

A quadratically convergent algorithm for finding the largest eigenvalue of a nonnegative homogeneous polynomial map

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Abstract In this paper we propose a quadratically convergent algorithm for finding the largest eigenvalue of a nonnegative homogeneous polynomial map where the Newton method is used to solve an equivalent system of nonlinear equations. The semi-symmetric tensor is introduced to reveal the relation between homogeneous polynomial map and its associated semi-symmetric tensor. Based on this relation a globally and quadratically convergent algorithm is established where the line search is inserted. Some numerical results of this method are reported.

Keywords Nonnegative homogenous polynomial mapping · Nonnegative tensors · Eigenvalue of polynomial mapping · Newton method

1 Introduction

Eigenvalues for higher order tensors were introduced in [7, 9, 13]. Meanwhile, the Perron-Frobenius theorem for nonnegative tensors is related to measuring higher order connectivity in hypergraphs [1, 2, 6]. The Perron-Frobenius theorem is further extended to multilinear forms and polynomial map with nonnegative coefficients [5, 17]. Applications of eigenvalues

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of tensors and multivariate polynomial include signal processing [14], medical resonance imaging [15,16], higher-order Markov chains [10], positive definiteness of even-order multivariate forms in automatic control [11]. Subsequently, an algorithm for finding the largest eigenvalue of a nonnegative tensor, extending the Collatz’s method for nonnegative matrices, was proposed by [10], its variations and linear convergence were studied in [8,18,19].

In this paper we consider the polynomial map $P = (p_1, p_2, \dots, p_n)^T : \Re^n \rightarrow \Re^n$ where each p_i is a homogeneous polynomial of degree $d \geq 1$ with nonnegative coefficients. For convenience, P is called a nonnegative homogeneous polynomial map of degree d .

Definition 1.1 Assume that P is a nonnegative homogeneous polynomial map of degree d . We say that $(\lambda, x) \in R \times (R^n \setminus \{0\})$ is an eigenvalue-eigenvector pair of P if

$$P(x) = \lambda x^{[d]}. \tag{1.1}$$

Here, $x^{[p]} = [x_1^p, x_2^p, \dots, x_n^p]^T$.

This definition is from [5] where a minor change is made. If we let $F(x) = P(x)^{[1/d]}$, then $(\lambda, x) \in R \times (R^n \setminus \{0\})$ is an eigenvalue-eigenvector pair of P if and only if $(\lambda^{1/d}, x) \in R \times (R^n \setminus \{0\})$ is an eigenvalue-eigenvector pair of F .

If P is a homogeneous polynomial map of degree d , then there exists a $(d + 1)$ -order n -dimensional tensor \mathcal{A} defined by

$$\mathcal{A} = (a_{i_0 i_1 \dots i_d}), \quad a_{i_0 i_1 \dots i_d} \in \Re, \quad 1 \leq i_0, i_1, \dots, i_d \leq n, \tag{1.2}$$

such that $P(x) = \mathcal{A}x^d$, where

$$\mathcal{A}x^d = \left(\sum_{i_1, i_2, \dots, i_d=1}^n a_{i_0 i_1 i_2 \dots i_d} x_{i_1} x_{i_2} \dots x_{i_d} \right)_{1 \leq i_0 \leq n}. \tag{1.3}$$

We call \mathcal{A} an associated tensor of P .

In addition, \mathcal{A} is called nonnegative (or, respectively, positive) if $a_{i_0 i_1 \dots i_d} \geq 0$ (or, respectively, $a_{i_0 i_1 \dots i_d} > 0$). A nonnegative tensor \mathcal{A} is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \dots, n\}$ such that

$$\mathcal{A}_{i_0 i_1 \dots i_d} = 0, \quad \forall i_0 \in I, \forall i_1, \dots, i_d \notin I. \tag{1.4}$$

If \mathcal{A} is not reducible, then we call \mathcal{A} irreducible. It is clear that if \mathcal{A} is nonnegative, then $P = \mathcal{A}x^d$ is a nonnegative homogeneous polynomial map of degree d .

In [5], the Perron-Frobenius theorem is extended to a nonnegative homogeneous polynomial map.

Theorem 1.1 ([5]) *Let P and \mathcal{A} be a nonnegative homogeneous polynomial map of degree d and its associated tensor, respectively. Assume that \mathcal{A} is irreducible. Then we have*

(1) *there are a unique scalar λ and a positive vector x which satisfy (1.1) and*

$$\lambda = \inf_{y \in \Re_{++}^n} \max_{1 \leq i \leq n} \frac{P_i(y)}{y_i^d} = \sup_{y \in \Re_+^n \setminus \{0\}} \min_{1 \leq i \leq n, y_i \neq 0} \frac{P_i(y)}{y_i^d},$$

where $P_i(y)$ is the i -th component of $P(x)$.

(2) *If $\omega \in C$ and $v = (v_1, \dots, v_n)^T \in C^n \setminus \{0\}$ are such that $P(v) = \omega v^{[d]}$, then $|\omega| \leq \lambda$.*

This theorem is a combination of Corollaries 4.2 and 4.3 in [5]. In [5], the results applied to a more general condition, called the weak irreducibility condition. Here we only state them

under the irreducibility condition. The definitions of irreducibility and weak irreducibility of a polynomial map may refer to [5].

In [5], an algorithm for finding the largest eigenvalue of a nonnegative polynomial map was also proposed by Friedland, Gauber and Han. We call it the FGH method for short. Local linear convergence of the FGH method was established in [5]. In this paper we want to study the relation between a nonnegative homogenous polynomial $\mathcal{A}x^d$ and its corresponding tensor \mathcal{A} , and propose a quadratically convergent algorithm for finding the largest eigenvalue of that nonnegative homogeneous polynomial map. The semi-symmetric tensor is introduced to reveal the relation between a homogeneous polynomial map and its associated tensor, while the property of semi-symmetric tensor gives a sound foundation for a new method for computing the eigenvalues of a homogenous polynomial map or a tensor.

This paper is organized as follows. We discuss the relation between a homogenous polynomial map and its associate semi-symmetric tensor in Sect. 2. A Newton algorithm for computing the largest eigenvalue of a nonnegative homogenous polynomial map is proposed in Sect. 3. We establish quadratical and global convergence of the algorithm in Sect. 4. In Sect. 5, we give some preliminary numerical test results.

Let $\mathfrak{R}_+^n = \{x \in \mathfrak{R}^n \mid x \geq 0\}$, $\mathfrak{R}_{++}^n = \{x \in \mathfrak{R}^n \mid x > 0\}$. Denote $x \circ y$ as the product of x and y by $x \circ y = (x_1y_1, \dots, x_ny_n)^T$ where $x, y \in \mathfrak{R}^n$. The tensor \mathcal{A} is called symmetric if its entries are invariant under any permutation of their indices, i.e.

$$a_{j_0j_1 \dots j_d} = a_{i_0i_1 \dots i_d},$$

$j_0j_1 \dots j_d$ is any permutation of $i_0i_1 \dots i_d$, $1 \leq i_0, i_1, \dots, i_d \leq n$. A nonnegative tensor \mathcal{A} is called primitive [3] if there exists a positive integer k such that $F^{(k)}(x) \in \mathfrak{R}_{++}^n$ for any nonzero $x \in \mathfrak{R}_+^n$ where $F^{(1)}(x) \equiv F(x) \equiv (\mathcal{A}x^d)^{1/d}$, $F^{(k)}(x) = F(F^{(k-1)}(x))$ for $k = 2, 3, \dots$.

2 A nonnegative homogeneous polynomial map and its associated tensor

In this section, we discuss the relation between a nonnegative homogeneous polynomial map and its associated tensor.

Example 2.1 Let $P(x) = \mathcal{A}x^3$, where $\mathcal{A} = (a_{ijkl})$ is defined by

$$\begin{aligned} a_{1111} &= a_{2222} = 2, \quad a_{3333} = 3; \\ a_{1112} &= 1, \quad a_{1121} = -2, \quad a_{1211} = 1; \\ a_{1113} &= 2, \quad a_{1131} = -1, \quad a_{1311} = 0; \\ a_{2123} &= -3, \quad a_{2132} = 2, \quad a_{2321} = 5; \\ a_{ijkl} &= 0 \text{ elsewhere.} \end{aligned}$$

Then

$$P(x) = \begin{pmatrix} 2x_1^3 + x_1^2x_3 \\ 2x_2^3 + 4x_1x_2x_3 \\ 3x_3^3 \end{pmatrix}.$$

The above example shows that P is nonnegative although its associated tensor \mathcal{A} is not nonnegative.

In order to reveal the relation of a nonnegative homogeneous polynomial map and its associated tensor, a semi-symmetric tensor is introduced in the following.

Definition 2.1 Assume that $d \geq 1$. An $(d + 1)$ -order n -dimensional tensor \mathcal{A} is called semi-symmetric if its entries are invariant under any permutation of their backward d indices, i.e.

$$a_{i_0 j_1 j_2 \dots j_d} = a_{i_0 i_1 \dots i_d}, \tag{2.1}$$

$1 \leq i_0 \leq n, j_1 \dots j_d$ is any permutation of $i_1 \dots i_d, 1 \leq i_1, \dots, i_d \leq n$.

From the definition, it is easy to see that a semi-symmetric tensor is the same as the symmetric tensor except the first subscript, a symmetric tensor is semi-symmetric, matrix is always semi-symmetric.

In order to describe semi-symmetric tensors, we denote $\Omega_{k,d}$ by

$$\Omega_{k,d} = \{(i_k, i_{k+1}, \dots, i_d) : i_k \leq i_{k+1} \leq \dots \leq i_d, i_k, i_{k+1}, \dots, i_d = 1, 2, \dots, n\}, \tag{2.2}$$

and define the set of all permutations of $(i_k \dots i_d)$ by

$$C_{i_k \dots i_d} = \{j_k j_{k+1} \dots j_d | : (j_k j_{k+1} \dots j_d) \text{ is a permutations of } (i_k \dots i_d)\}, \tag{2.3}$$

for $(i_k \dots i_d) \in \Omega_{k,d}$. For a semi-symmetric tensor, only entries with subscripts $i_0 i_1 \dots i_d$ for $i_0 = 1, 2, \dots, n$ and all $(i_1 \dots i_d) \in \Omega_{1,d}$ need to be given, other entries can be determined by (2.1).

Lemma 2.1 Let P be a homogeneous polynomial map of degree d . Then there is an unique $(d + 1)$ -order n -dimensional semi-symmetric tensor \mathcal{A}_s such that

$$P(x) = \mathcal{A}_s x^d. \tag{2.4}$$

Proof Each component in P is a homogeneous polynomial of d degree which includes

$$u_d = \binom{d+n-1}{n-1} \tag{2.5}$$

different monomials, all these monomials of degree d are ordered in a u_d -dimensional vector by the graded lexicographic order in the following

$$r(x) = (x_1^d, x_1^{(d-1)} x_2, \dots, x_1^{(d-1)} x_n, \dots, x_n^d)^T. \tag{2.6}$$

It is noted that u_d is also the cardinal number of $\Omega_{1,d}$ defined in (2.2). Let the coefficients of all monomials in P_i be $m_{ij}, j = 1, 2, \dots, u_d, i = 1, \dots, n$. Then

$$P(x) = Mr(x), \tag{2.7}$$

where $M = (m_{ij}) \in R^{n \times u_d}$. Now we define $(d + 1)$ -order n -dimensional semi-symmetric tensor \mathcal{A}_s where

$$a_{i i_1 \dots i_d} = \frac{m_{ij}}{|C_{i_1 \dots i_d}|}, \tag{2.8}$$

$i_1 \dots i_d \in \Omega_{1,d}, |C_{i_1 \dots i_d}|$ is the cardinal number of $C_{i_1 \dots i_d}$ defined in (2.3), j corresponds with the grlex order of $i_1 \dots i_d \in \Omega_{1,d}, j = 1, 2, \dots, u_d$. Other entries of \mathcal{A}_s are determined by (2.1). Hence we have

$$\begin{aligned}
 (\mathcal{A}_s x^d)_i &= \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_d=1}^n a_{ij_1 \dots j_d} x_{j_1} \cdots x_{j_d} \\
 &= \sum_{i_1=1}^n \sum_{i_2=i_1}^n \cdots \sum_{i_d=i_{d-1}}^n |C_{i_1 \dots i_d}| a_{ii_1 \dots i_d} x_{i_1} \cdots x_{i_d} \\
 &= \sum_{(i_1 i_2 \dots i_d) \in \Omega_{1d}} |C_{i_1 \dots i_d}| a_{ii_1 \dots i_d} x_{i_1} \cdots x_{i_d} \\
 &= e_i^T M r(x) = P_i(x),
 \end{aligned} \tag{2.9}$$

for $i = 1, \dots, n$. In (2.9), the second equality is obtained by combing same monomials with the same coefficient, and the fourth equality is from (2.6) and (2.8).

Now we prove the uniqueness. Let $\tilde{\mathcal{A}}$ be semi-symmetric such that $\tilde{\mathcal{A}}x^d = P(x)$. Since $\tilde{\mathcal{A}}$ is semi-symmetric, for $i_1 \cdots i_d \in \Omega_{1,d}$, j is the grlex order of $i_1 \cdots i_d \in \Omega_{1,d}$, $\tilde{a}_{ii_1 \dots i_d}$ is also determined by (2.8). Hence $\tilde{\mathcal{A}} = \mathcal{A}_s$. \square

Afterward we call \mathcal{A}_s in (2.4) the associated semi-symmetric tensor. From the definition of a nonnegative polynomial map and (2.7), we know that P is nonnegative if and only if the coefficients of $P(x)$, $m_{ij} \geq 0, i = 1, \dots, n; j = 1, \dots, u_d$ which implies by (2.8) that $\mathcal{A}_s \geq 0$. Hence P is nonnegative if and only if its associated semi-symmetric tensor \mathcal{A}_s is nonnegative.

Example 2.1 (continued) Let \mathcal{A}_s be the associated semi-symmetric tensor of P . Then

$$\begin{aligned}
 \tilde{a}_{1111} &= \tilde{a}_{2222} = 2, \quad \tilde{a}_{3333} = 3; \\
 \tilde{a}_{1113} &= \tilde{a}_{1131} = \tilde{a}_{1311} = \frac{1}{|C_{113}|} = \frac{1}{3}; \\
 \tilde{a}_{2123} &= \tilde{a}_{2132} = \tilde{a}_{2213} = \tilde{a}_{2231} = \tilde{a}_{2312} = \tilde{a}_{2321} = \frac{4}{|C_{113}|} = \frac{4}{6} = \frac{2}{3}; \\
 \tilde{a}_{ijkl} &= 0 \text{ elsewhere.}
 \end{aligned}$$

It is noted that \mathcal{A} is neither nonnegative nor semi-symmetric, while P and its associated semi-symmetric tensor \mathcal{A}_s are nonnegative.

Hence, by means of nonnegative homogenous polynomial map we can extend nonnegative tensor in the following.

Definition 2.2 An $(d + 1)$ -order n -dimensional tensor \mathcal{A} is called semi-nonnegative, if P is nonnegative where $P(x) = \mathcal{A}x^d$.

The following is a corollary of Theorem 1.1.

Corollary 2.1 Assume that $(d + 1)$ -order n -dimensional tensor \mathcal{A} is semi-nonnegative, and that P is weakly irreducible where $P(x) = \mathcal{A}x^d$. Then the pair (x^*, λ^*) in Eq. (1.1) satisfy (1) and (2) in Theorem 1.1.

In the next section, we propose an algorithm for computing the largest eigenvalue of nonnegative homogenous polynomial map.

3 Algorithm for finding the largest eigenvalue of a nonnegative homogenous polynomial map

In this section, we propose an algorithm for computing the largest eigenvalue of a nonnegative homogenous polynomial map.

Let \mathcal{U} be the $(d + 1)$ -order n -dimensional unit tensor whose entries are

$$u_{i_1 i_2 \dots i_d} = \begin{cases} 1, & \text{if } i_1 = i_2 = \dots = i_d, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Let P be a nonnegative homogeneous polynomial map of degree d , \mathcal{A} be the associated semi-symmetric tensor of P . Then

$$P(x) + tx^{[d]} = (\mathcal{A} + t\mathcal{U})x^d, \tag{3.2}$$

where $t > 0$.

Lemma 3.1 *Let P be a nonnegative homogeneous polynomial map of degree d , \mathcal{A} be the associated semi-symmetric tensor, $t > 0$, and $P(x) + tx^{[d]} = (\mathcal{A} + t\mathcal{U})x^{[d]}$.*

- (1) *If \mathcal{A} is irreducible, then $(\mathcal{A} + t\mathcal{U})$ is primitive.*
- (2) *If λ is the largest eigenvalue of $(\mathcal{A} + t\mathcal{U})$, then $\lambda - t$ is the largest eigenvalue of P .*

Proof This lemma follows from Corollary 3 in [13] and Theorem 2.4 in [8]. □

We choose $\sigma > 0$, define $F : \mathbb{R}_{++}^{n+1} \mapsto \mathbb{R}^{n+1}$

$$F(x, \lambda) = \begin{pmatrix} P(x) + \sigma x^{[d]} - \lambda x^{[d]} \\ \frac{1}{d+1} \left(1 - \sum_{i=1}^n x_i^{d+1} \right) \end{pmatrix}, \tag{3.3}$$

where P is nonnegative homogeneous of degree d , and get an important relation between the solution of $F(x, \lambda)$ and the largest eigenvalue of P .

Lemma 3.2 *Let P and \mathcal{A} be nonnegative homogeneous of degree d and its associated semi-symmetric tensor, respectively. Assume that \mathcal{A} is irreducible. Then there is a pair $(\lambda^*, x^*) > 0$ where λ^* is the largest eigenvalue of P and (λ^*, x^*) satisfies (1.1), and thus there are $\tilde{\lambda}^* > 0$ and $\tilde{x}^* > 0$ such that $F(\tilde{x}^*, \tilde{\lambda}^*) = 0$.*

On the contrary, if there are $\tilde{\lambda} > 0$ and $\tilde{x} > 0$ such that $F(\tilde{x}, \tilde{\lambda}) = 0$, then $(\tilde{\lambda} - \sigma)$ is the largest eigenvalue of P .

Proof From Theorem 1.1, it follows that there exists a pair $(\lambda^*, x^*) > 0$ where λ^* is the largest eigenvalue of P and (λ^*, x^*) satisfies (1.1). Let

$$c = \sum_{i=1}^n (x_i^*)^{d+1}, \tilde{x} = \frac{x^*}{\sqrt[d+1]{c}}.$$

Then from (3.3) and direct computation we have that $(\tilde{x}, \lambda^* + \sigma)$ satisfies $F(\tilde{x}, \lambda^* + \sigma) = 0$.

Assume that there exist $\tilde{\lambda} > 0$ and $\tilde{x} > 0$ such that $F(\tilde{x}, \tilde{\lambda}) = 0$. Then from the first equation of (3.3), we have

$$\mathcal{A}\tilde{x}^d = (\tilde{\lambda} - \sigma)\tilde{x}^{[d]}, \tag{3.4}$$

i.e. $(\tilde{\lambda} - \sigma, \tilde{x})$ satisfies (1.1). Because \mathcal{A} is the associated semi-symmetric tensor of P , by Lemma 3.1

$$P(\tilde{x}) = \mathcal{A}\tilde{x}^d.$$

Combining with (3.4) we have

$$P(\tilde{x}) = (\tilde{\lambda} - \sigma)\tilde{x}^{[d]}.$$

According to the assumption, \mathcal{A} is irreducible. From Lemma 2.2 in [10], it follows that $\tilde{\lambda} - \sigma > 0$ which implies that $\tilde{\lambda} - \sigma$ is the largest eigenvalue of P from Theorem 1.1. □

Based on Lemma 3.2, the problem for finding the largest eigenvalue of P is equivalent to solving (3.3). In the computation, $P(x)$ is replaced by $\mathcal{A}_s x^d$ where \mathcal{A}_s is the associated semi-symmetric tensor of P . The computation of the Jacobian of $F(x, \lambda)$ is determined in the following lemma.

Lemma 3.3 *The Jacobian of $F(x, \lambda)$ is*

$$F'(x, \lambda) = \begin{pmatrix} d((\sigma\mathcal{U} + \mathcal{A}_s)x^{d-1} - \lambda \operatorname{diag}(x^{[d-1]})) - x^{[d]} \\ -(x^{[d]})^T \\ 0 \end{pmatrix}, \tag{3.5}$$

where

$$(\mathcal{A}_s x^{(d-1)})_{ij} = \sum_{i_2, \dots, i_d=1}^n a_{ij i_2 \dots i_d} x_{i_2} \cdots x_{i_d}, \quad i, j = 1, \dots, n.$$

Proof From direct computation, we have

$$\begin{aligned} (\mathcal{A}x^d)_i &= \sum_{i_1, i_2, \dots, i_d=1}^n a_{i i_1 i_2 \dots i_d} x_{i_1} \cdots x_{i_d} \\ &= \sum_{i_1 i_2 \dots i_d \in \Omega_{1,d}} |c_{i_1 i_2 \dots i_d}| a_{i i_1 i_2 \dots i_d} x_{i_1} \cdots x_{i_d} \\ &= \sum_{k=0}^d \sum_{i_1 i_2 \dots i_d \in \Omega_{1,d}^{j,k}} |c_{i_1 i_2 \dots i_d}| a_{i i_1 i_2 \dots i_d} x_{i_1} \cdots x_{i_d}, \end{aligned} \tag{3.6}$$

where $\Omega_{1,d}^{j,k}$ is a subset of $\Omega_{1,d}$, and its entries include k j 's, $k = 0, 1, \dots, d$; $j = 1, \dots, n$. In (3.6), the second equality is obtained by the same deduction as (2.9). We compute the partial derivative and obtain that

$$\begin{aligned} \frac{\partial (\mathcal{A}x^d)_i}{\partial x_j} &= \sum_{k=1}^d \sum_{i_1 i_2 \dots i_d \in \Omega_{1,d}^{j,k}} |c_{i_1 i_2 \dots i_d}| a_{i i_1 i_2 \dots i_d} \frac{\partial (x_{i_1} \cdots x_{i_d})}{\partial x_j} \\ &= \sum_{k=0}^{d-1} \sum_{i_2 \dots i_d \in \Omega_{2,d}^{j,k}} |c_{i_2 \dots i_d}^j| a_{i j i_2 \dots i_d} (k+1) (x_{i_2} \cdots x_{i_d}), \end{aligned} \tag{3.7}$$

where $|c_{i_2 \dots i_d}^j| = |c_{j i_2 \dots i_d}|$, and the position of the first subscript j may be changed so that the permutation of subscripts satisfies the order in $\Omega_{1,d}$. Let $i_2 \cdots i_d \in \Omega_{2,d}^{j,k}$, $\beta_{j,k}$ be the number of permutation $i_2 \cdots i_d$ deleting k j 's. Then

$$(k+1)|c_{i_2 \dots i_d}^j| = (k+1)C_d^{k+1} \beta_{j,k} = dC_{d-1}^k \beta_{j,k} = d|c_{i_2 \dots i_d}|.$$

Substituting in (3.7) with the above equality, we have

$$\begin{aligned} \frac{\partial (\mathcal{A}x^d)_i}{\partial x_j} &= \sum_{k=0}^{d-1} \sum_{i_2 \dots i_d \in \Omega_{2,d}^{j,k}} d|c_{i_2 \dots i_d}| a_{i j i_2 \dots i_d} x_{i_2} \cdots x_{i_d} \\ &= d(\mathcal{A}x^{d-1})_{ij}, \end{aligned}$$

for $i, j = 1, \dots, n$. Hence we obtain (3.5) by the simple computation of other elements in $F'(x, \lambda)$. □

It is noted that the expression of $\frac{\partial(\mathcal{A}x^d)_i}{\partial x_j}$ will be complex if \mathcal{A} is not semi-symmetric. Hence the property of semi-symmetric benefits the computation of Jacobian matrix. In the following, we give Newton method to find $\lambda^* > 0$ and $x^* > 0$ such that $F(x^*, \lambda^*) = 0$.

Algorithm 3.1 *Newton Method*

*Step 0. Choose initial values. Choose $x_1 \in \mathfrak{R}_+^n, \lambda_0 > \sigma > 0$, let $k = 1$.
 Step 1. Determine $(\Delta x_k, \Delta \lambda_k)$ by solving*

$$F(x_k, \lambda_k) + F'(x_k, \lambda_k) \begin{pmatrix} \Delta x_k \\ \Delta \lambda_k \end{pmatrix} = 0, \tag{3.8}$$

where $F(x_k, \lambda_k)$ and $F'(x_k, \lambda_k)$ are defined by (3.3) and (3.5).
 Step 2. Compute new iterate point,

$$(x_{k+1}, \lambda_{k+1}) = (x_k, \lambda_k) + (\Delta x_k, \Delta \lambda_k).$$

Step 3. Check termination condition. If $\|F(x_{k+1}, \lambda_{k+1})\|_2 = 0$, then stop and set $(x^, \lambda^*) = (x_{k+1}, \lambda_{k+1})$; otherwise $k = k + 1$, go to Step 1.*

4 Convergence of algorithm

Now we discuss convergence of Algorithm 3.1. It is easy to see $F(x, \lambda)$ is continuously differentiable in R_{++}^{n+1} . From Lemma 3.2 we know that there exists an (x^*, λ^*) in R_{++}^{n+1} such that $F(x^*, \lambda^*) = 0$. The following conclusion is important for the feasibility and convergence of Algorithm 3.1.

Lemma 4.1 *Assume that P is nonnegative and its associated semi-symmetric tensor \mathcal{A}_s is irreducible, $F(x, \lambda)$ is defined by (3.3), $(x^*, \lambda^*) \in R_{++}^{n+1}$ such that $F(x^*, \lambda^*) = 0$. Then $F'(x^*, \lambda^*)$ is nonsingular.*

Proof Let $P(x) = \mathcal{A}_s x^d$. Then from Lemma 2.1, \mathcal{A}_s is nonnegative.

Let $(z^T, t)^T \in R^{n+1}$ and

$$0 = F'(x^*, \lambda^*) \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} d \left([(\sigma \mathcal{U} + \mathcal{A}_s)x^{*(d-1)}]^T - \lambda^* \text{diag}(x^{*[d-1]}) \right) z - t x^{*[d]} \\ -(x^{*[d]})^T z \end{pmatrix}, \tag{4.1}$$

where the second equality is determined by (3.5).

Multiplying the first equation by x^* in left-side, we have

$$d \left((\sigma \mathcal{U} + \mathcal{A}_s)x^{*d} - \lambda^* x^{*[d]} \right)^T z - t \sum_{i=1}^n (x_i^*)^{d+1} = 0.$$

Because $F(x^*, \lambda^*) = 0$, i.e. $(\sigma \mathcal{U} + \mathcal{A}_s)x^{*d} = \lambda^* x^{*[d]}$ and $x^* > 0$, we have $t = 0$. Substituting into the first equation of (4.1), we have

$$((\sigma \mathcal{U} + \mathcal{A}_s)x^{*(d-1)} - \lambda^* \text{diag}(x^{*[d-1]}))z = 0. \tag{4.2}$$

Let $Q = (\sigma \mathcal{U} + \mathcal{A}_s)x^{*(d-1)} - \lambda^* \text{diag}(x^{*[d-1]})$. Before proving $z = 0$, we prove that Q is irreducible. If Q is not irreducible, then there exists a nonempty proper index subset $I \subset \{1, 2, \dots, n\}$ such that

$$q_{ij} = 0, \forall i \in I, \forall j \notin I. \tag{4.3}$$

The off-diagonal elements of Q are those of $\mathcal{A}_s x^{*(d-1)}$, i.e.

$$q_{ij} = \sum_{i_2 \dots i_d}^n a_{ij i_2 \dots i_d} x_{i_2}^* \cdots x_{i_d}^*, \quad \forall i \neq j.$$

By $\mathcal{A}_s \geq 0$ and $x^* > 0$, (4.3) means that

$$a_{ij i_2 \dots i_d} = 0, \quad \forall i \in I, \forall j \notin I, i_2, \dots, i_d = 1, \dots, n,$$

which contradicts the fact that \mathcal{A}_s is irreducible (see (1.4)).

The diagonal elements of Q may be not positive, however there exists a positive number $\delta > 0$ such that $(\delta E + Q) \geq 0$, where E is unit matrix. Meanwhile, $\delta E + Q$ is irreducible, and by (4.2)

$$(\delta E + Q)z = \delta z.$$

If $z \neq 0$, then according to Perron-Frobenius theorem of a nonnegative irreducible matrix, $z > 0$ or $z < 0$. However the second equation of (4.1) means $z^T x^{*[d]} = 0$, thus $z = 0$.

Hence $(z, t) = 0$ which implies that $F'(x^*, \lambda^*)$ is nonsingular. □

Lemma 4.2 Assume $(x^*, \lambda^*) \in R_{++}^{n+1}$ such that $F(x^*, \lambda^*) = 0$. Then there exists a constant $c_1 > 0$ such that

$$\|F'(x, \lambda) - F'(x^*, \lambda^*)\| \leq c_1 \|(x, \lambda) - (x^*, \lambda^*)\|, \quad \forall (x, \lambda) \in D$$

where D is the neighborhood of (x^*, λ^*) .

Proof From the definition, we have

$$F'(x, \lambda) - F'(x^*, \lambda^*) = \begin{pmatrix} d\{(Ax^{(d-1)} - Ax^{*(d-1)}) - (\lambda \text{diag}(x^{[d-1]} - \lambda^* \text{diag}(x^{*[d-1]}))\} x^{*[d]} - x^{[d]} \\ (x^{*[d]} - x^{[d]})^T \hspace{15em} 0 \end{pmatrix}. \tag{4.4}$$

By direct computation, we obtain that

$$\begin{aligned} & Ax^{(d-1)} - Ax^{*(d-1)} \\ &= Ax^{(d-1)} - Ax^{(d-2)}x^* + Ax^{(d-2)}x^* - \dots + Ax x^{*(d-2)} - Ax^{*(d-1)} \\ &= [Ax^{(d-2)} + Ax^{(d-3)}x^* + \dots + Ax^{*(d-2)}](x - x^*), \end{aligned} \tag{4.5}$$

$$\begin{aligned} & x^{*[d]} - x^{[d]} \\ &= x^{*[d]} - x^{*[d-1]} \circ x + x^{*[d-1]} \circ x - \dots + x^{[d-1]} \circ x^* - x^{[d]} \\ &= \text{diag}(x^{*[d-1]} + x^{*[d-2]} \circ x + \dots + x^{[d-1]})(x^* - x), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} & \lambda \text{diag}(x^{[d-1]}) - \lambda^* \text{diag}(x^{*[d-1]}) \\ &= \lambda \text{diag}(x^{[d-1]}) - \lambda^* \text{diag}(x^{[d-1]}) + \lambda^* \text{diag}(x^{[d-1]}) - \lambda^* \text{diag}(x^{*[d-1]}) \\ &= (\lambda - \lambda^*) \text{diag}(x^{[d-1]}) + \lambda^* \text{diag}(x^{[d-1]} - x^{*[d-1]}), \\ &= (\lambda - \lambda^*) \text{diag}(x^{[d-1]}) + \lambda^* \text{diag}(x^{[d-2]} + x^{[d-3]} \circ x^* + \dots + x^{*[d-2]}) \text{diag}(x - x^*). \end{aligned} \tag{4.7}$$

Substituting into (4.4) with (4.5)–(4.7), we obtain the conclusion of this lemma from basic property of matrix and vector norm. □

Denote $w_k = (x_k, \lambda_k)$, $p_k = (\Delta x_k, \Delta \lambda_k)$. The quadratic convergence of Algorithm 3.1 is given below.

Theorem 4.1 *Let P and A be a nonnegative homogeneous map of degree d and its associated semi-symmetric tensor, respectively. Assume that A is irreducible. Then we have the following conclusions.*

- (1) *There is an open set S which contains $w^* = (x^*, \lambda^*)$ such that for any $w_0 \in S$ the iterates w_k generated by Algorithm 3.1 are well-defined, remain in S and converge to w^* .*
- (2) *There is a constant $\beta > 0$ such that*

$$\|w_{k+1} - w^*\| \leq \beta \|w_k - w^*\|^2, k = 0, 1, 2, \dots$$

Proof The conclusion in this theorem is obtained by Lemmas 4.1 and 4.2, Theorem 2.1 in [4].

In order to obtain a globally convergent algorithm, we use the global technique in Algorithm 3.1. A merit function $f(x) = \frac{1}{2} \|F(x)\|^2$ is chosen, p_k is modified such that

$$-p_k^T \nabla f(w_k) \geq \delta \|p_k\| \|\nabla f(w_k)\|, \tag{4.8}$$

a line search is used to find a steplength $t_k \in (0, 1]$ such that

$$f(w_k + t_k p_k) \leq f(w_k) + \rho t_k \nabla f(w_k)^T p_k, \tag{4.9}$$

$$\nabla f(w_k + t_k p_k)^T p_k \geq \bar{\sigma} \nabla f(w_k)^T p_k, \tag{4.10}$$

and

$$w_k + t_k p_k \in \mathfrak{R}_{++}^{n+1}, \tag{4.11}$$

where $\rho \in (0, 0.5)$, $\bar{\sigma} \in (\rho, 1)$, $\delta \in (0, 1)$.

Algorithm 4.1 *Global Newton method*

Step 0. *Choose initial values. Choose $\rho \in (0, 0.5)$, $\bar{\sigma} \in (\rho, 1)$, $\delta \in (0, 1)$, $\epsilon \geq 0$, $w_1 \in \mathfrak{R}_{++}^{n+1}$, where $w_{n+1}^{(1)} > \sigma > 0$, $k = 1$.*

Step 1. *Find $p_k = (\Delta x_k, \Delta \lambda_k)$ by solving (3.8). If (4.8) is not satisfied, then choose τ_k and compute*

$$p_k = -((F'_k)^T F'_k + \tau_k I)^{-1} \nabla f(w_k), \tag{4.12}$$

such that (4.8) is satisfied.

Step 2. *Find a steplength $t_k \in (0, 1]$ satisfying (4.9), (4.10) and (4.11). Set*

$$w_{k+1} = w_k + t_k p_k. \tag{4.13}$$

Step 3 *Check termination condition. If $f(w_{k+1}) \leq \epsilon$, then set $(x^*, \lambda^*) = w_{k+1}$ and stop; otherwise $k = k + 1$, go to Step 1.*

Remark In Step 1, if (4.8) is not satisfied, then the technique in [12, pages 264–265] is used to choose τ_k . According to Lemma 3.1 in [12], it is not difficult to find a t_k which can satisfy (4.9), (4.10) and (4.11).

In order to discuss the global convergence, we define the level set

$$L(w_0) = \{w \mid f(w) \leq f(w_0)\}$$

for $w_0 \in R^{n+1}$.

Theorem 4.2 Suppose that $F'(w)$ is Lipschitz continuous in $L(w_0)$. Then we have

$$\lim_{k \rightarrow \infty} \nabla f(w_k) = 0.$$

Proof This theorem follows Theorem 3.2 in [12] and the condition (4.8).

Theorem 4.3 Let P be nonnegative and its associated semi-symmetric tensor \mathcal{A}_s be irreducible, $F : \mathfrak{N}_{++}^{n+1} \rightarrow \mathfrak{N}^{n+1}$ be defined by (3.3), $w_0 \in \mathfrak{N}_{++}^{n+1}$. Assume that $F'(w)$ is Lipschitz continuous in $L(w_0)$, $t_k = 1$ is admissible and (4.8) is satisfied for all k greater than a certain index k_0 , $F'(w^*)$ is nonsingular where w^* is a limited point of $\{w_k\}$ generated by Algorithm 4.1. Then $\{w_k\}$ quadratically converges to w^* .

Proof By Theorem 4.2, we have

$$\lim_{k \rightarrow \infty} \nabla f(w_k) = 0,$$

where $\nabla f(w_k) = F'(w_k)F(w_k)$. Hence, $\{w_k\}$ is convergent. Let w^* be the limited point. Because $F'(w^*)$ is nonsingular, we obtain

$$\lim_{k \rightarrow \infty} F(w_k) = 0.$$

From the assumption, we know that there exists k_0 such that $t_k = 1$ is admissible and (4.8) is satisfied for all $k \geq k_0$. This means that w_k can be generated by Algorithm 3.1 for $k > k_0$. The conclusion is obtained by Theorem 4.1. □

5 Numerical test

In this section, we present some preliminary numerical tests. We use Algorithm 4.1 to compute some randomly generated problems. The computation was done on a personal computer (Pentium IV, 1.83GHz,512MB) running Matlab 7.0.

Because it is difficult to find test problems in the literature, we generate some problems by the random approaches for testing the performance of Algorithm 4.1. In the following problems, let $P(x) = \mathcal{A}x^d \in \mathfrak{N}^n$ where d is odd.

TP I and TP II are generated from some similar problems in [8].

TP I (special case) $P(x) = Ax^d$ is defined as

$$p(x) = \begin{pmatrix} (a_1 + \gamma)x_1^d + 0.5 \left(\sum_{i=1}^n b_i x_i \sum_{i=1}^n c_i x_i \right)^{\frac{d-1}{2}} \left(c_1 \sum_{i=1}^n b_i x_i + b_1 \sum_{i=1}^n c_i x_i \right) \\ \dots \\ (a_j + \gamma)x_j^d + 0.5 \left(\sum_{i=1}^n b_i x_i \sum_{i=1}^n c_i x_i \right)^{\frac{d-1}{2}} \left(c_j \sum_{i=1}^n b_i x_i + b_j \sum_{i=1}^n c_i x_i \right) \\ \dots \\ (a_n + \gamma)x_n^d + 0.5 \left(\sum_{i=1}^n b_i x_i \sum_{i=1}^n c_i x_i \right)^{\frac{d-1}{2}} \left(c_n \sum_{i=1}^n b_i x_i + b_n \sum_{i=1}^n c_i x_i \right) \end{pmatrix} \quad (5.1)$$

where $a_i, b_i, c_i (i = 1, \dots, n)$ are random number in $[0, 1]$, $\gamma > 0$ is a parameter.

Table 1 Results of Problem I

TPI			Algorithm 4.1			Power algorithm		
<i>d</i>	<i>n</i>	γ	iter	term	cpu	iter	term	cpu
3	20	10	8	8.1564e−6	0.0110	7	4.3884e−7	0.0050
3	20	10 ²	9	1.1396e−13	0.0094	14	4.2976e−6	0.0060
3	20	10 ³	8	6.8536e−6	0.0069	57	9.7904e−6	0.0113
3	20	10 ⁴	9	0.00	0.0060	171	9.5596e−6	0.0335
3	60	10 ²	10	3.6380e−12	0.0329	9	1.0862e−6	0.0160
3	60	10 ³	10	7.2760e−12	0.0172	10	3.9026e−6	0.0063
3	60	10 ⁴	10	3.6380e−12	0.0132	22	4.1141e−6	0.0115
3	60	10 ⁵	10	3.7774e−10	0.0123	160	9.7867e−6	0.0740
3	100	10 ³	10	2.1849e−7	0.0198	9	7.8849e−7	0.0088
3	100	10 ⁴	10	2.2774e−7	0.0186	13	2.9888e−6	0.0110
3	100	10 ⁵	10	2.8368e−7	0.0184	43	6.4058e−6	0.0335
3	100	10 ⁶	10	2.4005e−7	0.0192	291	9.7326e−6	0.2133

TP II (special case) $P(x) = Ax^d$ is defined as

$$p(x) = \begin{pmatrix} (a_1 + \gamma)x_1^d + 0.5 \left(\sum_{i=1}^n b_i x_i \right)^{\frac{d+1}{2}} x_1^{\frac{d-1}{2}} + 0.5 \left(\sum_{i=1}^n x_i^{\frac{d+1}{2}} \right) \left(\sum_{i=1}^n b_i x_i \right)^{\frac{d-1}{2}} b_1 \\ \dots \\ (a_j + \gamma)x_j^d + 0.5 \left(\sum_{i=1}^n b_i x_i \right)^{\frac{d+1}{2}} x_j^{\frac{d-1}{2}} + 0.5 \left(\sum_{i=1}^n x_i^{\frac{d+1}{2}} \right) \left(\sum_{i=1}^n b_i x_i \right)^{\frac{d-1}{2}} b_j \\ \dots \\ (a_n + \gamma)x_n^d + 0.5 \left(\sum_{i=1}^n b_i x_i \right)^{\frac{d+1}{2}} x_n^{\frac{d-1}{2}} + 0.5 \left(\sum_{i=1}^n x_i^{\frac{d+1}{2}} \right) \left(\sum_{i=1}^n b_i x_i \right)^{\frac{d-1}{2}} b_n \end{pmatrix} \tag{5.2}$$

where $a_i, b_i (i = 1, \dots, n)$ are random number in $[0, 1]$, $\gamma > 0$ is a parameter.

TP III (general case)

$$p(x) = Mr(x) + \gamma x^d, \tag{5.3}$$

where $p(x)$ is homogeneous polynomial map with the degree d , $M = (m_{ij}) \in \mathbb{R}^{n \times u_d}$, u_d is defined in (2.5), $m_{ij} (i = 1, \dots, n; j = 1, \dots, u_d)$ are random numbers in $[0, 1]$, $\gamma > 0$ is a parameter.

Algorithm 4.1 and the power algorithm in [5] are used to solve these problems. Tables 1, 2 and 3 list the numerical results of TP I-III where d, n are the number of dimension and order in tensor, γ is chosen from 10 to 10^7 , iter is the number of iterations, cpu is cpu time in seconds. In order to compare these two methods, we choose the same initial point $x_0 = (1, \dots, 1)^T$, and the same termination conditions

$$\|\lambda_k - \lambda_{k+1}\| \leq 10^{-5}, \quad \|P(x_{k+1}) - \lambda_{k+1} x_{k+1}^{[d]}\|_2 \leq 10^{-5}.$$

If the termination conditions are not satisfied in 500 iteration, then the algorithm stops. In addition, term is the last value of $\|\lambda_k - \lambda_{k+1}\|$.

The results reported in Tables 1, 2 and 3 show that Algorithm 4.1 performs well for these test problems. Although γ changes a lot, there is only subtle changes in the number of iterations of Algorithm 4.1. When γ is relative small, the number of iteration of the Power

Table 2 Results of Problem II

TPII			Algorithm 4.1			Power algorithm		
d	n	γ	iter	term	cpu	iter	term	cpu
3	20	10	9	3.5135e-8	0.0081	14	9.2605e-6	0.0065
3	20	10 ²	9	6.5457e-8	0.0075	30	9.6946e-6	0.0088
3	20	10 ³	9	5.9262e-8	0.0062	211	9.8138e-6	0.0359
3	20	10 ⁴	9	3.0865e-8	0.0065	>500		
3	60	10	9	4.5475e-13	0.0257	15	9.6892e-6	0.0097
3	60	10 ²	9	6.8212e-13	0.0130	17	5.0157e-6	0.0112
3	60	10 ³	9	6.4213e-13	0.0112	38	7.6581e-6	0.0111
3	60	10 ⁴	9	1.8190e-12	0.0110	178	9.3875e-6	0.0430
3	100	10	9	1.6151e-8	0.0288	16	6.6040e-6	0.0212
3	100	10 ²	9	1.1550e-8	0.0142	16	7.1325e-6	0.0105
3	100	10 ³	9	1.5506e-8	0.0147	25	6.5653e-6	0.0103
3	100	10 ⁴	9	1.4251e-8	0.0148	84	9.9054e-6	0.0333
7	100	10 ⁴	10	9.6392e-7	0.0226	25	9.9586e-6	0.0111
7	100	10 ⁵	11	9.3132e-10	0.0467	29	6.2184e-6	0.0223
7	100	10 ⁶	11	1.8626e-9	0.0277	34	6.0468e-6	0.0166
7	100	10 ⁷	11	3.7253e-9	0.0200	87	7.8995e-6	0.0355

Table 3 Results of Problem III

TPIII			Algorithm 4.1			Power algorithm		
d	n	γ	iter	term	cpu	iter	term	cpu
3	20	10 ²	8	4.5475e-13	0.1833	9	3.7559e-6	0.0806
3	20	10 ³	8	1.3188e-11	0.1864	27	6.9269e-6	0.2029
3	20	10 ⁴	8	2.3481e-7	0.1919	164	9.5877e-6	1.2369
3	20	10 ⁵	9	1.3097e-10	0.2127	> 500		
3	40	10 ²	8	9.2580e-8	2.2022	6	5.0420e-6	0.8786
3	40	10 ³	8	8.8839e-7	2.1601	11	2.5065e-6	1.5659
3	40	10 ⁴	8	8.9635e-6	2.1639	35	8.0440e-6	4.8113
3	40	10 ⁵	9	4.8021e-10	2.4448	225	9.6221e-6	32.228
3	60	10 ²	9	1.4522e-11	17.6957	6	1.7072e-7	8.3350
3	60	10 ³	9	1.0914e-11	17.7530	8	1.8095e-6	11.4625
3	60	10 ⁴	9	6.8556e-15	17.6028	18	3.6666e-06	25.2876
3	60	10 ⁵	9	1.1059e-9	17.7278	87	8.6087e-6	123.5715

algorithm is the same as or a little more than that of Algorithm 4.1, and uses less time. While γ increases, the Power algorithm needs more than two times of number of iteration for Algorithm 4.1, and uses more time.

From Corollary 5.2 in [5], it follows that the Power algorithm slowly converges, when $\frac{\lambda}{r}$ approaches to 1, where r is the maximal modulus of the eigenvalues of $p(x)$ distinct from λ . From TPI-III, it is easy to see that as γ is larger and larger, $\frac{\lambda}{r}$ is very close to 1. Hence, it is not strange that the Power algorithm needs more time and more number of iterations to solve TPI-III when γ is large.

Table 4 lists the numerical result of two algorithms which solves TPIII when $d = 3, n = 40, \gamma = 10,000$, which show the quadratic convergence of Algorithm 4.1 and linear convergence of the Power algorithm.

Table 4 Convergent rate

k	Algorithm 4.1 term	Power algorithm term
1	2.6880e+3	1.5818e+4
2	2.2477e+3	2.6090e+1
3	2.0965e+3	1.6462e+1
4	2.0965e+3	1.0412e+1
5	9.6079e+2	6.5965e+0
6	1.0508e+2	4.1832e+0
7	5.8060e−1	2.5645e+0
8	8.9635e−6	1.6851e+0
9		1.0700e+0
10		6.7950e−1
15		7.0301e−2
20		7.3150e−3
25		4.7758e−4
30		4.9415e−5
35		8.0440e−6

6 Comments

In this paper we propose a quadratically convergent algorithm for computing the largest eigenvalue of a nonnegative homogeneous polynomial map where the Newton method is used to solve an equivalent system of nonlinear equations. The semi-symmetric tensor is introduced to reveal the relation between homogeneous polynomial map and its associated semi-symmetric tensor. The technique of computation may be used to find the other eigenvalues of a nonnegative homogeneous polynomial map or a nonnegative tensor.

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