degree of the E-characteristic polynomial of a 4th order 3-dimensional supersymmetric tensor is strictly less than 13.

**Example 3.1.** Let *A* be a 4th order 3-dimensional supersymmetric tensor with  $a_{2222} = a_{3333} = 1$ ,  $a_{1122} = 1/6$  and other entries are zero. We have

$$\det(M_{\lambda}) = \frac{(1-2\lambda)^2(-1+\lambda)^8\lambda^8}{16384}$$
$$\det(M_{\lambda}') = \frac{(-1+\lambda)^2\lambda^5}{4}.$$

Hence, its E-characteristic polynomial is given by

$$\phi(\lambda) = \pm \frac{(1-2\lambda)^2(-1+\lambda)^6\lambda^3}{4096}$$

It is clear that the degree of  $\phi(\lambda)$  is less than 13.

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