Are There Sixth Order Three Dimensional PNS Hankel Tensors?

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Abstract

Are there positive semi-definite (PSD) but not sums of squares (SOS) Hankel tensors? If the answer to this question is "no", then the problem for determining an even order Hankel tensor is PSD or not can be solved in polynomial-time. By Hilbert, one of the cases of low order (degree) and dimension (number of variables), in which PSD non-SOS (PNS) symmetric tensors (homogeneous polynomials) exists, is of order six and dimension three. The famous Motzkin polynomial falls into this case. In this paper, we study the existence problem of sixth order three dimensional PNS Hankel tensors. We examine various important classes of sixth order three dimensional Hankel tensors. No PNS Hankel tensors are found in these cases. We then randomly generate several thousands of sixth order three dimensional Hankel tensors and make them PSD by adding adequate multiple of a fixed sixth order three dimensional positive definite Hankel tensor. Again, still no PNS Hankel tensors are found. Thus, we make a conjecture that there are no sixth order three dimensional PNS Hankel tensors. If this conjecture turns out to be true, this implies that the problem for determining a given sixth order three dimensional Hankel tensor is PSD or not can be solved by a semi-definite linear programming problem.

Key words: Hankel tensors, generating vectors, sum of squares, positive semi-definiteness, PNS Hankel tensors.

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1 Introduction

Hankel tensors arise from signal processing and some other applications [2, 6, 17, 19]. They are symmetric tensors. In [19], two classes of positive semi-definite (PSD) Hankel tensors were identified. They are even order strong Hankel tensors and even order complete Hankel tensors. In [13], it was proved that complete Hankel tensors are strong Hankel tensors, and even order strong Hankel tensors are SOS (sum of squares) Hankel tensors. Some other PSD Hankel tensors were identified in [13]. They are not strong Hankel tensors. But they are still SOS Hankel tensors. A question has been raised in [13]: Are there PSD non-SOS Hankel tensors? If there are no PSD non-SOS Hankel tensors, then the problem for determining a given even order Hankel tensor is PSD or not is polynomial time solvable [13]. In the following, as in [3], we further abbreviate "PSD non-SOS" to "PNS".

A symmetric tensor is uniquely corresponding to a homogeneous polynomial. It was proved by young Hilbert [8] that for homogeneous polynomials, only in the following three cases, a PSD polynomial definitely is an SOS polynomial: 1) $n = 2$; 2) $m = 2$; 3) $m = 4$ and $n = 3$, where m is the degree of the polynomial and n is the number of variables. For symmetric tensors, m is the order and n is the dimension. Hilbert proved that in all the other possible combinations of $m = 2k$ and n, there are PNS homogeneous polynomials. The most well-known PNS homogeneous polynomial is the Motzkin polynomial [16] given by

$$f_M(\mathbf{x}) = x_3^6 + x_1^2x_2^4 + x_1^4x_2^2 - 3x_1^2x_2^2x_3^2.$$

For the Motzkin polynomial, $m = 6$ and $n = 3$., i.e., the Motzkin polynomial is a ternary sextic [20, 21]. There are other examples of PNS homogeneous polynomials [1, 3, 4, 20, 21].

In a certain sense, the question raised in [13] is the Hilbert problem under the Hankel constraint.

Thus, an easier question has also been raised in [13]: Are there sixth order three dimensional PNS Hankel tensors? In the language of polynomials [20, 21], this question is: are there PNS Hankel ternary sextics? In this paper, we study this problem and provide some partial answers.

Let $\mathbf{v} = (v_0, v_1, \cdots, v_{12})^\top \in \Re^{13}$. A sixth order three dimensional **Hankel tensor** $\mathcal{A} = (a_{i_1\cdots i_6})$ is defined by

$$a_{i_1\cdots i_6} = v_{i_1+\cdots+i_6-6},$$

for $i_1, \cdots, i_6 = 1, 2, 3$. The corresponding vector $\mathbf{v}$ that defines the Hankel tensor $\mathcal{A}$ is called the **generating vector** of $\mathcal{A}$. For $\mathbf{x} = (x_1, x_2, x_3)^\top \in \Re^3$, $\mathcal{A}$ uniquely defines a homogeneous polynomial (a ternary sextic)

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes 6} = \sum_{i_1,\cdots,i_6=1}^{3} a_{i_1\cdots i_6} x_{i_1}\cdots x_{i_6} = \sum_{i_1,\cdots,i_6=1}^{3} v_{i_1+\cdots+i_6-6} x_{i_1}\cdots x_{i_6}. \tag{1}$$

We call such a polynomial a **(ternary sextic) Hankel polynomial**.

We study several special classes of sixth order three dimensional Hankel tensors.

The first class of Hankel tensors we examined is called truncated Hankel tensors. The generating vector $\mathbf{v}$ of a sixth order three dimensional truncated Hankel tensor $\mathcal{A}$ has only three nonzero entries: v_0, v_6 and v_{12}. We provide a sufficient and necessary condition that a sixth order three dimensional truncated Hankel tensor to be PSD. We show that such truncated Hankel tensors are PSD if and only if they are SOS. We also show that such SOS Hankel tensors are not strong Hankel tensors unless $v_6 = 0$.

The second class of Hankel tensors is called quasi-truncated Hankel tensors. The generating vector $\mathbf{v}$ of a sixth order three dimensional quasi-truncated Hankel tensor $\mathcal{A}$ has five nonzero entries: v_0, v_1, v_6, v_{11} and v_{12}. It is still true that such SOS Hankel tensors are not strong Hankel tensors unless $v_1 = v_6 = v_{11} = 0$. In this case, still no PNS Hankel tensors are found.

To motivate the third class of Hankel tensors, we recall that, beside the Motzkin polynomial, there is another well-known PNS homogeneous polynomial for $m = 6$ and $n = 3$. This is the Choi-Lam polynomial [4, 20]:

$$f_{CL}(\mathbf{x}) = x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2 - 3x_1^2x_2^2x_3^2.$$

An important property of the Choi-Lam polynomial is that

$$f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = f(x_3, x_1, x_2)$$

for any $\mathbf{x} \in \Re^3$. The generating vector $\mathbf{v}$ of a sixth order three dimensional Hankel tensor $\mathcal{A}$, associated with such a ternary sextic has the property

$$v_i = v_{i+3}, \tag{2}$$

for $i = 0, \cdots, 9$. By [6], a Hankel tensor satisfying (2) is called an anti-circulant tensor. The name “anti-circulant tensor” is an extension of the name “anti-circulant matrix” [5]. This is indeed the third class which we study in this paper. We show that a sixth order three dimensional anti-circulant tensor is PSD if and only if it is a nonnegative multiple of the all one tensor, which is an SOS Hankel tensor. Thus, no PNS Hankel tensors are found in this case.

The fourth class of Hankel tensors is defined that the generating vectors $\mathbf{v}$ of such Hankel tensors satisfy

$$v_i = v_{i+2},$$

for $i = 0, \cdots, 10$. We call such Hankel tensors alternatively anti-circulant tensors. We give a sufficient and necessary condition for a sixth order three dimensional alternatively anti-circulant tensor to be PSD, and show that a sixth order three dimensional PSD alternatively anti-circulant tensor is a strong Hankel tensor, hence an SOS Hankel tensor. Thus, still no PNS Hankel tensors are found.

Since we cannot find sixth order three dimensional PNS Hankel tensors in all the above four special cases, we turn our search to numerical tests. To conduct the numerical tests, we randomly generate several thousands of sixth order three dimensional Hankel tensors and make them PSD but not positive definite by adding adequate multiple of a fixed sixth order three dimensional positive definite Hankel tensor. Again, still no PNS Hankel tensors are found. Thus, we make a conjecture that there are no sixth order three dimensional PNS Hankel tensors. If this conjecture is true, then the problem for determining a given sixth order three dimensional Hankel tensor is PSD or not can be solved by a semi-definite linear programming problem.

The remainder of this paper is organized as follows. In the next section, we first review the definitions of sixth order three dimensional PSD Hankel tensors, SOS Hankel tensors and strong Hankel tensors. Then we write out the corresponding homogeneous polynomial of a sixth order three dimensional Hankel tensor explicitly. This will be helpful for our further discussion. Some simple properties of sixth order three dimensional Hankel tensors are obtained from this form. We study sixth order three dimensional truncated Hankel, quasi-truncated Hankel, anti-circulant and alternatively anti-circulant tensors in Sections 3-6 respectively. Numerical tests on randomly generated sixth order three dimensional Hankel tensors are reported in Section 7.

2 Sixth Order Three Dimensional Hankel Tensors

In the introduction, we have already defined sixth order three dimensional Hankel tensors and their associated homogeneous polynomials (ternary sextics). In (1), if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Re^3$, then f is called a **PSD** (positive semi-definite) **Hankel polynomial** and $\mathcal{A}$ is called a **PSD Hankel tensor** [18]. Denote $\mathbf{0} = (0, 0, 0)^\top \in \Re^3$. If $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Re^3, \mathbf{x} \neq \mathbf{0}$, then f and $\mathcal{A}$ are called **positive definite**. If f can be decomposed to the sum of squares of polynomials of degree three, then f is called an **SOS Hankel polynomial** and $\mathcal{A}$ is called an **SOS Hankel tensor** [9, 10, 13, 15]. Clearly, an SOS Hankel tensor is a PSD Hankel tensor but not vice versa. By [19], a necessary condition for $\mathcal{A}$ to be PSD is that

$$v_0 \geq 0,\ v_6 \geq 0,\ v_{12} \geq 0. \tag{3}$$

Let $\mathbf{e}_1 = (1, 0, 0)^\top, \mathbf{e}_2 = (0, 1, 0)^\top$ and $\mathbf{e}_3 = (0, 0, 1)^\top$. Substitute them to (1). Then we get

(3) directly. The generating vector $\mathbf{v}$ may also generate a 7×7 Hankel matrix $A = (a_{ij})$ by

$$a_{ij} = v_{i+j-2},$$

for $i, j = 1, \cdots, 7$. If the associated Hankel matrix A is PSD, then the Hankel tensor $\mathcal{A}$ is called a **strong Hankel tensor** [19]. In [13], it was proved that an even order strong Hankel tensor is an SOS Hankel tensor. On the other hand, the converse is not true in general [19, 13]. A necessary condition for $\mathcal{A}$ to be a strong Hankel tensor is that

$$v_0 \geq 0,\ v_2 \geq 0,\ v_4 \geq 0,\ v_6 \geq 0,\ v_8 \geq 0,\ v_{10} \geq 0,\ v_{12} \geq 0. \tag{4}$$

A simple example of Hankel tensor is the **Hilbert tensor**. The sixth order three dimensional Hilbert tensor $\mathcal{H}$ has the form $\mathcal{H} = (\frac{1}{i_1+\cdots+i_6-5})$. Its generating vector is $\mathbf{v} = (1, \frac{1}{2}, \cdots, \frac{1}{13})^\top$. It was shown in [22] that $\mathcal{H}$ is positive definite. It is easy to see that the associated Hankel matrix of the Hilbert tensor is a Hilbert matrix, which is positive definite. Thus, the sixth order three dimensional Hilbert tensor $\mathcal{H}$ is a strong Hankel tensor, and hence is an SOS tensor.

We may write out (1) explicitly in terms of the coordinates of its generating vector $\mathbf{v}$. Then we have

$$\begin{aligned} f(\mathbf{x}) \ = \ & v_0 x_1^6 + 6v_1 x_1^5 x_2 + v_2(15x_1^4 x_2^2 + 6x_1^5 x_3) + v_3(20x_1^3 x_2^3 + 30x_1^4 x_2 x_3) \\ & + v_4(15x_1^2 x_2^4 + 60x_1^3 x_2^2 x_3 + 15x_1^4 x_3^2) + v_5(6x_1 x_2^5 + 60x_1^2 x_2^3 x_3 + 60x_1^3 x_2 x_3^2) \\ & + v_6(x_2^6 + 30x_1 x_2^4 x_3 + 90x_1^2 x_2^2 x_3^2 + 20x_1^3 x_3^3) \\ & + v_7(6x_2^5 x_3 + 60x_1 x_2^3 x_3^2 + 60x_1^2 x_2 x_3^3) + v_8(15x_2^4 x_3^2 + 60x_1 x_2^2 x_3^3 + 15x_1^2 x_3^4) \\ & + v_9(20x_2^3 x_3^3 + 30x_1 x_2 x_3^4) + v_{10}(15x_2^2 x_3^4 + 6x_1 x_3^5) + 6v_{11} x_2 x_3^5 + v_{12} x_3^6. \end{aligned} \tag{5}$$

Let $g(\mathbf{y}) = \mathbf{y}^\top A \mathbf{y}$, where $\mathbf{y} = (y_1, \cdots, y_7)^\top \in \Re^7$ and A is the associated Hankel matrix of $\mathcal{A}$. Then

$$\begin{aligned} g(\mathbf{y}) \ = \ & v_0 y_1^2 + 2v_1 y_1 y_2 + v_2(y_2^2 + 2y_1 y_3) + v_3(2y_1 y_4 + 2y_2 y_3) \\ & + v_4(y_3^2 + 2y_1 y_5 + 2y_2 y_3) + v_5(2y_1 y_6 + 2y_2 y_5 + 2y_3 y_4) \\ & + v_6(y_4^2 + 2y_1 y_7 + 2y_2 y_6 + 2y_3 y_5) \\ & + v_7(2y_2 y_7 + 2y_3 y_6 + 2y_4 y_5) + v_8(y_5^2 + 2y_3 y_7 + 2y_4 y_6) \\ & + v_9(2y_4 y_7 + 2y_5 y_6) + v_{10}(y_6^2 + 2y_5 y_7) + 2v_{11} y_6 y_7 + v_{12} y_7^2. \end{aligned} \tag{6}$$

Thus, $\mathcal{A}$ is a strong Hankel tensor if and only if g is PSD.

These will be helpful for our further discussion.

If $\mathbf{v} = (1, 1, \cdots, 1)^\top$, then $\mathcal{A}$ is the all one tensor. By (5), in this case, $f(\mathbf{x}) = (x_1 + x_2 + x_3)^6$. Thus, the all one tensor is an SOS Hankel tensor, but not a positive definite tensor. By (6), it is a strong Hankel tensor.

Now we may have some simple properties of sixth order three dimensional Hankel tensors.

Theorem 1 *Suppose that $\mathcal{A} = (a_{i_1 \cdots i_6})$ is a Hankel tensor generated by its generating vector $\mathbf{v} = (v_0, v_1, \cdots, v_{12})^\top \in \Re^{13}$. If $\mathcal{A}$ is a PSD (or positive definite, or SOS, or strong) Hankel tensor, then the Hankel tensors $\mathcal{B}, \mathcal{C}, \mathcal{D}$, generated by $(v_{12}, v_{11}, \cdots, v_0)^\top$, $(v_0, -v_1, v_2, -v_3, \cdots, v_{12})^\top$, $(v_0, 0, v_2, 0, \cdots, v_{12})^\top$ are also a PSD (or positive definite, or SOS, or strong) Hankel tensor.*

Proof In (5) and (6), changing $\mathbf{x} = (x_1, x_2, x_3)^\top$ and $\mathbf{y} = (y_1, \cdots, y_7)^\top$ to $(x_3, x_2, x_1)^\top$ and $(y_7, \cdots, y_1)^\top$ respectively, we see that the conclusions on $\mathcal{B}$ hold.

In (5) and (6), changing $\mathbf{x} = (x_1, x_2, x_3)^\top$ and $\mathbf{y} = (y_1, \cdots, y_7)^\top$ to $(x_1, -x_2, x_3)^\top$ and $(y_1, -y_2, y_3, -y_4, \cdots, y_7)^\top$ respectively, we get the conclusions on $\mathcal{C}$.

Since $\mathcal{D} = \frac{\mathcal{A}+\mathcal{C}}{2}$, the conclusions on $\mathcal{D}$ follow. $\square$

3 Sixth Order Three Dimensional Truncated Hankel Tensors

In this section, we consider the case that the Hankel tensor $\mathcal{A}$ is generated by $\mathbf{v} = (v_0, 0, 0, 0, 0, 0, v_6, 0, 0, 0, 0, 0, v_{12})^\top$. Now, (5) and (6) have the simple form

$$f(\mathbf{x}) = v_0 x_1^6 + v_6(x_2^6 + 30x_1x_2^4x_3 + 90x_1^2x_2^2x_3^2 + 20x_1^3x_3^3) + v_{12}x_3^6 \tag{7}$$

and

$$g(\mathbf{y}) = v_0 y_1^2 + v_6(y_4^2 + 2y_1y_7 + 2y_2y_6 + 2y_3y_5) + v_{12}y_7^2. \tag{8}$$

We call such a Hankel tensor a **truncated Hankel tensor**. Since we are only concerned about PSD Hankel tensors, we may assume that (3) holds. From (7) and (8), we have the following proposition.

Proposition 1 *Suppose that (3) holds. If $v_6 = 0$, then the truncated Hankel tensor $\mathcal{A}$ is a strong Hankel tensor and an SOS Hankel tensor. If $v_6 > 0$, then $\mathcal{A}$ is not a strong Hankel tensor.*

Proof When $v_6 = 0$, from (7) and (8), we see that the truncated Hankel tensor $\mathcal{A}$ is a strong Hankel tensor and an SOS Hankel tensor. If $v_6 > 0$, consider $\bar{\mathbf{y}} = (0, 0, 1, 0, -1, 0, 0)^\top$. We see that $g(\bar{\mathbf{y}}) = -2v_6 < 0$. Hence $\mathcal{A}$ is not a strong Hankel tensor in this case. $\square$

We now give the main result of this section.

Theorem 2 *The following statements are equivalent:*

(i) *The truncated Hankel tensor $\mathcal{A}$ is a PSD Hankel tensor;*

(ii) *The truncated Hankel tensor $\mathcal{A}$ is an SOS Hankel tensor;*

(iii) *The relation (3) holds and*

$$\sqrt{v_0 v_{12}} \geq (560 + 70\sqrt{70}) v_6. \tag{9}$$

Furthermore, the truncated Hankel tensor $\mathcal{A}$ *is positive definite if and only if* $v_0, v_6, v_{12} > 0$ *and strict inequality holds in (9).*

Proof [(i) $\Rightarrow$ (iii)] Suppose that $\mathcal{A}$ is PSD, then clearly (3) holds. To see (iii), we only need to show (9) holds. Let $t \geq 0$ and let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^\top$, where

$$\bar{x}_1 = v_{12}^{\frac{1}{6}}, \ \bar{x}_2 = \sqrt{t}(v_0 v_{12})^{\frac{1}{12}}, \ \bar{x}_3 = -v_0^{\frac{1}{6}}.$$

Substitute them to (7). If $\mathcal{A}$ is PSD, then $f(\bar{\mathbf{x}}) \geq 0$. It follows from (7) that

$$v_0 v_{12} + v_6(t^3 - 30t^2 + 90t - 20)\sqrt{v_0 v_{12}} + v_0 v_{12} \geq 0.$$

From this, we have

$$\sqrt{v_0 v_{12}} \geq \frac{-t^3 + 30t^2 - 90t + 20}{2} v_6.$$

Substituting $t = 10 + \sqrt{70}$ to it, we have (9).

[(iii) $\Rightarrow$ (ii)] We now assume that (3) and (9) hold. We will show that $\mathcal{A}$ is SOS. If $v_6 = 0$, then by Proposition 1, $\mathcal{A}$ is an SOS Hankel tensor. Assume that $v_6 > 0$. By (9), $v_0 > 0$ and $v_{12} > 0$. We now have

$$f(\mathbf{x}) = 10v_6 \left(\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{4}} x_1^3 + \left(\frac{v_{12}}{v_0}\right)^{\frac{1}{4}} x_3^3 \right)^2 + v_6 \left(\sqrt{\frac{10 - \sqrt{70}}{2}} x_2^3 + \sqrt{150 + 15\sqrt{70}} x_1 x_2 x_3 \right)^2 + f_1(\mathbf{x}),$$

where

$$f_1(\mathbf{x}) = \left(v_0 - 10v_6 \left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}} \right) x_1^6 + \frac{\sqrt{70} - 8}{2} v_6 x_2^6 + \left(v_{12} - 10v_6 \left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}} \right) x_3^6 - (60 + 15\sqrt{70}) v_6 x_1^2 x_2^2 x_3^2. \tag{10}$$

We see that $f_1(\mathbf{x})$ is a diagonal minus tail form [7]. By the arithmetic-geometric inequality, we have

$$\left(v_0 - 10v_6 \left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}} \right) x_1^6 + \frac{\sqrt{70} - 8}{2} v_6 x_2^6 + \left(v_{12} - 10v_6 \left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}} \right) x_3^6$$

$$\geq 3 \left(\frac{\sqrt{70} - 8}{2} v_6 (\sqrt{v_0 v_{12}} - 10v_6)^2 \right)^{\frac{1}{3}} x_1^2 x_2^2 x_3^2.$$

By (9),

$$3 \left(\frac{\sqrt{70} - 8}{2} v_6 (\sqrt{v_0 v_{12}} - 10v_6)^2 \right)^{\frac{1}{3}} x_1^2 x_2^2 x_3^2 \geq (60 + 15\sqrt{70}) v_6 x_1^2 x_2^2 x_3^2. \tag{11}$$

Thus, f_1 is a PSD diagonal minus tail form. By [7], f_1 is an SOS polynomial. Hence, f is also an SOS polynomial if (3) and (9) hold.

[(ii) $\Rightarrow$ (i)] This implication is direct by the definition.

We now prove the last conclusion of this theorem. First, we assume that $\mathcal{A}$ is positive definite. Then, $v_6 = f(\mathbf{e}_2) > 0$ as $\mathbf{e}_2 \neq \mathbf{0}$. Similarly, $v_0 = f(\mathbf{e}_1) > 0$ and $v_{12} = f(\mathbf{e}_3) > 0$. Note that in the above [(i) $\Rightarrow$ (iii)] part, $f(\bar{\mathbf{x}}) > 0$ as $\bar{\mathbf{x}} \neq \mathbf{0}$. Then strict inequality holds for the last two inequalities in the above [(i) $\Rightarrow$ (iii)] part. This implies that strict inequality holds in (9).

On the other hand, assume that $v_0, v_6, v_{12} > 0$ and strict inequality holds in (9). Let $\mathbf{x} = (x_1, x_2, x_3)^\top \neq \mathbf{0}$. If $x_1 \neq 0, x_2 \neq 0$ and $x_3 \neq 0$, then strict inequality holds in (11) as $v_6 > 0$ and strict inequality holds in (9). Then $f_1(\mathbf{x}) > 0$. If $x_2 \neq 0$ but $x_1 x_3 = 0$, then from (10), we still have $f_1(\mathbf{x}) > 0$. If $x_2 = 0$ and one of x_1 and x_3 are nonzero, then we still have $f_1(\mathbf{x}) > 0$ by (10). Thus, we always have $f_1(\mathbf{x}) > 0$ as long as $\mathbf{x} \neq \mathbf{0}$. This implies $f(\mathbf{x}) > 0$ as long as $\mathbf{x} \neq \mathbf{0}$. Hence, $\mathcal{A}$ is positive definite. $\square$

4 Sixth Order Three Dimensional Quasi-Truncated Hankel Tensors

In this section, we consider the case that the Hankel tensor $\mathcal{A}$ is generated by $\mathbf{v} = (v_0, v_1, 0, 0, 0, 0, v_6, 0, 0, 0, 0, v_{11}, v_{12})^\top \in \Re^{13}$. Adding v_1 and v_{11} to the case in the last section, we get this case. We call such a Hankel tensor a **quasi-truncated Hankel tensor**. Hence, truncated Hankel tensors are quasi-truncated Hankel tensors.

Since we are only concerned about PSD Hankel tensors, we may assume that (3) holds. Now, (5) and (6) have the simple form

$$f(\mathbf{x}) = v_0 x_1^6 + 6v_1 x_1^5 x_2 + v_6(x_2^6 + 30x_1x_2^4x_3 + 90x_1^2x_2^2x_3^2 + 20x_1^3x_3^3) + 6v_{11}x_2x_3^5 + v_{12}x_3^6, \quad (12)$$

and

$$g(\mathbf{y}) = v_0 y_1^2 + 2v_1 y_1 y_2 + v_6(y_4^2 + 2y_1y_7 + 2y_2y_6 + 2y_3y_5) + 2v_{11}y_6y_7 + v_{12}y_7^2. \quad (13)$$

We first show that a result with the form of Proposition 1 continues to hold in this case.

Proposition 2 *Suppose that (3) holds. If $v_6 = 0$, then the quasi-truncated Hankel tensor $\mathcal{A}$ is PSD if and only if $v_1 = v_{11} = 0$. In this case, $\mathcal{A}$ is a strong Hankel tensor and an SOS Hankel tensor. If $v_6 > 0$, then $\mathcal{A}$ is not a strong Hankel tensor.*

Proof Suppose that $v_6 = 0$. Assume that $v_1 \neq 0$. If $v_0 = 0$, consider $\hat{\mathbf{x}} = (1, -v_1, 0)^\top$. Then $f(\hat{\mathbf{x}}) < 0$. If $v_0 > 0$, consider $\tilde{\mathbf{x}} = (1, -\frac{v_0}{v_1}, 0)^\top$. Then $f(\tilde{\mathbf{x}}) < 0$. Thus, $\mathcal{A}$ is not PSD in these two cases. Similar discussion holds for the case that $v_{11} = 0$. Assume now that $v_1 = v_{11} = 0$.

By Proposition 1, we see that the truncated Hankel tensor $\mathcal{A}$ is a strong Hankel tensor and an SOS Hankel tensor in this case. This proves the first part of this proposition.

Suppose that $v_6 > 0$. Consider $\bar{\mathbf{y}} = (0, 0, 1, 0, -1, 0, 0)^\top \in \Re^7$. We see that $g(\bar{\mathbf{y}}) = -2v_6 < 0$. Hence $\mathcal{A}$ is not a strong Hankel tensor in this case. $\square$

To present a necessary condition for a sixth order three dimensional quasi-truncated Hankel tensor to be PSD, we first prove the following lemma.

Lemma 1 *Consider*

$$\hat{f}(x_1, x_2) = v_0 x_1^6 + 6v_1 x_1^5 x_2 + v_6 x_2^6.$$

Then $\hat{f}$ is PSD if and only if $v_0 \geq 0$, $v_6 \geq 0$ and

$$|v_1| \leq \left(\frac{v_0}{5}\right)^{\frac{5}{6}} v_6^{\frac{1}{6}}. \tag{14}$$

Proof Suppose that $v_0 \geq 0$, $v_6 \geq 0$ and (14) holds. Then, by the arithmetic-geometric inequality, one has

$$\begin{aligned} v_0 x_1^6 + v_6 x_2^6 &= \frac{1}{5} v_0 x_1^6 + \frac{1}{5} v_0 x_1^6 + \frac{1}{5} v_0 x_1^6 + \frac{1}{5} v_0 x_1^6 + \frac{1}{5} v_0 x_1^6 + v_6 x_2^6 \\ &\geq 6 \left(\left(\frac{v_0}{5}\right)^5 x_1^{30} v_6 x_2^6 \right)^{\frac{1}{6}} \\ &\geq 6 |v_1 x_1^5 x_2|. \end{aligned}$$

This implies that $\hat{f}(x_1, x_2) \geq 0$ for any $(x_1, x_2)^\top \in \Re^2$, i.e., $\hat{f}(x_1, x_2)$ is PSD.

Suppose that $\hat{f}(x_1, x_2)$ is PSD. It is easy to see that $v_0 \geq 0$ and $v_6 \geq 0$. Assume now that (14) does not hold, i.e.,

$$|v_1| > \left(\frac{v_0}{5}\right)^{\frac{5}{6}} v_6^{\frac{1}{6}}. \tag{15}$$

If $v_0 = v_6 = 0$, let $x_1 = 1$ and $x_2 = -v_1$. Then $\hat{f}(x_1, x_2) < 0$. We get a contradiction. If $v_0 = 0$ and $v_6 \neq 0$, let $x_1 = v_6^{\frac{1}{5}}$ and $x_2 = -v_1^{\frac{1}{5}}$. Again, $\hat{f}(x_1, x_2) < 0$. We get a contradiction. Similarly, if $v_0 \neq 0$ and $v_6 = 0$, we may get a contradiction. If $v_0 \neq 0$ and $v_6 \neq 0$, let $x_1 = (5v_6)^{\frac{1}{6}}$ and $x_2 = -\frac{v_1}{|v_1|} v_0^{\frac{1}{6}}$. Then by (15),

$$\hat{f}(x_1, x_2) = 6v_0 v_6 - 6|v_1|(5v_6)^{\frac{5}{6}} v_0^{\frac{1}{6}} < 0.$$

We still get a contradiction. This completes the proof. $\square$

We now present a necessary condition for a sixth order three dimensional quasi-truncated Hankel tensor to be PSD.

Proposition 3 *Suppose that (3) holds. If $\mathcal{A}$ is a PSD quasi-truncated Hankel tensor, then (14) and the following inequalities*

$$|v_1| \leq \left(\frac{v_{12}}{5}\right)^{\frac{5}{6}} v_6^{\frac{1}{6}} \tag{16}$$

and

$$\sqrt{v_0 v_{12}} \geq 10 v_6 \tag{17}$$

hold. If furthermore

$$v_1 v_{12}^{\frac{5}{6}} = v_{11} v_0^{\frac{5}{6}}, \tag{18}$$

then (9) also holds.

Proof Suppose that $\mathcal{A}$ is PSD. In (12), let $x_3 = 0$. By Lemma 1, (14) holds. In (12), let $x_1 = 0$. By an argument similar to Lemma 1, (16) holds. In (12), let $x_2 = 0$. Since $\mathcal{A}$ is PSD, we may easily get (17).

Suppose further that (18) holds. As in [(i) $\Rightarrow$ (iii)] part of the proof of Theorem 2, we let $t \geq 0$ and let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^\top$, where $\bar{x}_1 = v_{12}^{\frac{1}{6}}$, $\bar{x}_2 = \sqrt{t}(v_0 v_{12})^{\frac{1}{12}}$, $\bar{x}_3 = -v_0^{\frac{1}{6}}$. It follows from (18) that

$$6 v_1 \bar{x}_1^5 \bar{x}_2 + 6 v_{11} \bar{x}_2 \bar{x}_3^5 = 0. \tag{19}$$

This together with (12) implies that

$$f(\bar{\mathbf{x}}) = v_0 v_{12} + v_6 (t^3 - 30t^2 + 90t - 20)\sqrt{v_0 v_{12}} + v_0 v_{12} \geq 0.$$

Proceed as in [(i) $\Rightarrow$ (iii)] part of the proof of Theorem 2, we see that (9) holds in this case. This completes the proof. □

We may also present a sufficient condition for a sixth order three dimensional quasi-truncated Hankel tensor to be SOS.

Proposition 4 *Let $\mathcal{A}$ be a quasi-truncated Hankel tensor. Suppose that $v_0, v_6, v_{12} > 0$. Let $t_1, t_2 > 0$. If*

$$|v_1| \leq \frac{1}{t_1} - \frac{10 v_6}{t_1 \sqrt{v_0 v_{12}}} \tag{20}$$

$$|v_{11}| \leq \frac{1}{t_2} - \frac{10 v_6}{t_2 \sqrt{v_0 v_{12}}} \tag{21}$$

$$|v_1| \left(\frac{5}{t_1 v_0}\right)^5 + |v_{11}| \left(\frac{5}{t_2 v_{12}}\right)^5 \leq \frac{\sqrt{70} - 8}{2} v_6 \tag{22}$$

and

$$\begin{aligned} &\left(v_0 - 10 v_6 \left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}} - |v_1| t_1 v_0\right)\left(v_{12} - 10 v_6 \left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}} - |v_{11}| t_2 v_{12}\right) \\ &\quad \times \left(\frac{\sqrt{70} - 8}{2} v_6 - |v_1| \left(\frac{5}{t_1 v_0}\right)^5 - |v_{11}| \left(\frac{5}{t_2 v_{12}}\right)^5\right) \\ &\quad \geq \frac{1}{27} v_6^3 (60 + 15\sqrt{70})^3 \end{aligned} \tag{23}$$

hold, then $\mathcal{A}$ is SOS.

Proof We write $f(\mathbf{x}) = \sum_{i=1}^5 f_i(\mathbf{x})$, where

$$f_2(\mathbf{x}) = 10v_6\left(\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{4}}x_1^3 + \left(\frac{v_{12}}{v_0}\right)^{\frac{1}{4}}x_3^3\right)^2,$$

$$f_3(\mathbf{x}) = |v_1|t_1v_0x_1^6 + 6v_1x_1^5x_2 + |v_1|\left(\frac{5}{t_1v_0}\right)^5 x_2^6,$$

$$f_4(\mathbf{x}) = |v_{11}|t_2v_{12}x_3^6 + 6v_{11}x_3^5x_2 + |v_{11}|\left(\frac{5}{t_2v_{12}}\right)^5 x_2^6,$$

$$f_5(\mathbf{x}) = v_6\left(\sqrt{\frac{10-\sqrt{70}}{2}}x_2^3 + \sqrt{150+15\sqrt{70}}x_1x_2x_3\right)^2$$

and

$$\begin{aligned} f_1(\mathbf{x}) \;\; = \;\; & \left(v_0 - 10v_6\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}} - |v_1|t_1v_0\right)x_1^6 + \left(\frac{\sqrt{70}-8}{2}v_6 - |v_1|\left(\frac{5}{t_1v_0}\right)^5 - |v_{11}|\left(\frac{5}{t_2v_{12}}\right)^5\right)x_2^6 \\ & - \left(v_{12} - 10v_6\left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}} - |v_{11}|t_2v_{12}\right)x_3^6 - v_6(60+15\sqrt{70})x_1^2x_2^2x_3^2. \end{aligned}$$

Clearly, f_2 and f_5 are squares. From Lemma 1, we may show that f_3 and f_4 are PSD. Since each of f_3 and f_4 has only two variables, they are SOS. If (20-23) hold, by the arithmetic-geometric inequality, f_1 is PSD. In this case, f_1 is a PSD diagonal minus tail form. By [7], f_1 is SOS. Thus, if (20-23) hold, then f, hence $\mathcal{A}$, is SOS. □

To get more insights for quasi-truncated Hankel tensors, we conduct some numerical tests for sixth order three dimensional quasi-truncated Hankel tensors. For simplicity purpose, we let $v_0 = v_{12}$ and $v_1 = v_{11}$. Note that in this case (18) holds. Thus, by Proposition 3, (9) holds, i.e., a necessary condition for $\mathcal{A}$ to be PSD is that $v_0 \geq 560 + 70\sqrt{70}$. Numerically, we observe that there is a function $\phi(\theta) \geq 0$, defined for $\theta \geq 560 + 70\sqrt{70}$ such that in this case, $\mathcal{A}$ is PSD if and only if $v_0 \geq 560 + 70\sqrt{70}$ and $|v_1| \leq \phi(v_0)$. In this case, $\mathcal{A}$ is also SOS. In the following, we give a table and a figure to sketch the graph of the function ϕ.

We get the value of ϕ by using the toolbox (Gloptipoly3 and SeDuMi) to confirm whether a sixth order three dimensional Hankel tensor is PSD or not and use the toolbox (YALMIP) [14] to test whether a sixth order three dimensional Hankel tensor is SOS or not. We tested ten different values of v_0 and the corresponding values of ϕ are in Table 1 and Figure1. Note that approximately

$$560 + 70\sqrt{70} \approx 1145.7.$$

Hence, no PNS Hankel tensors are found in this case.

Table 1: The values of ϕ for different v_0

v_0	ϕ	v_0	ϕ
1146	1.3034	1160	8.4925
1147	2.5853	1170	11.0947
1148	3.4183	1180	13.2144
1149	4.0855	1190	15.0563
1150	4.6585	1200	16.7130

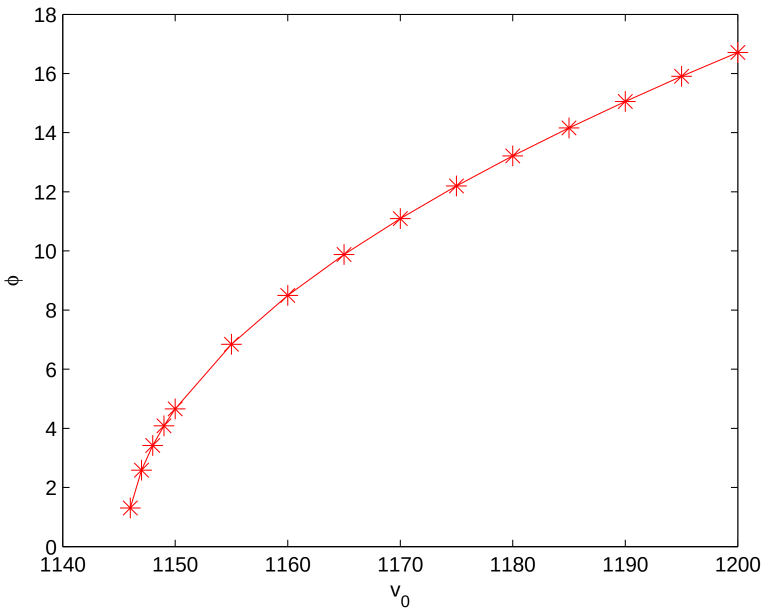

Figure 1: The value of ϕ

5 Sixth Order Three Dimensional Anti-Circulant Tensors

In this section, we consider sixth order three dimensional Hankel tensors $\mathcal{A}$, satisfying

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes 6} \equiv f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = f(x_3, x_1, x_2). \tag{24}$$

Notably, the all one tensor satisfies (24). Comparing (24) with (5), we find that the entries of the generating vector of a sixth order three dimensional anti-circulant tensor $\mathcal{A}$ satisfy

$$v_i = v_{i+3},$$

for $i = 0, \cdots, 9$. By [6], such a Hankel tensor is called an **anti-circulant tensor**. Thus, the generating vector of such a Hankel tensor has the following form

$$\mathbf{v} = (v_0, v_1, v_2, v_0, v_1, v_2, v_0, v_1, v_2, v_0, v_1, v_2, v_0)^\top \in \Re^{13}.$$

There are only three independent entries v_0, v_1 and v_2. Now, (5) has the simple form:

$$\begin{aligned} f(\mathbf{x}) \;=\; & v_0\left[x_1^6 + x_2^6 + x_3^6 + 20(x_1^3x_2^3 + x_2^3x_3^3 + x_1^3x_3^3) + 30(x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) + 90x_1^2x_2^2x_3^2\right] \\ & +v_1\left[6(x_1^5x_2 + x_2^5x_3 + x_1x_3^5) + 15(x_1^2x_2^4 + x_2^2x_3^4 + x_1^4x_3^2) + 60(x_1^3x_2^2x_3 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3)\right] \\ & +v_2\left[6(x_1x_2^5 + x_2x_3^5 + x_1^5x_3) + 15(x_1^4x_2^2 + x_2^4x_3^2 + x_1^2x_3^4) + 60(x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1x_2^2x_3^3).\right] \end{aligned} \tag{25}$$

Since we are only concerned about PSD Hankel tensors, we may assume that (3) holds which, in this case, means $v_0 \geq 0$.

Let us write

$$f(\mathbf{x}) = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x}) + v_2 f_2(\mathbf{x}),$$

where f_0, f_1 and f_2 are given by

$$f_0(\mathbf{x}) = x_1^6 + x_2^6 + x_3^6 + 20(x_1^3x_2^3 + x_2^3x_3^3 + x_1^3x_3^3) + 30(x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) + 90x_1^2x_2^2x_3^2, \tag{26}$$

$$f_1(\mathbf{x}) = 6(x_1^5x_2 + x_2^5x_3 + x_1x_3^5) + 15(x_1^2x_2^4 + x_2^2x_3^4 + x_1^4x_3^2) + 60(x_1^3x_2^2x_3 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3), \tag{27}$$

$$f_2(\mathbf{x}) = 6(x_1x_2^5 + x_2x_3^5 + x_1^5x_3) + 15(x_1^4x_2^2 + x_2^4x_3^2 + x_1^2x_3^4) + 60(x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1x_2^2x_3^3). \tag{28}$$

Next, we provide a characterization for a sixth order three dimensional anti-circulant tensor $\mathcal{A}$ to be PSD.

Theorem 3 *Suppose that $\mathcal{A}$ is a sixth order three dimensional anti-circulant tensor. Then $\mathcal{A}$ is PSD if and only $v_0 = v_1 = v_2 \geq 0$. In this case,*

$$f(\mathbf{x}) = v_0(x_1 + x_2 + x_3)^6. \tag{29}$$

This implies that $\mathcal{A}$ is SOS if only if it is PSD.

Proof Suppose that $\mathcal{A}$ is PSD. Then $f(1,-1,0) \geq 0$ and $f(1,1,-2) \geq 0$. From (25), we derive that $v_1+v_2 \geq 2v_0$ and $v_1+v_2 \leq 2v_0$ respectively. So, $v_1+v_2 = 2v_0$. Let $v_1 = v_0(1+\alpha)$ with $\alpha \in \Re$. Then $v_2 = v_0(1-\alpha)$ and

$$f(\mathbf{x}) = v_0(x_1+x_2+x_3)^6 + v_0\alpha(f_1(\mathbf{x}) - f_2(\mathbf{x})),$$

where f_1 and f_2 are defined as in (27) and (28) respectively. From this and $f(1,2,-3) \geq 0$, we have $\alpha \geq 0$. From this and $f(1,-3,2) \geq 0$, we have $\alpha \leq 0$. Thus $\alpha = 0$ and (29) follows. $\square$

Thus, there are no sixth order three dimensional PNS anti-circulant tensors.

6 Sixth Order Three Dimensional Alternatively Anti-Circulant Tensors

In this section, we consider sixth order three dimensional Hankel tensors $\mathcal{A}$, whose generating vector has the form $\mathbf{v} = (v_0, v_1, v_0, v_1, v_0, v_1, v_0, v_1, v_0, v_1, v_0, v_1, v_0)^\top$. We call such a Hankel tensor an **alternatively anti-circulant tensor**. Since we are only concerned about PSD Hankel tensors, we may assume that (3) holds, i.e., $v_0 \geq 0$. Now, (5) and (6) have the simple form

$$\begin{aligned} f(\mathbf{x}) \;=\;& v_0\left[x_1^6+x_2^6+x_3^6+6(x_1^5x_3+x_1x_3^5)+15(x_1^4x_2^2+x_1^2x_2^4+x_1^4x_3^2+x_2^4x_3^2+x_1^2x_3^4+x_2^2x_3^4)\right.\\ &\left.+20x_1^3x_3^3+30x_1x_2^4x_3+60(x_1^3x_2^2x_3+x_1x_2^2x_3^3)+90x_1^2x_2^2x_3^2\right]\\ &+v_1\left[6(x_1^5x_2+x_1x_2^5+x_2^5x_3+x_2x_3^5)+20(x_1^3x_2^3+x_2^3x_3^3)+30(x_1^4x_2x_3+x_1x_2x_3^4)\right.\\ &\left.+60(x_1^2x_2^3x_3+x_1^3x_2x_3^2+x_1x_2^3x_3^2+x_1^2x_2x_3^3)\right] \end{aligned} \tag{30}$$

and

$$\begin{aligned} g(\mathbf{y}) \;=\;& v_0(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2+y_7^2)\\ &+2v_0(y_1y_3+y_1y_5+y_2y_3+y_1y_7+y_2y_6+y_3y_5+y_3y_7+y_4y_6+y_5y_7)\\ &+2v_1(y_1y_2+y_1y_4+y_2y_3+y_1y_6+y_2y_5+y_3y_4+y_2y_7+y_3y_6+y_4y_5+y_4y_7+y_5y_6+y_6y_7). \end{aligned} \tag{31}$$

We have the following theorem which provides a characterization for a sixth order three dimensional alternatively anti-circulant tensor $\mathcal{A}$ to be PSD.

Theorem 4 *Suppose that $\mathcal{A}$ is a sixth order three dimensional alternatively anti-circulant tensor defined above. Then $\mathcal{A}$ is PSD if and only if $|v_1| \leq v_0$. In this case, $\mathcal{A}$ is a strong Hankel tensor, and thus an SOS Hankel tensor.*

Proof Suppose that $\mathcal{A}$ is PSD. From $f(1,1,0) \geq 0$ and (30), we have $v_0 + v_1 \geq 0$. From $f(1,-1,0) \geq 0$ and (30), we have $v_0 - v_1 \geq 0$. This implies that $v_0 \geq |v_1|$. On the other hand, suppose that $v_0 \geq |v_1|$. We may write $v_1 = v_0(2t-1)$, where $t \in [0,1]$. Write $f(\mathbf{x}) = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x})$ where

$$\begin{aligned} f_0(\mathbf{x}) &= x_1^6 + x_2^6 + x_3^6 + 6(x_1^5 x_3 + x_1 x_3^5) + 15(x_1^4 x_2^2 + x_1^2 x_2^4 + x_1^4 x_3^2 + x_2^4 x_3^2 + x_1^2 x_3^4 + x_2^2 x_3^4) \\ &\quad +20 x_1^3 x_3^3 + 30 x_1 x_2^4 x_3 + 60(x_1^3 x_2^2 x_3 + x_1 x_2^2 x_3^3) + 90 x_1^2 x_2^2 x_3^2 \end{aligned}$$

and

$$\begin{aligned} f_1(\mathbf{x}) &= 6(x_1^5 x_2 + x_1 x_2^5 + x_2^5 x_3 + x_2 x_3^5) + 20(x_1^3 x_2^3 + x_2^3 x_3^3) + 30(x_1^4 x_2 x_3 + x_1 x_2 x_3^4) \\ &\quad +60(x_1^2 x_2^3 x_3 + x_1^3 x_2 x_3^2 + x_1 x_2^3 x_3^2 + x_1^2 x_2 x_3^3). \end{aligned}$$

It can be verfied that $f_0(\mathbf{x}) + f_1(\mathbf{x}) = (x_1 + x_2 + x_3)^6$ and $f_0(\mathbf{x}) - f_1(\mathbf{x}) = (x_1 - x_2 + x_3)^6$ for all $\mathbf{x} = (x_1, x_2, x_3)^\top \in \Re^{13}$. It then follows from (30) that

$$\begin{aligned} f(\mathbf{x}) = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x}) &= v_0 f_0(\mathbf{x}) + (2t-1) v_0 f_1(\mathbf{x}) \\ &= t v_0 (f_0(\mathbf{x}) + f_1(\mathbf{x})) + (1-t) v_0 (f_0(\mathbf{x}) - f_1(\mathbf{x})) \\ &= t v_0 (x_1 + x_2 + x_3)^6 + (1-t) v_0 (x_1 - x_2 + x_3)^6. \end{aligned}$$

Similarly, we have

$$g(\mathbf{y}) = t v_0 (y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)^2 + (1-t) v_0 (y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7)^2.$$

The conclusions now follow from the definitions of PSD, SOS and strong Hankel tensors.
$\square$

Thus, there are no sixth order three dimensional PNS alternatively anti-circulant tensors. We also note that the above theorem can be easily extended to general even order alternatively anti-circulant tensors.

7 Numerical Tests and A Conjecture

In this section, we conduct numerical experiments to search sixth order three dimensional PNS Hankel tensors. We first explain how to generate a positive semi-definite Hankel tensor $\mathcal{A}_\alpha$ with a parameter α randomly, and determine a value α_0 such that $\mathcal{A}_\alpha$ is PSD if and only if $\alpha \geq \alpha_0$.

We first generate a vector $\mathbf{v} \in \Re^{13}$ randomly. We form a Hankel tensor $\mathcal{A}_0$ by using $\mathbf{v}$ as its generating vector. Then, we consider a parameterized tensor $\mathcal{A}_\alpha = \mathcal{A}_0 + \frac{\alpha}{2}(\mathcal{H} + \tilde{\mathcal{H}})$ where

$\alpha \in \Re$, $\mathcal{H}$ is the sixth order three dimensional Hilbert tensor and $\tilde{\mathcal{H}}$ is the Hankel tensor generating by $\tilde{\mathbf{v}} = (\frac{1}{13}, \frac{1}{12}, \cdots, 1)^\top \in \mathbb{R}^{13}$. As $\mathcal{H}$ is a positive definite Hankel tensor, $\tilde{\mathcal{H}}$ is also positive definite Hankel tensor by Theorem 1. So, $\mathcal{A}_\alpha$ is also a Hankel tensor and $\mathcal{A}_\alpha$ is positive definite if α is large enough. We then find α_0 to be the smallest number α such that $\mathcal{A}_\alpha$ is positive semi-definite. Here, α_0 can be negative if $\mathcal{A}_0$ is positive definite. Now we test if $\mathcal{A}_{\alpha_0}$ is SOS or not. If $\mathcal{A}_{\alpha_0}$ is not SOS, then we find a sixth order three dimensional PNS Hankel tensor. If $\mathcal{A}_{\alpha_0}$ is SOS, then we see that $\mathcal{A}_\alpha$ is also an SOS Hankel tensor if $\alpha > \alpha_0$, as

$$\mathcal{A}_\alpha = \mathcal{A}_{\alpha_0} + \frac{\alpha - \alpha_0}{2}(\mathcal{H} + \tilde{\mathcal{H}})$$

and $\frac{\alpha-\alpha_0}{2}(\mathcal{H} + \tilde{\mathcal{H}})$ is also an SOS Hankel tensor. Thus, if $\mathcal{A}_{\alpha_0}$ is SOS, then there is no α with $\alpha \geq \alpha_0$ such that $\mathcal{A}_\alpha$ is a PNS Hankel tensor.

We use $\frac{1}{2}(\mathcal{H} + \tilde{\mathcal{H}})$ as the reference positive definite Hankel tensor instead of using $\mathcal{H}$, as the entries of the generating vector of $\frac{1}{2}(\mathcal{H} + \tilde{\mathcal{H}})$ is distributed somewhat evenly. This makes our numerical tests more efficient in terms of finding $\mathcal{A}_{\alpha_0}$.

Due to numerical inaccuracy, instead of finding α_0, we find α_1 such that $\alpha_1 \geq \alpha_0$ and $\alpha_1 - \alpha_0 \leq \epsilon$, where ϵ is a given very small positive number. Then we test if $\mathcal{A}_{\alpha_1}$ is SOS or not.

Here, we use the toolbox (Gloptipoly3 and SeDuMi) to confirm whether a sixth order three dimensional Hankel tensor is PSD or not and use the toolbox (YALMIP) to test whether a sixth order three dimensional Hankel tensor is SOS or not. All codes were written by MATLAB 2014a and run on a Lenovo desktop computer with Core processor 2.83 GHz and 4 GB memory. We have not found any PNS Hankel tensor in six thousand tests.

Taking into account of the four special classes we examined and our numerical experiment, we now make the following conjecture:

There are no sixth order three dimensional PNS Hankel tensors.

If this conjecture turns out to be true, then determining a given sixth order three dimensional Hankel tensor is PSD or not can be solved by a semi-definite linear programming problem.

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