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An irreducible function basis of isotropic invariants of a third order three-dimensional symmetric tensor

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In this paper, we present an eleven invariant isotropic irreducible function basis of a third order three-dimensional symmetric tensor. This irreducible function basis is a proper subset of the Olive-Auffray minimal isotropic integrity basis of that tensor. The octic invariant and a sextic invariant in the Olive-Auffray integrity basis are dropped out. This result is of significance to the further research of irreducible function bases of higher order tensors. *Published by AIP Publishing.* https://doi.org/10.1063/1.5028307

I. INTRODUCTION

Tensor function representation theory constitutes an important fundamental of theoretical and applied mechanics. Representations of a complete and irreducible basis for isotropic invariants could predict the available nonlinear constitutive theories by the formulation of the energy term. Since irreducible representations for tensor-valued functions can be immediately yielded from known irreducible representations for invariants (scalar-valued functions),[1] the studies of isotropic function basis have most priority. Perhaps we may trace back the modern development of tensor representation theory to the great mathematician Hermann Weyl's book.[2] This book was first published in 1939. Here, we cite its new edition in 2016. Then, since the 1955 paper of Rivilin and Ericksen,[3] many researchers, such as Smith, Pipkin, Spencer, Boehler, Betten, Pennisi, and Zheng,[4–10] to name only a few of them here, have made important contributions to tensor representation theory. For the literature of tensor representation theory before 1994, people may find it in the 1994 survey paper of Zheng.[1] The development of tensor representation theory after 1994 paid more attentions to minimal integrity bases of isotropic invariants of third and fourth order three-dimensional tensors.[9–14] The polynomial basis of anisotropic invariants of the elasticity tensor was studied by Boehler, Kirillov, and Onat[11] in 1994. Zheng and Betten[9] and Zheng[10] studied the tensor function representations involving tensors of orders higher than two. Smith and Bao[14] presented minimal integrity bases of isotropic invariants for third and fourth order three-dimensional symmetric and traceless tensors in 1997. Note that Boehler, Kirillov, and Onat[11] had already given a minimal integrity basis for a fourth order three-dimensional symmetric and traceless tensor in 1994. But the minimal integrity basis given by Smith and Bao[14] for the same tensor is slightly different.[15] In 2014, an integrity basis with thirteen isotropic invariants of a (completely) symmetric third order three-dimensional tensor was presented by Olive and Auffray.[12] Olive[16] (p. 1409) stated that this integrity basis is a minimal integrity basis. Olive, Kolev, and

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Auffray[13] presented a minimal integrity basis of the elasticity tensor, with 297 invariants, in 2017. Very recently, Chen, Qi, and Zou[15] showed that the four invariant Smith-Bao minimal isotropic integrity basis of a third order three-dimensional symmetric and traceless tensor is also an irreducible function basis of that tensor.

In this paper, the summation convention is used. If an index is repeated twice in a product, then it means that this product is summed up with respect to this index from 1 to 3.

Suppose that a tensor $\mathbf{A}$ has the form $A_{i_1\ldots i_m}$ under an orthonormal basis $\{\mathbf{e}_i\}$. A scalar function of $\mathbf{A}$ $f(\mathbf{A}) = f(A_{i_1\ldots i_m})$ is said to be an isotropic invariant of $\mathbf{A}$ if for any orthogonal matrix q_{ij} we have

$$f(A_{i_1\ldots i_m}) = f(A_{j_1\ldots j_m} q_{i_1 j_1} \ldots q_{i_m j_m}).$$

By Refs. 1 and 9, a set of isotropic polynomial invariants $f_1, \ldots, f_r$ of $\mathbf{A}$ is said to be an *integrity basis* of $\mathbf{A}$ if any isotropic polynomial invariant is a polynomial of $f_1, \ldots, f_r$, and a set of isotropic invariants $f_1, \ldots, f_m$ of $\mathbf{A}$ is said to be a *function basis* of $\mathbf{A}$ if any isotropic invariant is a function of $f_1, \ldots, f_m$. An integrity basis is always a function basis but not vice versa.[13] A set of isotropic polynomial invariants $f_1, \ldots, f_r$ of $\mathbf{A}$ is said to be *polynomially irreducible* if none of them can be a polynomial of the others. Similarly, a set of isotropic invariants $f_1, \ldots, f_m$ of $\mathbf{A}$ is said to be *functionally irreducible* if none of them can be a function of the others. An integrity basis of $\mathbf{A}$ is said to be a *minimal integrity basis* of $\mathbf{A}$ if it is polynomially irreducible, and a function basis of $\mathbf{A}$ is said to be an *irreducible function basis* of $\mathbf{A}$ if it is functionally irreducible.

In this paper, we present an eleven invariant isotropic irreducible function basis of a third order three-dimensional symmetric tensor. This irreducible function basis is a proper subset of the Olive-Auffray minimal isotropic integrity basis of that tensor. The octic invariant and a sextic invariant in the Olive-Auffray integrity basis are dropped out.

In Sec. II, some preliminary results are given. These include the minimal integrity basis result of Smith and Bao[14] for a third order three-dimensional symmetric and traceless tensor, the consequent result of Chen, Qi, and Zou[15] to confirm that it is also an irreducible function basis, and the result of Olive and Auffray[12] for a minimal integrity basis of a third order three-dimensional symmetric tensor.

In Sec. III, we present an eleven invariant isotropic function basis of a third order three-dimensional symmetric tensor. This function basis is obtained by using two syzygy relations to drop out the octic invariant and a sextic invariant from the Olive-Auffray integrity basis. Note that *a syzygy relation* is a set of coefficients in the polynomial ring such that the corresponding element generated by the function basis vanishes in the module.

Then in Sec. IV, we show that this function basis is indeed an irreducible function basis of a third order three-dimensional symmetric tensor.

This result is significant to the further research of irreducible function bases of higher order tensors. First, this is the first time to give an irreducible function basis of isotropic invariants of a third order three-dimensional symmetric tensor. Second, there are still three syzygy relations among these eleven invariants. This shows that an irreducible function basis consisting of polynomial invariants may not be algebraically minimal. We discuss this in Sec. V.

From now on, we use $\mathbf{A}$ to denote a third order three-dimensional tensor and assume that it is represented by A_{ijk} under an orthonormal basis $\{\mathbf{e}_i\}$. We consider the three-dimensional physical space. Hence $i, j, k \in \{1, 2, 3\}$. We say that $\mathbf{A}$ is a symmetric tensor if for $i, j, k = 1, 2, 3$ we have

$$A_{ijk} = A_{jik} = A_{ikj} = A_{kji}.$$

We say that $\mathbf{A}$ is traceless if

$$A_{iij} = A_{ijj} = A_{iji} = 0.$$

We use $\mathbf{0}$ to denote the zero vector and $\mathcal{O}$ to denote the third order three-dimensional zero tensor.

II. PRELIMINARIES

In this section, we review the minimal integrity basis result of Smith and Bao[14] for a third order three-dimensional symmetric and traceless tensor, the consequent result of Chen, Qi, and Zou[15] to confirm that it is also an irreducible function basis, and the minimal integrity basis result of Olive and Auffray[12] for a third order three-dimensional symmetric tensor.

A. An irreducible function basis of a third order three-dimensional symmetric and traceless tensor

In 1997, Smith and Bao[14] presented the following theorem.

Theorem II.1. *Let* $\mathbf{D}$ *be an irreducible (i.e., symmetric and traceless) third order three-dimensional tensor. Denote* $v_p := D_{ijk}D_{ij\ell}D_{k\ell p}$, $I_2 := D_{ijk}D_{ijk}$, $I_4 := D_{ijk}D_{ij\ell}D_{pqk}D_{pq\ell}$, $I_6 := v_iv_i$, *and* $I_{10} := D_{ijk}v_iv_iv_k$. *Then* $\{I_2, I_4, I_6, I_{10}\}$ *is a minimal integrity basis of* $\mathbf{D}$.

Very recently, Chen, Qi, and Zou[15] proved the following theorem.

Theorem II.2. *Under the notation of Theorem II.1, the Smith-Bao minimal integrity basis* $\{I_2, I_4, I_6, I_{10}\}$ *is also an irreducible function basis of* $\mathbf{D}$.

B. The Olive-Auffray integrity basis of a third order three-dimensional symmetric tensor

According to Ref. 10, we decompose a third order three-dimensional symmetric tensor $\mathbf{A}$ into a third order three-dimensional symmetric and traceless tensor $\mathbf{D}$ and a vector $\mathbf{u}$, with

$$u_i = A_{i\ell\ell}$$

and

$$D_{ijk} = A_{ijk} - \frac{1}{5}\left(u_k\delta_{ij} + u_j\delta_{ik} + u_i\delta_{jk}\right),$$

where $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$.

In 2014, Olive and Auffray[12] presented the following theorem.

Theorem II.3. *Let* $\mathbf{A}$ *be a third order three-dimensional symmetric tensor with the above decomposition. The following thirteen invariants*

$$\begin{aligned}
&I_2 := D_{ijk}D_{ijk}, && J_2 := u_iu_i,\\
&I_4 := D_{ijk}D_{ij\ell}D_{pqk}D_{pq\ell}, && J_4 := D_{ijk}u_kD_{ij\ell}u_\ell,\\
&K_4 := D_{ijk}D_{ij\ell}D_{k\ell p}u_p, && L_4 := D_{ijk}u_ku_ju_i,\\
&I_6 := v_iv_i, && J_6 := D_{ijk}D_{ij\ell}u_kD_{\ell pq}u_pu_q,\\
&K_6 := v_kw_k, && L_6 := D_{ijk}D_{ij\ell}u_kv_\ell,\\
&M_6 := D_{ijk}D_{pqk}u_iu_ju_pu_q, && I_8 := D_{ijk}D_{ij\ell}u_kD_{pq\ell}D_{pqr}v_r,\\
&I_{10} := D_{ijk}v_iv_jv_k,
\end{aligned}$$

where $v_p := D_{ijk}D_{ij\ell}D_{k\ell p}$ *and* $w_k := D_{ijk}u_iu_j$, *form an integrity basis of* $\mathbf{A}$.

As an integrity basis is always a function basis, we may start from the Olive-Auffray integrity basis

$$\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, K_6, L_6, M_6, I_8, I_{10}\}$$

to find an irreducible function basis of $\mathbf{A}$.

III. AN ELEVEN INVARIANT FUNCTION BASIS

In this section, we show that the following eleven invariant set

$$\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$$

is a function basis of the third order three-dimensional symmetric tensor $\mathbf{A}$. Note that this set is obtained by dropping K_6 and I_8 from the Olive-Auffray integrity basis

$$\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, K_6, L_6, M_6, I_8, I_{10}\}.$$

Thus, the task of this section is to show that K_6 and I_8 can be dropped out for a function basis.

We first prove the following proposition.

Proposition III.1. In the Olive-Auffray integrity basis, we have

$$2I_2J_2 - 3J_4 \geq 0,$$

where the equality holds if and only if either $\mathbf{D} = \mathcal{O}$ *or* $\mathbf{u} = \mathbf{0}$.

Proof. By definition, if either $\mathbf{D} = \mathcal{O}$ or $\mathbf{u} = \mathbf{0}$, we have $I_2J_2 = 0$ and $J_4 = 0$. Hence $2I_2J_2 - 3J_4 = 0$ in this case.

Consider the optimization problem

$$\min\{2I_2J_2 - 3J_4 : D_{ijk}D_{ijk} = 1, u_iu_i = 1\},$$

where the variables are the seven independent components of $\mathbf{D}$ and the three components of $\mathbf{u}$. Using GloptiPoly 3[17] and SeDuMi,[18] we compute the minimum value of this optimization problem to be 0.2, where the minimizer is $D_{111} = 0.2829$, $D_{112} = D_{113} = 0$, $D_{122} = -0.2828$, $D_{123} = -0.2450$, $D_{222} = 0$, $D_{223} = -0.2828$, $u_1 = -0.4471$, $u_2 = -0.7746$, and $u_3 = -0.4474$. Hence, the minimum value is positive. This implies that if $2I_2J_2 - 3J_4 = 0$, then either $\mathbf{D} = \mathcal{O}$ or $\mathbf{u} = \mathbf{0}$. □

We are now ready to prove the following theorem.

Theorem III.2. *The eleven invariant set* $\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ *is a function basis of the third order three-dimensional symmetric tensor* $\mathbf{A}$.

Proof. Consider all possible tenth degree powers and products of these thirteen invariants I_2, $J_2, I_4, J_4, K_4, L_4, I_6, J_6, K_6, L_6, M_6, I_8, I_{10}$ in the Olive-Auffray minimal integrity basis of $\mathbf{A}$. Find linear relations among these tenth degree powers and products. Then we have two syzygy relations among these thirteen invariants as follows:

$$\begin{aligned}6J_2I_8 = {} & -I_2^2J_2K_4 - I_2^3L_4 + 3I_2I_4L_4 - 3I_2J_4K_4 + 4J_2I_4K_4 \\ & + 2I_2^2J_6 + 3I_2J_2L_6 - 3L_4I_6 - 6I_4J_6 + 3J_4L_6 + 6K_4K_6,\end{aligned} \tag{1}$$

and

$$\begin{aligned}& 2I_2J_2K_6 + I_2^2J_2J_4 - I_2J_4^2 + 2I_2K_4L_4 + 3J_2K_4^2 - 2J_2I_4J_4 \\ & + J_2^2I_6 - 2I_2^2M_6 - 12K_4J_6 + 6L_4L_6 + 6I_4M_6 - 3J_4K_6 = 0,\end{aligned}$$

i.e.,

$$\begin{aligned}(2I_2J_2 - 3J_4)K_6 = {} & -I_2^2J_2J_4 + I_2J_4^2 - 2I_2K_4L_4 - 3J_2K_4^2 + 2J_2I_4J_4 \\ & - J_2^2I_6 + 2I_2^2M_6 + 12K_4J_6 - 6L_4L_6 - 6I_4M_6.\end{aligned} \tag{2}$$

We first use the syzygy relation (1). If $\mathbf{u} = \mathbf{0}$, then $J_2 = u_iu_i = 0$, and the right-hand side of (1) is also equal to zero. In this case, we have $I_8 = D_{ijk}D_{ij\ell}u_kD_{pq\ell}D_{pqr}v_r = 0$, where $v_p := D_{ijk}D_{ij\ell}D_{k\ell p}$. If $\mathbf{u} \neq \mathbf{0}$, then $J_2 = u_iu_i \neq 0$. By the syzygy relation (1), we have

$$\begin{aligned}I_8 = {} & -\frac{1}{6}I_2^2K_4 + \frac{2}{3}I_4K_4 + \frac{1}{2}I_2L_6 + \frac{1}{6J_2}\big(-I_2^3L_4 + 3I_2I_4L_4 \\ & - 3I_2J_4K_4 + 2I_2^2J_6 - 3L_4I_6 - 6I_4J_6 + 3J_4L_6 + 6K_4K_6\big).\end{aligned}$$

Then I_8 is a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, K_6, L_6, M_6, I_{10}$.

We now use the syzygy relation (2). If $2I_2J_2 - 3J_4 = 0$, by Proposition III.1, either $\mathbf{D} = \mathcal{O}$ or $\mathbf{u} = \mathbf{0}$. This implies that $K_6 = 0$. Note that in this case, the right-hand side of (2) is also equal to zero. If $2I_2J_2 - 3J_4 \neq 0$, we have

$$K_6 = \frac{1}{2I_2J_2 - 3J_4}\big(- I_2^2 J_2 J_4 + I_2 J_4^2 - 2I_2K_4L_4 - 3J_2K_4^2 + 2J_2I_4J_4 - J_2^2 I_6 + 2I_2^2 M_6 + 12K_4J_6 - 6L_4L_6 - 6I_4M_6\big).$$

This shows that K_6 is a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}$.

Hence, $\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is a function basis of the third order three-dimensional symmetric tensor $\mathbf{A}$. □

IV. THIS FUNCTION BASIS IS AN IRREDUCIBLE FUNCTION BASIS

To show that $\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is an irreducible function basis of the third order three-dimensional symmetric tensor $\mathbf{A}$, we only need to show that each of these eleven invariants is not a function of the other ten invariants.

To show that each of K_4, L_4, J_6, and L_6 is not a function of the other ten invariants in this function basis, we may use a tactic, which is stated in the following proposition.

Proposition IV.1. We have the following four conclusions.

(a) *If there is a third order three-dimensional tensor* $\mathbf{A}$ *such that* $K_4 = L_4 = J_6 = 0$ *but* $L_6 \neq 0$, *then* L_6 *is not a function of* $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, M_6$, *and* I_{10}.
(b) *If there is a third order three-dimensional tensor* $\mathbf{A}$ *such that* $K_4 = L_4 = L_6 = 0$ *but* $J_6 \neq 0$, *then* J_6 *is not a function of* $I_2, J_2, I_4, J_4, K_4, L_4, I_6, L_6, M_6$, *and* I_{10}.
(c) *If there is a third order three-dimensional tensor* $\mathbf{A}$ *such that* $K_4 = J_6 = L_6 = 0$ *but* $L_4 \neq 0$, *then* L_4 *is not a function of* $I_2, J_2, I_4, J_4, K_4, I_6, J_6, L_6, M_6$, *and* I_{10}.
(d) *If there is a third order three-dimensional tensor* $\mathbf{A}$ *such that* $L_4 = J_6 = L_6 = 0$ *but* $K_4 \neq 0$, *then* K_4 *is not a function of* $I_2, J_2, I_4, J_4, L_4, I_6, J_6, L_6, M_6$, *and* I_{10}.

Proof. By the definition of invariants $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$, and I_{10}, if we keep $\mathbf{D}$ unchanged but change $\mathbf{u}$ to $-\mathbf{u}$, then $I_2, J_2, I_4, J_4, I_6, M_6$, and I_{10} are unchanged, but K_4, L_4, J_6, and L_6 change their signs.

We now prove conclusion (a). If there is a third order three-dimensional tensor $\mathbf{A}$ such that $K_4 = L_4 = J_6 = 0$ but $L_6 \neq 0$, we may keep $\mathbf{D}$ unchanged but change $\mathbf{u}$ to $-\mathbf{u}$, then I_2, J_2, I_4, J_4, I_6, M_6, and I_{10} are unchanged, K_4, L_4, and J_6 are still zeros, but L_6 changes its sign and value as it is not zero. This implies that L_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, M_6$, and I_{10}. The other three conclusions (b), (c), and (d) can be proved similarly. □

We now present the main theorem of this section.

Theorem IV.2. *The eleven invariant set* $\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ *is an irreducible function basis of the third order three-dimensional symmetric tensor* $\mathbf{A}$.

Proof. By Theorem III.2, $\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is a function basis of $\mathbf{A}$. It suffices to show that each of these eleven invariants is not a function of the other ten invariants.

We divide the proof into three parts.

Part (i). In this part, we show that each of I_2, I_4, I_6, I_{10}, and J_2 is not a function of the other ten invariants. The first four invariants form an irreducible function basis of the symmetric and traceless tensor $\mathbf{D}$. The fifth invariant J_2 forms an irreducible function basis of the vector $\mathbf{u}$. Using this property, we may prove that each of them is not a function of the other ten invariants easily.

By Theorem II.2, $\{I_2, I_4, I_6, I_{10}\}$ is an irreducible function basis of $\mathbf{D}$. This implies that each of these four invariants is not a function of the other three invariants. Hence, each of these four invariants is not a function of the other ten invariants of $\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$.

Let $\mathbf{D} = \mathcal{O}$, and $\mathbf{u} \neq \mathbf{u}'$ such that $u_iu_i \neq u_i'u_i'$. Then J_2 takes two different values, but the other ten invariants $I_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$, and I_{10} are all zero. This shows that J_2 is not a function of the other ten invariants $I_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$, and I_{10}.

Part (ii). In this part, we show that each of K_4, L_4, J_6, and L_6 is not a function of the other ten invariants. We use Proposition IV.1 to realize this purpose.

We first show that L_6 is not a function of the other ten invariants. Let A_{111}, A_{112}, A_{113}, A_{122}, A_{123}, A_{133}, A_{222}, A_{223}, A_{233}, and A_{333} be the representatives of the components of $\mathbf{A}$. If the values of these ten components are fixed, then the other components of $\mathbf{A}$ also fixed by symmetry. Let $A_{111} = \frac{3}{5}, A_{122} = \frac{6}{5}, A_{133} = -\frac{4}{5}, A_{223} = \frac{1}{2}, A_{333} = -\frac{1}{2}$, and $A_{112} = A_{113} = A_{123} = A_{222} = A_{233} = 0$, then we have $K_4 = L_4 = J_6 = 0$ and $L_6 = -2$ such that we may use Proposition IV.1 (a). The values of the other invariants are $I_2 = 7, J_2 = 1, I_4 = \frac{37}{2}, J_4 = 2, I_6 = 4, M_6 = 0, I_{10} = 4$. By Proposition IV.1 (a), L_6 is not a function of I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, M_6, and I_{10}.

Then we show that J_6 is not a function of the other ten invariants. Let

$$A_{111} = \frac{1}{6}\sqrt{\frac{1}{2}(149 - \sqrt{313})} - \frac{18(-215 + 7\sqrt{313})}{5\sqrt{8\,053\,043 - 308\,071\sqrt{313}}},$$

$$A_{112} = \frac{121(2963 - 103\sqrt{313})}{10(-215 + 7\sqrt{313})}\sqrt{\frac{298 - 2\sqrt{313}}{648\,164\,815 - 26\,977\,811\sqrt{313}}},$$

$$A_{113} = \frac{3\,966\,519 - 219\,867\sqrt{313}}{5\sqrt{648\,164\,815 - 26\,977\,811\sqrt{313}}(-215 + 7\sqrt{313})},$$

$$A_{122} = -\frac{6(-215 + 7\sqrt{313})}{5\sqrt{8\,053\,043 - 308\,071\sqrt{313}}}, \qquad A_{123} = 1,$$

$$A_{133} = -\frac{1}{6}\sqrt{\frac{1}{2}(149 - \sqrt{313})} - \frac{6(-215 + 7\sqrt{313})}{5\sqrt{8\,053\,043 - 308\,071\sqrt{313}}},$$

$$A_{222} = \frac{363(2963 - 103\sqrt{313})}{10(-215 + 7\sqrt{313})}\sqrt{\frac{298 - 2\sqrt{313}}{648\,164\,815 - 26\,977\,811\sqrt{313}}},$$

$$A_{223} = 1 + \frac{3\,966\,519 - 219\,867\sqrt{313}}{5\sqrt{648\,164\,815 - 26\,977\,811\sqrt{313}}(-215 + 7\sqrt{313})},$$

$$A_{233} = \frac{121(2963 - 103\sqrt{313})}{10(-215 + 7\sqrt{313})}\sqrt{\frac{298 - 2\sqrt{313}}{648\,164\,815 - 26\,977\,811\sqrt{313}}},$$

$$A_{333} = -1 + \frac{3(3\,966\,519 - 219\,867\sqrt{313})}{5\sqrt{648\,164\,815 - 26\,977\,811\sqrt{313}}(-215 + 7\sqrt{313})}.$$

These values are solutions of $K_4 = L_4 = L_6 = 0$ and $J_6 \neq 0$. Except that $A_{123} = 1$, the approximate digit values of the other independent components are as follows:

$$A_{111} = 1.554,\ A_{112} = -0.1877,\ A_{113} = -0.01\,287,$$

$$A_{122} = 0.06\,780,\ A_{133} = -1.283,\ A_{222} = -0.5631,$$

$$A_{223} = 0.9871,\ A_{233} = -0.1877,\ A_{333} = -1.039.$$

Then we have $K_4 = L_4 = L_6 = 0$ and $J_6 = 0.5112$, satisfying the condition of Proposition IV.1 (b). The values of the other invariants are $I_2 = 17.29$, $J_2 = 1$, $I_4 = 132.6$, $J_4 = 2.547$, $I_6 = 83.81$, $M_6 = 0.1687$ and $I_{10} = -831$. By Proposition IV.1 (b), J_6 is not a function of I_2, J_2, I_4, J_4, K_4, L_4, I_6, L_6, M_6, and I_{10}.

We now show that L_4 is not a function of the other ten invariants. Similarly, we can find a symmetric third order three-dimensional tensor **A** such that $K_4 = J_6 = L_6 = 0$ and $L_4 \neq 0$. To be specific, except that $A_{123} = 1$, the approximate digit values of the other independent components are as follows:

$$A_{111} = 1.0358,\ A_{112} = 0.06\,373,\ A_{113} = -0.06\,357,$$
$$A_{122} = 1.8269,\ A_{133} = -1.9697,\ A_{222} = 0.1912,$$
$$A_{223} = 0.9364,\ A_{233} = 0.06\,373,\ A_{333} = -1.1907.$$

Then we have $K_4 = J_6 = L_6 = 0$ and $L_4 = -0.3843$, satisfying the condition of Proposition IV.1 (c). We also have $I_2 = 32.2465$, $J_2 = 1$, $I_4 = 394.69$, $J_4 = 9.1213$, $I_6 = 509.67$, $M_6 = 3.2506$, and $I_{10} = 17\,825.1$. By Proposition IV.1 (c), L_4 is not a function of $I_2, J_2, I_4, J_4, K_4, I_6, J_6, L_6, M_6$, and I_{10}.

We further show that K_4 is not a function of the other ten invariants. Let

$$A_{111} = \frac{3}{5\sqrt{2}}, \qquad A_{112} = \frac{\sqrt{3}}{10}, \qquad A_{113} = \frac{1}{10},$$
$$A_{122} = \frac{4\sqrt{2}}{15} - \frac{1}{\sqrt{3}},\ A_{123} = \frac{1}{3} + \frac{1}{\sqrt{6}},\ A_{133} = -\frac{\sqrt{2}}{15} + \frac{1}{\sqrt{3}},$$
$$A_{222} = \frac{3\sqrt{3}}{10}, \qquad A_{223} = -\frac{9}{10}, \qquad A_{233} = \frac{\sqrt{3}}{10},$$
$$A_{333} = \frac{13}{10}.$$

Then we have $L_4 = J_6 = L_6 = 0$ and $K_4 = \frac{8}{9}$, satisfying the condition of Proposition IV.1 (d). We also have

$$I_2 = 8,\quad J_2 = \frac{3}{2},\quad I_4 = \frac{88}{3},\qquad J_4 = \frac{8}{3},$$
$$I_6 = \frac{64}{9},\ M_6 = \frac{11}{9},\ I_{10} = \frac{11\,776}{729}.$$

By Proposition IV.1 (d), K_4 is not a function of $I_2, J_2, I_4, J_4, L_4, I_6, J_6, L_6, M_6$, and I_{10}.

Part (iii). In this part, we show that each of M_6 and J_4 is not a function of the other ten invariants. We cannot use Proposition IV.1 here. However, we may use another tactic. We try to find a tensor **A** there such that $K_4 = L_4 = J_6 = L_6 = 0$ to reduce the influence of these four invariants. Then we change the values of some independent components of **A** such that the values of K_4, L_4, J_6 and L_6 keep to be zero, the value of M_6 or J_4 is changed and the values of the remaining other six invariants unchanged.

We first show that M_6 is not a function of the other ten invariants. Let $u_1 = 5a$, $u_2 = 5b$, $u_3 = 5c$, $D_{123} = d$ and the other six independent components of **D** be zeros. Let $a = b = 0$ and $c = d = 1$. Then $I_2 = 6$, $J_2 = 25$, $I_4 = 12$, $J_4 = 50$, $K_4 = L_4 = I_6 = J_6 = L_6 = I_{10} = 0$, and $M_6 = 0$. Let $a = b = \frac{\sqrt{2}}{2}$, $c = 0$, and $d = 1$. We still have $I_2 = 6$, $J_2 = 25$, $I_4 = 12$, $J_4 = 50$, $K_4 = L_4 = I_6 = J_6 = L_6 = I_{10} = 0$, but $M_6 = 625$. Hence, M_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6$, and I_{10}.

Finally, we show that J_4 is not a function of the other ten invariants. Let $A_{111} = \frac{3}{5}\cos\theta$, $A_{112} = \frac{1}{5}\sin\theta$, $A_{113} = 0$, $A_{122} = \frac{1}{5}\cos\theta$, $A_{123} = 1$, $A_{133} = \frac{1}{5}\cos\theta$, $A_{222} = \frac{3}{5}\sin\theta$, $A_{223} = 1$, $A_{233} = \frac{1}{5}\sin\theta$, and $A_{333} = -1$, Then we really have $K_4 = L_4 = J_6 = L_6 = 0$. We also have $I_2 = 10$, $J_2 = 1$, $I_4 = 44$, $I_6 = 16$, $I_{10} = -64$, and

$$J_4(\theta) = 2 + 4\cos\theta\sin\theta + 2\sin^2\theta,\ M_6(\theta) = \sin^2\theta(2\cos\theta + \sin^2\theta).$$

Clearly, $J_4(\frac{3}{4}\pi) = 1$, $M_6(\frac{3}{4}\pi) = \frac{1}{4}$, $M_6(0) = 0$, and $M_6(\frac{\pi}{4}) = \frac{9}{4}$. Since $M_6(\theta)$ is continuous in the interval $[0, \frac{\pi}{4}]$, there exists $\theta_0 \in (0, \frac{\pi}{4})$ such that $M_6(\theta_0) = M_6(\frac{3}{4}\pi) = \frac{1}{4}$. On the other hand, we have

$$J_4'(\theta) = 4\cos(2\theta) + 2\sin(2\theta) \geq 0, \qquad \forall\theta \in \left[0, \frac{\pi}{4}\right].$$

It follows that $J_4(\theta_0) \geq J_4(0) = 2 > J_4(\frac{3}{4}\pi) = 1$. Hence, J_4 is not a function of $I_2, J_2, I_4, K_4, L_4, I_6, J_6, L_6, M_6$, and I_{10}.

Combining the results of these three parts, each of these eleven invariants is not a function of the other ten invariants. Therefore, this eleven invariant set $\{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is indeed an irreducible function basis of **A**. □

Part (i) and the first part of Part (iii) of this proof show that each of $I_2, J_2, I_4, I_6, M_6, I_{10}$ is not a function of the other ten invariants. This follows Theorem 3.1 of Ref. 15. For self-sufficiency and completeness of this paper, we give this part of the proof directly. The organization of the proof to three parts also makes the proof an integral entity.

V. SIGNIFICANCE OF THIS RESULT

This result is significant to the further research of irreducible function bases of higher order tensors. First, this is the first result on irreducible function bases of a third order three-dimensional symmetric tensor. Second, there are still at least three syzygy relations among these eleven invariants; see (3)–(5). This shows that an irreducible function basis consisting of polynomial invariants may not be algebraically minimal in the sense that the basis consists of polynomial invariants and there is no algebraic relations in these invariants.[19] The second point is observed as there are still some syzygy relations among these eleven invariants.

Consider all possible sixteenth degree powers or products of the eleven invariants $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}$. Find linear relations among these sixteenth-degree powers or products. Then we have the following three syzygy relations among these eleven invariants:

$$\begin{aligned}
&2I_2^3J_2^3J_4 - 4I_2J_2^3I_4J_4 - 6J_2^3J_4I_6 - 9I_2^2J_2^2J_4^2 + 18J_2^2I_4J_4^2 + 9J_4^4 + 36I_2J_2J_6^2 - 54J_4J_6^2 \\
&- 48I_2J_2^2K_4J_6 + 144J_2J_4K_4J_6 + 12I_2J_2^3K_4^2 - 36J_2^2J_4K_4^2 - 24I_2^2J_2L_4J_6 + 36I_2J_4L_4J_6 \\
&+ 12I_2^2J_2^2K_4L_4 - 18I_2J_2J_4K_4L_4 - 18J_4^2K_4L_4 + 6I_2^3J_2L_4^2 - 6I_2J_2I_4L_4^2 - 9I_2^2J_4L_4^2 + 9I_4J_4L_4^2 \\
&- 36J_2J_4L_4L_6 - 6I_2^3J_2^2M_6 + 12I_2J_2^2I_4M_6 + 9J_2^2I_6M_6 + 36I_2^2J_2J_4M_6 - 72J_2I_4J_4M_6 \\
&- 18I_2J_4^2M_6 - 108K_4J_6M_6 + 27J_2K_4^2M_6 + 18I_2K_4L_4M_6 + 54L_4L_6M_6 - 18I_2^2M_6^2 + 54I_4M_6^2 \\
&= 0,
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\frac{4}{9}I_2^3J_2^3K_4 + \frac{2}{9}I_2^4J_2^2L_4 + \frac{4}{3}I_2^3J_2J_4L_4 - \frac{8}{9}I_2J_2^3I_4K_4 - \frac{4}{9}I_2^2J_2^2I_4L_4 - \frac{4}{3}I_2^2J_2^2J_4K_4 - 2I_2^2J_4^2L_4 \\
&+ 2I_2^2K_4L_4^2 + 2J_2^2K_4^3 + 4I_2J_2J_4^2K_4 + 5I_2J_2K_4^2L_4 - 4I_2J_2I_4J_4L_4 - \frac{4}{3}I_2^3J_2^2J_6 + \frac{2}{3}J_2^3K_4I_6 \\
&+ \frac{1}{3}I_2J_2^2L_4I_6 + \frac{8}{3}I_2J_2^2I_4J_6 + \frac{4}{3}I_2^2J_2J_4J_6 - 2I_2^3L_4M_6 + J_2J_4L_4I_6 - 16I_2K_4L_4J_6 - 14J_2K_4^2J_6 \\
&+ 6I_2L_4^2L_6 + 4J_2K_4L_4L_6 + 6I_2I_4L_4M_6 - 2I_2J_4K_4M_6 + 4J_2I_4K_4M_6 + 4I_2^2J_6M_6 - 2J_2^2I_6J_6 \\
&- 4I_2J_2L_6M_6 - 12I_4J_6M_6 + 6J_4L_6M_6 + 24K_4J_6^2 - 12L_4J_6L_6 - 4J_4^3K_4 + 4I_4J_4^2L_4 - J_4K_4^2L_4 \\
&= 0,
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\frac{1}{18}I_2^5J_2^3 - \frac{2}{9}I_2^3J_2^3I_4 + \frac{2}{9}I_2J_2^3I_4^2 + \frac{1}{12}I_2^2J_2^3I_6 - \frac{1}{6}J_2^3I_4I_6 - \frac{1}{6}I_2^4J_2^2J_4 + \frac{1}{3}I_2^2J_2^2I_4J_4 + \frac{1}{2}I_2J_2^2J_4I_6 \\
&+ \frac{1}{2}I_2^3J_2J_4^2 - I_2J_2I_4J_4^2 - \frac{3}{4}J_2J_4^2I_6 - \frac{1}{2}I_2^2J_4^3 + I_4J_4^3 - I_2^2J_2K_4J_6 + 2J_2I_4K_4J_6 + \frac{1}{4}I_2^2J_2^2K_4^2 \\
&- \frac{1}{2}J_2^2I_4K_4^2 + \frac{3}{2}I_2J_2J_4K_4^2 - \frac{9}{4}J_4^2K_4^2 + \frac{1}{2}I_2^3J_2K_4L_4 - I_2J_2I_4K_4L_4 - \frac{1}{2}I_2^2J_4K_4L_4 + I_4J_4K_4L_4 \\
&+ 2I_2J_2J_6L_6 - 3J_4J_6L_6 - 2I_2J_2^2K_4L_6 + 3J_2J_4K_4L_6 - \frac{1}{2}I_2^2J_2L_4L_6 - J_2I_4L_4L_6 + \frac{3}{2}I_2J_4L_4L_6 \\
&- \frac{1}{6}I_2^4J_2M_6 + \frac{5}{6}I_2^2J_2I_4M_6 - J_2I_4^2M_6 - I_2J_2I_6M_6 + \frac{3}{2}J_4I_6M_6 = 0.
\end{aligned} \tag{5}$$

As shown by Theorem IV.2, these three syzygy relations do not imply any single-valued function relation of any of these eleven invariants, with respect to the other ten invariants.

The second point is meaningful to the further research of irreducible function bases of higher order tensors. For example, for the nine invariant Smith-Bao minimal integrity basis of a fourth order three-dimensional symmetric and traceless tensor, there are five syzygy relations.[15,19] These five syzygy relations are not so well-structured like (1) and (2), but even more complicated than (3)–(5). However, it is still possible that the nine invariant Smith-Bao minimal integrity basis is indeed an irreducible function basis of a fourth order three-dimensional symmetric and traceless tensor, just like the four invariant Smith-Bao minimal integrity basis is indeed an irreducible function basis of a third order three-dimensional symmetric and traceless tensor, which was proved in Ref. 15.

In Sec. IV, we show that the eleven invariant function basis is indeed an irreducible function basis, by showing that each of these eleven invariants is not a function of the other ten invariants. This is the method proposed by Pennisi and Trovato.[7] However, we divide the proof into three parts. In Part (i), we show that each of the five invariants I_2, I_4, I_6, I_{10}, and J_2, which form the irreducible function bases of the composition tensors $\mathbf{D}$ and $\mathbf{u}$, is not a function of the other ten invariants. In Part (ii), we use Proposition IV.1 to show that each of K_4, L_4, L_4, and J_6 is not a function of the other ten invariants. In Part (iii), we use another tactic to show that each of the remaining two invariants M_6 and J_4 is not a function of the other ten invariants. Such tactics may also be instructive for the further research of irreducible function bases of higher order tensors.

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