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On the largest eigenvalue of a symmetric nonnegative tensor

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SUMMARY

In this paper, some important spectral characterizations of symmetric nonnegative tensors are analyzed. In particular, it is shown that a symmetric nonnegative tensor has the following properties: (i) its spectral radius is zero if and only if it is a zero tensor; (ii) it is weakly irreducible (respectively, irreducible) if and only if it has a unique positive (respectively, nonnegative) eigenvalue–eigenvector; (iii) the minimax theorem is satisfied without requiring the weak irreducibility condition; and (iv) if it is weakly reducible, then it can be decomposed into some weakly irreducible tensors. In addition, the problem of finding the largest eigenvalue of a symmetric nonnegative tensor is shown to be equivalent to finding the global solution of a convex optimization problem. Subsequently, algorithmic aspects for computing the largest eigenvalue of symmetric nonnegative tensors are discussed. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Let $\Re$ be the real field. In this paper, we consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\Re$:

$$\mathcal{A} = (a_{i_1 i_2 \ldots i_m}), \quad a_{i_1 i_2 \ldots i_m} \in \Re, \quad 1 \leqslant i_1, i_2, \ldots, i_m \leqslant n. \tag{1.1}$$

$\mathcal{A}$ is called nonnegative (or, respectively, positive) if $a_{i_1 i_2 \ldots i_m} \geqslant 0$ (or, respectively, $a_{i_1 i_2 \ldots i_m} > 0$). When $m = 2$, $\mathcal{A}$ is a matrix. When $m \geqslant 3$, $\mathcal{A}$ is called a higher-order tensor. Tensors play an important role in physics, engineering, and mathematics. Applications of tensors include data analysis and mining, information science, signal and image processing, and computational biology, and so on. See [1–3] and references therein.

To an n-dimensional column vector $\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n]^T \in \Re^n$, real or complex, we define an n-dimensional column vector:

$$\mathcal{A}\boldsymbol{x}^{m-1} := \left(\sum_{i_2, \ldots, i_m = 1}^{n} a_{i\, i_2 \ldots i_m} \boldsymbol{x}_{i_2} \cdots \boldsymbol{x}_{i_m} \right)_{1 \leqslant i \leqslant n}. \tag{1.2}$$

Definition 1.1
Let $\mathcal{A}$ be an m-order n-dimensional tensor and $\mathcal{C}$ be the set of all complex numbers. Assume that $\mathcal{A}\boldsymbol{x}^{m-1}$ is not identical to 0. We say that $(\lambda, \boldsymbol{x}) \in \mathcal{C} \times (\mathcal{C}^n \backslash \{0\})$ is an eigenvalue–eigenvector of $\mathcal{A}$ if

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{[m-1]}. \tag{1.3}$$

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Here, $\boldsymbol{x}^{[\alpha]} = [\boldsymbol{x}_1^{\alpha}, \boldsymbol{x}_2^{\alpha}, \ldots, \boldsymbol{x}_n^{\alpha}]^T$.

This definition was introduced in [4–6]. Unlike matrices, eigenvalue problems for high-order tensors are nonlinear. Applications of eigenvalues of high-order tensors include medical resonance imaging [7, 8], higher-order Markov chains [9], positive definiteness of even-order multivariate forms in automatical control [10], and best-rank one approximation in data analysis [1, 11, 12], and so on.

Recently, eigenvalue problems for high-order tensors have gained special attention in the realm of numerical multilinear algebra. In particular, the Perron–Frobenius theorem for eigenvalues of nonnegative tensors has been established in [4, 13, 14]. Friedland, Gaubert, and Han [15] established the Perron–Frobenius theorem for homogeneous monotone maps. The Perron–Frobenius theorem for nonnegative tensors is related to measuring higher-order connectivity in linked objects [16] and hyper-graphs [17–19]. Subsequently, on the basis of the minimax theorem for nonnegative tensors in [4], Ng, Qi, and Zhou [9] proposed a power-type method for computing the largest eigenvalue and the corresponding eigenvector for nonnegative tensors. This method is an extension of a method of Collatz [20, 21] for calculating the spectral radius of a nonnegative matrix. The convergence of the Ng–Qi–Zhou method [9] for *primitive* (respectively *weakly primitive*) nonnegative tensors has been established in [22] (respectively, [15]). In [23–25], the authors proposed an updated version of the Ng–Qi–Zhou method [9], and it has been proven that this algorithm is always convergent for any irreducible nonnegative tensor. Recently, it has been shown in [26] that this algorithm is convergent for any weakly irreducible nonnegative tensor. Furthermore, global linear convergence of the Ng–Qi–Zhou method and its updated version has been studied in [24, 26, 27].

The main purpose of this paper is to analyze some important spectral characterizations of symmetric nonnegative tensors. After some preliminaries on nonnegative tensors in Section 2, we give in Section 3 some important properties of the largest eigenvalue of a symmetric nonnegative tensor $\boldsymbol{\mathcal{A}}$, including (i) the spectral radius of $\boldsymbol{\mathcal{A}}$ is 0 if and only if $\boldsymbol{\mathcal{A}} = 0$; (ii) $\boldsymbol{\mathcal{A}}$ is weakly irreducible if and only if $\boldsymbol{\mathcal{A}}$ has a unique positive eigenvalue–eigenvector; (iii) $\boldsymbol{\mathcal{A}}$ is irreducible if and only if $\boldsymbol{\mathcal{A}}$ has a unique nonnegative eigenvalue–eigenvector; (iv) if $\boldsymbol{\mathcal{A}}$ is weakly reducible, then $\boldsymbol{\mathcal{A}}$ can be decomposed into some weakly irreducible tensors; and (v) the minimax theorem is satisfied without requiring the weak irreducibility condition. Some counterexamples are given to show that these properties may not be satisfied if tensor $\boldsymbol{\mathcal{A}}$ is not symmetric. In addition, we show that the problem of finding the largest eigenvalue for symmetric tensors is equivalent to finding the global solution of a convex optimization problem. Subsequently, we discuss in Section 4 some algorithmic aspects for the largest eigenvalue of any symmetric nonnegative tensor.

Throughout this paper, we use $\Re^n$ ($\mathcal{C}^n$) to denote the n-dimensional real (complex) space, respectively. Let $\boldsymbol{P}_n = \{\boldsymbol{x} \in \Re^n : \boldsymbol{x}_i \geqslant 0,\ 1 \leqslant i \leqslant n\}$ and $int(\boldsymbol{P}_n) = \{\boldsymbol{x} \in \Re^n : \boldsymbol{x}_i > 0,\ 1 \leqslant i \leqslant n\}$. Matrices are denoted by italic capitals ($\boldsymbol{A}, \boldsymbol{B}, \ldots$), and high-order tensors are written as calligraphic capitals ($\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \ldots$).

2. PRELIMINARIES

In this section, we gather some definitions and results about nonnegative matrices and nonnegative tensors, which will be used in the following sections.

2.1. Nonnegative matrices

Let $\boldsymbol{M}$ be an $n \times n$ nonnegative matrix. The graph associated to $\boldsymbol{M}$ (see Chapter 2, [21]), $\boldsymbol{\mathcal{G}}(\boldsymbol{M})$, is the directed graph with vertices $1, 2, \ldots, n$ and an edge from i to j if and only if $\boldsymbol{M}_{ij} \neq 0$. A directed graph is said to be *strongly connected* if there is a directed path between any two distinct vertices. The matrix $\boldsymbol{M}$ is called *irreducible* if the graph $\boldsymbol{\mathcal{G}}(\boldsymbol{M})$ is strongly connected. We say that $\boldsymbol{M}$ is *primitive* if the graph $\boldsymbol{\mathcal{G}}(\boldsymbol{M})$ is strongly connected and the greatest common divisor of the lengths of its circuits is equal to one. An irreducible matrix with a nonzero main diagonal is primitive (Corollary 3.2, [28]). Let $\sigma(\boldsymbol{M})$ denote the spectrum of $\boldsymbol{M}$, the set of all eigenvalues of $\boldsymbol{M}$. The *spectral radius* of $\boldsymbol{M}$, denoted by $\rho(\boldsymbol{M})$, is the maximum distance of an eigenvalue from the origin,

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that is, $\rho(\boldsymbol{M}) = \max\{|\lambda| : \lambda \in \sigma(\boldsymbol{M})\}$. The classical Perron–Frobenius theorem for nonnegative matrices may be stated as follows (see Chapter 2, [21]).

Theorem 2.1
If $\boldsymbol{M}$ is an irreducible nonnegative matrix, then $\boldsymbol{M}$ has an eigenvector $\boldsymbol{u} \in int(\boldsymbol{P}_n)$, unique up to a scale multiple, whose associated eigenvalue is the spectral radius of $\boldsymbol{M}$, $\rho(\boldsymbol{M})$. Moreover, $\rho(\boldsymbol{M})$ is a simple root of the characteristic equation of $\boldsymbol{M}$. Furthermore, if the nonnegative matrix $\boldsymbol{M}$ is primitive, then

$$\rho(\boldsymbol{M}) > |\lambda|, \ \forall \lambda \in \sigma(\boldsymbol{M}) \backslash \{\rho(\boldsymbol{M})\}.$$

2.2. Nonnegative tensors

Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. The spectral radius of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$.

Definition 2.2 ([4]*)*
An m-order n-dimensional tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1 i_2 \ldots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. The graph associated to $\mathcal{A}$, $\mathcal{G}(\mathcal{A})$, is the directed graph with vertices $1, 2, \ldots, n$ and an edge from i to j if and only if $a_{i\, i_2 \ldots i_m} \neq 0$ for some $i_l = j, l = 2, 3, \ldots, m$.

Definition 2.3 ([15]*)*
An m-order n-dimensional tensor $\mathcal{A}$ is called *weakly irreducible* if $\mathcal{G}(\mathcal{A})$ is strongly connected. If $\mathcal{G}(\mathcal{A})$ is strongly connected and the greatest common divisor of the lengths of its circuits is equal to 1, then $\mathcal{A}$ is called weakly primitive.

We have the following proposition.

Proposition 2.1 ([15]*)*
If the nonnegative tensor $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible. For $m = 2$, $\mathcal{A}$ is irreducible if and only if $\mathcal{A}$ is weakly irreducible.

Let $\mathcal{I}$ be the m-order n-dimensional unit tensor whose entries are

$$I_{i_1 i_2 \ldots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \ldots = i_m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Proposition 2.2 ([26]*)*
If the nonnegative tensor $\mathcal{A}$ is weakly irreducible, then $\mathcal{A} + \mathcal{I}$ is weakly primitive.

Let $\mathcal{A}$ be a nonnegative tensor. For any vector $\boldsymbol{x} \in \boldsymbol{P}_n$, we define the following sequence $\{\mathcal{A}^{(k)}\boldsymbol{x}\}$:

$$\begin{aligned}
&\mathcal{A}^{(1)}\boldsymbol{x} = \mathcal{A}(\boldsymbol{x})^{m-1}, z^{(1)} = \left(\mathcal{A}^{(1)}\boldsymbol{x}\right)^{\left[\frac{1}{m-1}\right]} \\
&\mathcal{A}^{(2)}\boldsymbol{x} = \mathcal{A}\left(z^{(1)}\right)^{m-1}, \ z^{(2)} = \left(\mathcal{A}^{(2)}\boldsymbol{x}\right)^{\left[\frac{1}{m-1}\right]} \\
&\qquad \vdots \\
&\mathcal{A}^{(k)}\boldsymbol{x} = \mathcal{A}\left(z^{(k-1)}\right)^{m-1}, \ z^{(k)} = \left(\mathcal{A}^{(k)}\boldsymbol{x}\right)^{\left[\frac{1}{m-1}\right]}, \ k \geqslant 2.
\end{aligned}$$

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Definition 2.4 ([22]*)*
A nonnegative tensor $\mathcal{A}$ is primitive if there exists a positive integer k such that $\mathcal{A}^{(k)}\boldsymbol{x} \in int(\boldsymbol{P}_n)$ for any nonzero $\boldsymbol{x} \in \boldsymbol{P}_n$.

Clearly, positive tensors and essentially positive tensors [22, 29] are primitive. A primitive nonnegative tensor $\mathcal{A}$ is irreducible, but the converse is false [22]. We have the following result.

Theorem 2.2 ([22, 23]*)*
Suppose $\mathcal{A}$ is an irreducible nonnegative tensor. Let $\mathcal{B} = \mathcal{A} + \mathcal{I}$. Then,

(i) $\mathcal{B}$ is primitive.
(ii) If λ is the largest eigenvalue of $\mathcal{B}$ and $\boldsymbol{u}$ is a positive eigenvector of $\mathcal{B}$ associated with λ, then $\lambda - 1$ is the largest eigenvalue of $\mathcal{A}$, and $\boldsymbol{u}$ is a positive eigenvector of $\mathcal{A}$ associated with $\lambda - 1$.

In [4, 15], the Perron–Frobenius theorem and the well-known Collatz [20] minimax theorem for nonnegative matrices have been extended to nonnegative tensors, and further results for Perron–Frobenius theorem for nonnegative tensors have been given in [30, 31]. In the following, we state these results for reference.

Theorem 2.3
Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n and $\rho(\mathcal{A})$ be the spectral radius of $\mathcal{A}$.

(i) [30] $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector.
(ii) [15] If $\mathcal{A}$ is weakly irreducible, then $\mathcal{A}$ has a positive eigenvector $\boldsymbol{u} \in int(\boldsymbol{P}_n)$, unique up to a scale multiple, whose associated eigenvalue is $\rho(\mathcal{A})$.
(iii) [4] If $\mathcal{A}$ is irreducible, then $\mathcal{A}$ has a positive eigenvector $\boldsymbol{u} \in int(\boldsymbol{P}_n)$ whose associated eigenvalue is $\rho(\mathcal{A})$. Moreover, if λ is an eigenvalue with nonnegative eigenvector, then $\lambda = \rho(\mathcal{A})$, and the nonnegative eigenvector is unique up to a multiplicative constant.

2.3. The Ng–Qi–Zhou algorithm

In this subsection, we state a power-type algorithm for calculating the largest eigenvalue of a nonnegative tensor $\mathcal{A}$ by applying the algorithm proposed by Ng, Qi, and Zhou in [9] to tensor $\mathcal{B} = \mathcal{A} + \mathcal{I}$ and establish the Q-linear convergence of this power algorithm under a weak irreducibility condition. This algorithm has been studied in [23].

For any nonnegative column vector $\boldsymbol{x} \in \Re^n$, we define $\phi : \boldsymbol{P}_n \to P_1$ by

$$\phi(\boldsymbol{x}) = \sum_{i=1}^{n} \boldsymbol{x}_i. \tag{2.5}$$

Algorithm 2.1
Step 0. Choose $\boldsymbol{x}^{(1)} \in int(\boldsymbol{P}_n)$. Let $\mathcal{B} = \mathcal{A} + \mathcal{I}$, and set $k := 1$.
Step 1. Compute

$$y^{(k)} = \mathcal{B}\left(\boldsymbol{x}^{(k)}\right)^{m-1},$$
$$\underline{\lambda}_k = \min_{\boldsymbol{x}_i^{(k)}>0} \frac{\left(y^{(k)}\right)_i}{\left(\boldsymbol{x}_i^{(k)}\right)^{m-1}},$$
$$\bar{\lambda}_k = \max_{\boldsymbol{x}_i^{(k)}>0} \frac{\left(y^{(k)}\right)_i}{\left(\boldsymbol{x}_i^{(k)}\right)^{m-1}}.$$

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Step 2. If $\bar{\lambda}_k = \underline{\lambda}_k$, then let $\lambda = \bar{\lambda}_k$, and stop. Otherwise, compute

$$\boldsymbol{x}^{(k+1)} = \frac{\left(y^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\phi\left(\left(y^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right)},$$

replace k by $k+1$, and go to Step 1.

Let

$$F(\boldsymbol{x}) = \mathcal{B}\boldsymbol{x}^{m-1}, \;\; G(\boldsymbol{x}) = F(\boldsymbol{x})^{\left[\frac{1}{m-1}\right]} \text{ and } \; H(\boldsymbol{x}) = \frac{G(\boldsymbol{x})}{\phi\left(G(\boldsymbol{x})\right)}. \tag{2.6}$$

Clearly, the sequence $\{\boldsymbol{x}^{(k)}\}$ generated by Algorithm 2.1 satisfies

$$\boldsymbol{x}^{(k+1)} = H(\boldsymbol{x}^{(k)}), \; k = 1, 2, \ldots, \tag{2.7}$$

and $\phi(\boldsymbol{x}^{(k)}) = 1$ for all $k = 1, 2, \cdots$.

Theorem 2.4 ([22, 23]*)*
Suppose the nonnegative tensor $\mathcal{A}$ is irreducible and $(\rho(\mathcal{A}); \mathbf{u})$ is a positive eigenvalue–eigenvector of $\mathcal{A}$ satisfying $\phi(\mathbf{u}) = 1$. Then, starting from any $\boldsymbol{x}^{(1)} \in \boldsymbol{P}\backslash\{0\}$, Algorithm 2.1 produces a value of λ and a corresponding eigenvector $\boldsymbol{u}$ in a finite number of steps or generates three convergent sequences $\{\underline{\lambda}_k\}$, $\{\bar{\lambda}_k\}$, and $\{\boldsymbol{x}^{(k)}\}$ such that $\lim_{k\to\infty} \underline{\lambda}_k = \lim_{k\to\infty} \bar{\lambda}_k = \lambda$ and $\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \mathbf{u}$. Furthermore, $\rho(\mathcal{A}) = \lambda - 1$.

Theorem 2.5 ([15, 26]*)*
Suppose the nonnegative tensor $\mathcal{A}$ is weakly irreducible and $(\rho(\mathcal{A}), \mathbf{u})$ is a positive eigenvalue–eigenvector of $\mathcal{A}$ satisfying $\phi(\mathbf{u}) = 1$. Then, starting from any $\boldsymbol{x}^{(1)} \in int(\boldsymbol{P})$, Algorithm 2.1 generates a sequence $\{\boldsymbol{x}^{(k)}\}$, which converges to the vector $\mathbf{u}$.

Proof
By Proposition 2.2, $\mathcal{B}$ is weakly primitive. Hence, from Corollary 5.1 [15] and the result (ii) of Theorem 2.2, this theorem holds. □

Next, we give the Q-linear convergence of Algorithm 2.1.

Theorem 2.6
Let $\mathcal{A}$, $\mathcal{B}$, and $\{\boldsymbol{x}^{(k)}\}$ be as in Theorem 2.5. If $\mathcal{A}$ is weakly irreducible, then the sequence $\{\boldsymbol{x}^{(k)}\}$ is convergent, and the convergence rate of $\{\boldsymbol{x}^{(k)}\}$ is Q-linear, that is, there exists a vector norm $\|\cdot\|$ such that

$$\limsup_{k\to\infty} \frac{\|\boldsymbol{x}^{(k+1)} - \boldsymbol{u}\|}{\|\boldsymbol{x}^{(k)} - \boldsymbol{u}\|} < 1.$$

Theorem 2.6 can be proved in a similar argument as in Corollary 5.2 [15], so we omit it. In Theorem 2.6, we establish the Q-linear convergence of Algorithm 2.1 under the weak irreducibility condition. A power algorithm for polynomial eigenvalue problems has been recently introduced in [15], and the R-linear convergence of the power algorithm has been established under the *weak primitivity* condition. In addition, it is shown recently in [26] that Algorithm 2.1 is globally R-linearly convergent for weakly irreducible nonnegative tensors.

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3. SYMMETRIC NONNEGATIVE TENSORS

In this section, we give some important properties of symmetric nonnegative tensors; see Theorems 3.1–3.5. In addition, we show that the largest eigenvalue of a symmetric nonnegative tensor connects with the global solution of a *convex polynomial optimization* problem.

Tensor $\mathcal{A}$ is called *symmetric* if its entries $a_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of their indices $\{i_1, i_2, \ldots, i_m\}$ [6]. Let $f(\boldsymbol{x})$, an mth degree homogeneous polynomial form of n variables, be defined by

$$f(\boldsymbol{x}) := \mathcal{A}\boldsymbol{x}^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \ldots i_m} \boldsymbol{x}_{i_1} \boldsymbol{x}_{i_2} \cdots \boldsymbol{x}_{i_m}. \tag{3.8}$$

By simple computation,

$$\nabla f(\boldsymbol{x}) := m\mathcal{A}\boldsymbol{x}^{m-1} = m \left(\sum_{i_2, \ldots, i_m = 1}^{n} a_{i\, i_2 \ldots i_m} \boldsymbol{x}_{i_2} \cdots \boldsymbol{x}_{i_m} \right)_{1 \leqslant i \leqslant n}. \tag{3.9}$$

We consider the following constrained optimization problem:

$$\begin{array}{ll} \max & f(\boldsymbol{x}) \\ s.t & \sum_{i=1}^{n} \boldsymbol{x}_i^m = 1 \\ & \boldsymbol{x} \geqslant 0. \end{array} \tag{3.10}$$

A generalized form of Problem (3.10) has been extensively studied in [32], and it is shown in [32] that this generalized optimization problem can be used for solving some hardly tractable combinational optimization problems such as the labeling problem.

Lemma 3.1
Suppose the nonnegative tensor $\mathcal{A}$ is symmetric. If $\boldsymbol{x}$ is a local solution of Problem (3.10), then $(f(\boldsymbol{x}), \boldsymbol{x})$ is a nonnegative eigenvalue–eigenvector of $\mathcal{A}$.

Proof
If $\boldsymbol{x}$ is a local solution of Problem (3.10), then, by Theorem 12.1 of [33], there exist $\boldsymbol{c} \in \Re$ and $z \in \Re^n$ such that

$$\mathcal{A}\boldsymbol{x}^{m-1} = \boldsymbol{c}\boldsymbol{x}^{[m-1]} - z, \tag{3.11}$$

$$\boldsymbol{x}_i z_i = 0,\ \boldsymbol{x}_i \geqslant 0,\ z_i \geqslant 0,\ i = 1, 2, \ldots, n, \tag{3.12}$$

$$\sum_{i=1}^{n} \boldsymbol{x}_i^m = 1. \tag{3.13}$$

From (3.11)–(3.13), we have

$$f(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^m = \boldsymbol{x}^T(\mathcal{A}\boldsymbol{x}^{m-1}) = \boldsymbol{c}\boldsymbol{x}^T(\boldsymbol{x}^{[m-1]}) - \boldsymbol{x}^T z = \boldsymbol{c} \sum_{i=1}^{n} \boldsymbol{x}_i^m = \boldsymbol{c}.$$

For $i = 1, 2, \ldots, n$, by (3.12), $z_i = 0$ if $\boldsymbol{x}_i > 0$. If $\boldsymbol{x}_i = 0$, by (3.11), $(\mathcal{A}\boldsymbol{x}^{m-1})_i = -z_i$. Because $(\mathcal{A}\boldsymbol{x}^{m-1})_i \geqslant 0$ and $z_i \geqslant 0$, we obtain $z_i = 0$. Hence, $z_i = 0$ for all $i = 1, 2, \ldots, n$. Therefore, from (3.11), $\mathcal{A}\boldsymbol{x}^{m-1} = \boldsymbol{c}\boldsymbol{x}^{[m-1]}$, which implies that $(f(\boldsymbol{x}), \boldsymbol{x}) = (\boldsymbol{c}, \boldsymbol{x})$ is a nonnegative eigenvalue–eigenvector of $\mathcal{A}$. □

Definition 3.5
$(\lambda, \boldsymbol{x})$ is called a KKT pair of Problem (3.10) if it satisfies

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{[m-1]},\ \sum_{i=1}^{n} \boldsymbol{x}_i^m = 1,\ \lambda \geqslant 0,\ \boldsymbol{x} \geqslant 0. \tag{3.14}$$

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Clearly, $(\lambda, \boldsymbol{x})$ is a KKT pair of Problem (3.10) if and only if $(\lambda, \boldsymbol{x})$ is a nonnegative eigenvalue–eigenvector of $\mathcal{A}$.

Theorem 3.1
Let $\mathcal{A}$ be a symmetric nonnegative tensor. Then, $\rho(\mathcal{A}) = 0$ if and only if $\mathcal{A} = 0$.

Proof
Clearly, if $\mathcal{A} = 0$, then $\rho(\mathcal{A}) = 0$. Suppose $\mathcal{A} \neq 0$. Then, there exists $\boldsymbol{x} \in \Re^n$ satisfying $\sum_{i=1}^n \boldsymbol{x}_i^m = 1$ such that $f(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^m > 0$. This means that there exists $\boldsymbol{y} \in \Re^n$ satisfying $f(\boldsymbol{y}) > 0$ such that $\boldsymbol{y}$ is a local solution of Problem (3.10). By Lemma 3.1, $(f(\boldsymbol{y}), \boldsymbol{y})$ is a nonnegative eigenvalue–eigenvector of $\mathcal{A}$. Hence, $\rho(\mathcal{A}) \geqslant f(\boldsymbol{y}) > 0$. Therefore, if $\rho(\mathcal{A}) = 0$, then $\mathcal{A} = 0$. This completes the proof. □

Remark 1
Theorem 3.1 is not true if $\mathcal{A}$ is not symmetric. Consider a three-order two-dimensional tensor $\mathcal{F}$ given by $f_{121} = 1$ and 0 elsewhere. By simple computation, we obtain $\rho(\mathcal{F}) = 0$, but $\mathcal{F} \neq 0$. In [26], it is shown that $\rho(\mathcal{T}) > 0$ if the nonnegative tensor $\mathcal{T}$ is strictly nonnegative. In Theorem 3.1, it is shown that if the nonnegative tensor $\mathcal{T}$ is symmetric and $\mathcal{T} \neq 0$, then $\rho(\mathcal{T}) > 0$.

Definition 3.6 ([32]*)*
A homogeneous polynomial function $f(\boldsymbol{x})$ defined in (3.8) is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$f(\boldsymbol{x}) = f_1(\boldsymbol{x}_I) + f_2(\boldsymbol{x}_J),$$

where $J = \{1, 2, \ldots, n\} \backslash I$, $f_1(\boldsymbol{x}_I)$, and $f_2(\boldsymbol{x}_J)$ are homogeneous polynomial functions, respectively. If $f(\boldsymbol{x})$ is not reducible, then we call $f(\boldsymbol{x})$ irreducible.

We have the following result.

Lemma 3.2
Let $f(\boldsymbol{x})$ and $\mathcal{A}$ be as in (3.8). Then, $\mathcal{A}$ is weakly irreducible if and only if $f(\boldsymbol{x})$ is irreducible.

Proof
Suppose $\mathcal{A}$ is weakly irreducible, and assume to the contrary that $f(\boldsymbol{x})$ is reducible. Then, there exists a partition $\{I, J\}$ of $\{1, 2, \ldots, n\}$ such that $f(\boldsymbol{x}) = f_1(\boldsymbol{x}_I) + f_2(\boldsymbol{x}_J)$. This implies that $a_{i_1 i_2 \ldots i_m} = 0$ if $i_1 \in I$ and some $i_j \in J$, $j = 2, 3, \ldots, m$,, and $a_{i_1 i_2 \ldots i_m} = 0$ if $i_1 \in J$, and some $i_j \in I$, $j = 2, 3, \ldots, m$. Hence, the graph associated to $\mathcal{A}$, $\mathcal{G}(\mathcal{A})$, is not strongly connected because there is not an edge for any $i \in I$ and $j \in J$, which contradicts the weak irreducibility of $\mathcal{A}$.

Now we suppose $f(\boldsymbol{x})$ is irreducible and assume to the contrary that $\mathcal{A}$ is weakly reducible. By Definition 2.3, the graph associated to $\mathcal{A}$, $\mathcal{G}(\mathcal{A})$, is not strongly connected. Then, there exists a partition $\{I, J\}$ of $\{1, 2, \ldots, n\}$ such that there is not an edge for any $i \in I$ and $j \in J$. This implies that $a_{i_1 i_2 \ldots i_m} = 0$ if there exist some $i_k \in I$ and $i_j \in J, k \neq j, k, j = 1, 2, \ldots, m$. Hence, by (3.8), $f(\boldsymbol{x})$ can be written as the sum of two functions $f_1(\boldsymbol{x}_I)$ and $f_2(\boldsymbol{x}_J)$, which contradicts the irreducibility of $f(\boldsymbol{x})$. □

Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n, and $I \subset \{1, 2, \ldots, n\}$. We define $\mathcal{A}_I = (d_{i_1 i_2 \ldots i_m}), i_j \in I, j = 1, 2, \ldots, m$, a tensor induced from $\mathcal{A}$, by

$$d_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m}, i_j \in I, j = 1, 2, \ldots, m.$$

Let $|I|$ be the number of elements of I. Then, $\mathcal{A}_I$ is a tensor of order m and dimension $|I|$.

By using Lemma 3.2, it is easy to obtain that if $\mathcal{A}$ is weakly reducible, then there exists a partition $\{I_1, I_2, \ldots, I_k\}$ of $\{1, 2, \ldots, n\}$ such that the function $f(\boldsymbol{x})$ defined in (3.8) can be written as

$$f(\boldsymbol{x}) = \sum_{j=1}^{k} f_j(\boldsymbol{x}_{I_j}), \tag{3.15}$$

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where for each $j = 1, 2, \ldots, k$, $f_j(\boldsymbol{x}_{I_j})$ is irreducible or it is a zero function. Moreover, for each $j = 1, 2, \ldots, k$, the corresponding induced tensor $\boldsymbol{\mathcal{A}}_{I_j}$ is either weakly irreducible or a zero tensor. From this observation, we have the following theorem.

Theorem 3.2
Let $\boldsymbol{\mathcal{A}}$ be a symmetric nonnegative tensor of order m and dimension n. Suppose $\boldsymbol{\mathcal{A}}$ is weakly reducible. Then, we have the following results.

(1) There exists a partition $\{I_1, I_2, \ldots, I_k\}$ of $\{1, 2, \ldots, n\}$ such that each induced tensor $\boldsymbol{\mathcal{A}}_{I_i}, i = 1, 2, \ldots, k$ is either weakly irreducible or a zero tensor.
(2) There exists an $n \times n$ permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}(\boldsymbol{\mathcal{A}}\boldsymbol{x}^{m-1}) = \begin{bmatrix} \boldsymbol{\mathcal{A}}_{I_1}(\boldsymbol{x}_{I_1})^{m-1} \\ \boldsymbol{\mathcal{A}}_{I_2}(\boldsymbol{x}_{I_2})^{m-1} \\ \vdots \\ \boldsymbol{\mathcal{A}}_{I_k}(\boldsymbol{x}_{I_k})^{m-1} \end{bmatrix}. \tag{3.16}$$

(3) The eigenvalues of $\boldsymbol{\mathcal{A}}_{I_i}, i = 1, 2, \ldots, k$ are the eigenvalues of $\boldsymbol{\mathcal{A}}$. If $(\lambda, \boldsymbol{x}) \in \mathcal{C} \times (\mathcal{C}^n \backslash \{\boldsymbol{0}\})$ is an eigenvalue–eigenvector of $\boldsymbol{\mathcal{A}}$, then λ is an eigenvalue of some $\boldsymbol{\mathcal{A}}_{I_i}, 1 \leqslant i \leqslant k$.
(4) $\rho(\boldsymbol{\mathcal{A}}) = \max_{1 \leqslant i \leqslant k} \rho(\boldsymbol{\mathcal{A}}_{I_i})$.

Proof
It follows from (3.15) that (1) holds, and then (2) is satisfied. By (3.16), we can easily obtain that (3) holds. Hence, (4) is satisfied. □

By Theorem 3.2, we have the following minimax theorem for symmetric nonnegative tensors, without the weak irreducibility condition.

Theorem 3.3
Assume that $\boldsymbol{\mathcal{A}}$ is a symmetric nonnegative tensor of order m and dimension n. Then,

$$\mathrm{Min}_{\boldsymbol{x} \in int(\boldsymbol{P})} \mathrm{Max}_{1 \leqslant i \leqslant n} \frac{(\boldsymbol{\mathcal{A}}\boldsymbol{x}^{m-1})_i}{\boldsymbol{x}_i^{m-1}} = \rho(\boldsymbol{\mathcal{A}}) = \mathrm{Max}_{\boldsymbol{x} \in \boldsymbol{P} \backslash \{0\}} \mathrm{Min}_{\boldsymbol{x}_i \neq 0, 1 \leqslant i \leqslant n} \frac{(\boldsymbol{\mathcal{A}}\boldsymbol{x}^{m-1})_i}{\boldsymbol{x}_i^{m-1}}. \tag{3.17}$$

Proof
It has been proven in [4, 15] that (3.17) holds if $\boldsymbol{\mathcal{A}}$ is weakly irreducible or it is irreducible. Now we will show that (3.17) also holds if $\boldsymbol{\mathcal{A}}$ is weakly reducible. By using Theorem 3.2, if $\boldsymbol{\mathcal{A}}$ is weakly reducible, then there exists a partition $\{I_1, I_2, \ldots, I_k\}$ of $\{1, 2, \ldots, n\}$ such that each induced tensor $\boldsymbol{\mathcal{A}}_{I_i}, i = 1, 2, \ldots, k$ is either weakly irreducible or a zero tensor. Moreover, without loss of generality, we suppose

$$\boldsymbol{\mathcal{A}}\boldsymbol{x}^{m-1} = \begin{bmatrix} \boldsymbol{\mathcal{A}}_{I_1}(\boldsymbol{x}_{I_1})^{m-1} \\ \boldsymbol{\mathcal{A}}_{I_2}(\boldsymbol{x}_{I_2})^{m-1} \\ \vdots \\ \boldsymbol{\mathcal{A}}_{I_k}(\boldsymbol{x}_{I_k})^{m-1} \end{bmatrix}. \tag{3.18}$$

For each $i = 1, 2, \ldots, k$, by Corollary 4.2 [15], we obtain

$$\mathrm{Min}_{\boldsymbol{x}_{I_i} > 0} \mathrm{Max}_{j \in I_i} \frac{(\boldsymbol{\mathcal{A}}_{I_i}(\boldsymbol{x}_{I_i})^{m-1})_j}{\boldsymbol{x}_j^{m-1}} = \rho(\boldsymbol{\mathcal{A}}_{I_i}) = \mathrm{Max}_{\boldsymbol{x}_{I_i} \geqslant 0, \boldsymbol{x}_{I_i} \neq 0} \mathrm{Min}_{\boldsymbol{x}_j \neq 0, j \in I_i} \frac{(\boldsymbol{\mathcal{A}}_{I_i}(\boldsymbol{x}_{I_i})^{m-1})_j}{\boldsymbol{x}_j^{m-1}}. \tag{3.19}$$

Note that (3.19) holds for a zero tensor. For each $\boldsymbol{x} \in int(\boldsymbol{P})$, by the left equality of (3.19), we have

$$\mathrm{Max}_{j \in I_i} \frac{(\boldsymbol{\mathcal{A}}_{I_i}(\boldsymbol{x}_{I_i})^{m-1})_j}{\boldsymbol{x}_j^{m-1}} \geqslant \rho(\boldsymbol{\mathcal{A}}_{I_i}), i = 1, 2, \ldots, k.$$

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So, by (3.18) and the statement (4) of Theorem 3.2,

$$\mathrm{Max}_{1\leqslant i\leqslant n}\frac{(\mathcal{A}\boldsymbol{x}^{m-1})_i}{\boldsymbol{x}_i^{m-1}}\geqslant\max_{1\leqslant i\leqslant k}\rho(\mathcal{A}_{I_i})=\rho(\mathcal{A}). \tag{3.20}$$

Let $\boldsymbol{u}_{I_i}(>0)$ be an eigenvector of $\mathcal{A}_{I_i}$ with the associated eigenvalue $\rho(\mathcal{A}_{I_i})$, $k=1,2,\ldots,k$, and define $\boldsymbol{u}=(\boldsymbol{u}_{I_i},1\leqslant i\leqslant k)$. Clearly, we obtain

$$\mathrm{Max}_{1\leqslant i\leqslant n}\frac{(\mathcal{A}\boldsymbol{u}^{m-1})_i}{\boldsymbol{u}_i^{m-1}}=\rho(\mathcal{A}). \tag{3.21}$$

It follows from (3.20) and (3.21) that the left equality of (3.17) is satisfied. We now prove that the right equality of (3.17) holds. For each $\boldsymbol{x}\in\boldsymbol{P}\backslash\{0\}$, by the right equality of (3.19), we have

$$\mathrm{Min}_{\boldsymbol{x}_j\neq 0,j\in I_i}\frac{(\mathcal{A}_{I_i}(\boldsymbol{x}_{I_i})^{m-1})_j}{\boldsymbol{x}_j^{m-1}}\leqslant\rho(\mathcal{A}_{I_i}),\ \text{for some } 1\leqslant i\leqslant k.$$

So, by (3.18) and the statement (4) of Theorem 3.2,

$$\mathrm{Min}_{\boldsymbol{x}_i\neq 0,1\leqslant i\leqslant n}\frac{(\mathcal{A}\boldsymbol{x}^{m-1})_i}{\boldsymbol{x}_i^{m-1}}\leqslant\rho(\mathcal{A}_{I_i})\leqslant\rho(\mathcal{A}). \tag{3.22}$$

Let $\boldsymbol{v}_{I_j}(>0)$ be an eigenvector of $\mathcal{A}_{I_j}$ with the associated eigenvalue $\rho(\mathcal{A}_{I_j})=\rho(\mathcal{A})$, and define $\boldsymbol{v}\in\Re^n$ by $\boldsymbol{v}_i=(\boldsymbol{v}_{I_j})_i$ if $i\in I_j$ and zero otherwise. Clearly, we obtain

$$\mathrm{Min}_{\boldsymbol{v}_i\neq 0,1\leqslant i\leqslant n}\frac{(\mathcal{A}\boldsymbol{v}^{m-1})_i}{\boldsymbol{v}_i^{m-1}}=\rho(\mathcal{A}). \tag{3.23}$$

It follows from (3.22) and (3.23) that the right equality of (3.17) is satisfied. So we completes the proof. □

Remark 2

Theorem 3.3 does not hold if $\mathcal{A}$ is not symmetric. Consider a three-order two-dimensional tensor $\mathcal{S}$ given by $s_{111}=2, s_{122}=s_{222}=1$ and zero elsewhere. By simple computation, we obtain $\rho(\mathcal{S})=2$ with associated eigenvector $\boldsymbol{x}=(1,0)^T$. However, the left equality of (3.17) is not satisfied for tensor $\mathcal{S}$. Recently, it has been shown in Theorems 5.2 and 5.3 of [30] that the right equality of (3.17) is satisfied for any nonnegative tensor and the left equality of (3.17) holds under the condition that the nonnegative tensor $\mathcal{A}$ has a positive eigenvector corresponding to some eigenvalue. Theorem 3.3 shows that (3.17) is satisfied for any nonnegative symmetric tensor.

Lemma 3.3

Let $\mathcal{A}$ and $\mathcal{A}_{I_i}, i=1,2,\ldots,k$ be as in Theorem 3.2. Then, the following two statements are equivalent.

(a) $\mathcal{A}_{I_i}, i=1,2,\ldots,k$ are weakly irreducible and $\rho(\mathcal{A}_{I_1})=\rho(\mathcal{A}_{I_2})=\ldots=\rho(\mathcal{A}_{I_k})$.
(b) $\mathcal{A}$ has a positive eigenvector $\boldsymbol{u}\in int(\boldsymbol{P}_n)$ whose associated eigenvalue is $\rho(\mathcal{A})$.

Proof

By Theorem 3.2, without loss of generality, we suppose that (3.18) holds. Assume that (a) holds. Then, for each $i=1,2,\ldots,k$, $\mathcal{A}_{I_i}$ has a positive eigenvalue–eigenvector $(\rho(\mathcal{A}_{I_i}),\boldsymbol{u}_{I_i})$. From Theorem 3.2, we obtain $\rho(\mathcal{A})=\rho(\mathcal{A}_{I_1})=\rho(\mathcal{A}_{I_2})=\ldots=\rho(\mathcal{A}_{I_k})$. Let $\boldsymbol{u}=(\boldsymbol{u}_{I_i},i=1,2,\ldots,k)\in\Re^n$ be formed by $\boldsymbol{u}_{I_i}, i=1,2,\ldots,k$. Then, by (3.18), $(\rho(\mathcal{A}),\boldsymbol{u})$ is a positive eigenvalue–eigenvector of $\mathcal{A}$.

Now, we suppose that (b) holds. Then, by (3.18), $(\rho(\mathcal{A}),\boldsymbol{u}_{I_i})$ is a positive eigenvalue–eigenvector of $\mathcal{A}_{I_i}, i=1,2,\ldots,k$. Hence, $\mathcal{A}_{I_i}, i=1,2,\ldots,k$ are not zero tensors, and (a) is satisfied. □

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Theorem 3.4
Let $\mathcal{A}$ be a symmetric nonnegative tensor of order m and dimension n. Then, the following two statements are equivalent.

(i) $\mathcal{A}$ is weakly irreducible.
(ii) $\mathcal{A}$ has a positive eigenvector $\boldsymbol{u} \in int(\boldsymbol{P}_n)$, unique up to a scale multiple, whose associated eigenvalue is $\rho(\mathcal{A})$.

Proof
If $\mathcal{A}$ is weakly irreducible, then, by Theorem 2.3, (ii) holds. Now we suppose that (ii) holds and assume to the contrary that $\mathcal{A}$ is weakly reducible. By Theorem 3.2 and Lemma 3.3, there exists a partition $\{I_1, I_2, \ldots, I_k\}$ of $\{1, 2, \ldots, n\}$ such that each induced tensor $\mathcal{A}_{I_i}, i = 1, 2, \ldots, k$ is weakly irreducible, and $\boldsymbol{\rho(\mathcal{A})} = \rho(\mathcal{A}_{I_i}), 1 \leqslant i \leqslant k$. Moreover, without loss of generality, we suppose that (3.18) holds. For each $i = 1, 2, \ldots, k$, $\mathcal{A}_{I_i}$ has a positive eigenvalue–eigenvector $(\rho(\mathcal{A}_{I_i}), \boldsymbol{u}_{I_i})$. Let $\boldsymbol{u} = (\boldsymbol{u}_{I_i}, i = 1, 2, \ldots, k) \in \Re^n$ be formed by $\boldsymbol{u}_{I_i}, i = 1, 2, \ldots, k$, and $\boldsymbol{v} = (i \times \boldsymbol{u}_{I_i}, i = 1, 2, \ldots, k) \in \Re^n$. Then, by (3.18), $(\rho(\mathcal{A}), \boldsymbol{u})$ and $(\rho(\mathcal{A}), \boldsymbol{v})$ are positive eigenvalue–eigenvectors of $\mathcal{A}$, and clearly there does not exist a constant $\boldsymbol{c} \neq 0$ such that $\boldsymbol{u} = \boldsymbol{c}\boldsymbol{v}$, which contradicts (ii). So we complete the proof. □

Remark 3
Theorem 3.4 does not hold if $\mathcal{A}$ is not symmetric. Consider a three-order two-dimensional tensor $\mathcal{D}$ given by $d_{122} = d_{222} = 1$ and zero elsewhere. By simple computation, we obtain $\mathcal{D}$ with a positive eigenvector $\boldsymbol{x} = (1, 1)^T$, unique up to a scale multiple, whose associated eigenvalue is 1. However, $\mathcal{D}$ is weakly reducible.

Theorem 3.5
Let $\mathcal{A}$ be a symmetric nonnegative tensor of order m and dimension n. Then, the following two statements are equivalent.

(a) $\mathcal{A}$ is irreducible.
(b) Problem (3.10) has a unique KKT pair $(\lambda, \boldsymbol{x})$, and λ and $\boldsymbol{x}$ are positive.

Proof
If $\mathcal{A}$ is irreducible, then, by the statement (iii) of Theorem 2.3 and Definition 3.5, (b) holds.

Now we suppose that (b) holds and assume to the contrary that $\mathcal{A}$ is reducible, that is, there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1 i_2 \ldots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \in J, \; J = \{1, 2, \ldots, n\} \backslash I. \tag{3.24}$$

Let $\mathcal{A}_J$ be the induced tensor from $\mathcal{A}$ and $\boldsymbol{x} = (0_I, \boldsymbol{x}_J) \in \Re^n$. Then, it follows from (3.24) that there exists a permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}(\mathcal{A}\boldsymbol{x}^{m-1}) = \begin{bmatrix} 0_I \\ \mathcal{A}_J(\boldsymbol{x}_J)^{m-1} \end{bmatrix}. \tag{3.25}$$

By the statement (i) of Theorem 2.3, $\mathcal{A}_J$ has a nonnegative eigenvalue–eigenvector $(l, \boldsymbol{x}_J^*)$ such that

$$\mathcal{A}_J(\boldsymbol{x}_J^*)^{m-1} = l(\boldsymbol{x}_J^*)^{[m-1]}, \; \sum_{i \in J}(\boldsymbol{x}_i^*)^m = 1, \; l \geqslant 0, \; \boldsymbol{x}_J^* \geqslant 0.$$

Let $\boldsymbol{x}^* = (0_I, \boldsymbol{x}_J^*)$. Then, clearly, $(l, \boldsymbol{x}^*)$ is a nonnegative eigenvalue–eigenvector of $\mathcal{A}$, which contradicts with (b). This completes the proof. □

Remark 4
The statement (b) of Theorem 3.5 does not hold for weakly irreducible tensors. Consider a three-order two-dimensional tensor $\mathcal{Q}$ given by $q_{111} = 1, q_{222} = 2, q_{112} = q_{121} = q_{211} = 1$ and

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zero elsewhere. By simple computation, we obtain $\mathcal{Q}$ is weakly irreducible and it has a positive eigenvector $\boldsymbol{x} = (1, 1)^T$ with associated eigenvalue 3. However, $\mathcal{Q}$ also has an eigenvalue 2 with corresponding eigenvector $\boldsymbol{y} = (0, 1)^T$. This also means that Problem (3.10) may have more than one KKT pairs for weakly irreducible tensors. In addition, in a similar argument as in Theorem 3.5, we can prove that a nonnegative tensor (not necessarily symmetric) $\mathcal{A}$ is irreducible if and only if the following system has a unique solution $(\lambda, \boldsymbol{x})$:

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{[m-1]},\ \sum_{i=1}^{n} \boldsymbol{x}_i^m = 1,\ \lambda \geqslant 0, \boldsymbol{x} \geqslant 0,$$

and $\lambda > 0, \boldsymbol{x} > 0$.

3.1. Convex optimization reformulation

In this subsection, we show that the largest eigenvalue of a symmetric nonnegative tensor is linked to a convex polynomial optimization problem.

Theorem 3.6
Suppose nonnegative tensor $\mathcal{A}$ is symmetric. If $\boldsymbol{x}^*$ is a global solution of Problem (3.10), then $f(\boldsymbol{x}^*)$ is the largest eigenvalue of $\mathcal{A}$ with corresponding eigenvector $\boldsymbol{x}^*$. In addition, if λ is the largest eigenvalue of $\mathcal{A}$ associated with the eigenvector $\boldsymbol{x}^*$ satisfying $\sum_{i=1}^{n}(x_i^*)^m = 1$, then $\boldsymbol{x}^*$ is a global solution of Problem (3.10).

Proof
It follows from Lemma 3.1 that if $\boldsymbol{x}^*$ is a global solution of Problem (3.10), then $f(\boldsymbol{x}^*)$ is an eigenvalue of $\mathcal{A}$ with corresponding eigenvector $\boldsymbol{x}^*$. Assume to the contrary that $f(\boldsymbol{x}^*)$ is not the largest eigenvalue of $\mathcal{A}$ and $\lambda(> f(\boldsymbol{x}^*))$ is the largest eigenvalue of $\mathcal{A}$ associated with the nonnegative eigenvector $\boldsymbol{y}$ satisfying $\sum_{i=1}^{n} y_i^m = 1$. Then, we obtain

$$\mathcal{A}\boldsymbol{y}^{m-1} = \lambda \boldsymbol{y}^{[m-1]}, \tag{3.26}$$

$$\sum_{i=1}^{n} \boldsymbol{y}_i^m = 1. \tag{3.27}$$

From (3.11) and (3.13), we have

$$f(\boldsymbol{y}) = \mathcal{A}\boldsymbol{y}^m = \boldsymbol{y}^T\left(\mathcal{A}\boldsymbol{y}^{m-1}\right) = \lambda \boldsymbol{y}^T\left(\boldsymbol{y}^{[m-1]}\right) = \lambda \sum_{i=1}^{n} \boldsymbol{y}_i^m = \lambda > f(\boldsymbol{x}^*).$$

This is a contradiction because $\boldsymbol{x}^*$ is a global solution of Problem (3.10). In a similar argument, we can prove that if λ is the largest eigenvalue of $\mathcal{A}$ associated with the eigenvector $\boldsymbol{x}^*$ satisfying $\sum_{i=1}^{n}(\boldsymbol{x}_i^*)^m = 1$, then $\boldsymbol{x}^*$ is a global solution of Problem (3.10). □

In the following, we will show that Problem (3.10) can be reformulated as a convex optimization problem and moreover, Problem (3.10) has a unique positive solution if $\mathcal{A}$ is weakly irreducible.

Lemma 3.4
Suppose the nonnegative tensor $\mathcal{A}$ is symmetric and weakly irreducible. Then, Problem (3.10) has a unique solution $\boldsymbol{x}^*$ and $\boldsymbol{x}^* \in int(\boldsymbol{P}_n)$.

Proof
Because $\mathcal{A}$ is weakly irreducible, $f(\boldsymbol{x})$ is an irreducible polynomial function. By Theorem 5.4 [32], this theorem holds. □

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Lemma 3.5
Suppose the nonnegative tensor $\mathcal{A}$ is symmetric and weakly irreducible. Then, λ is the largest eigenvalue of $\mathcal{A}$ associated with the eigenvector x^* satisfying $\sum_{i=1}^{n}(x_i^*)^m = 1$ if and only if x^* is the unique solution of Problem (3.10) and $\lambda = f(x^*)$.

Proof
By Lemma 3.4 and Theorem 3.6, this lemma holds. □

Let $y = \left(x_1^m, x_2^m, \ldots, x_n^m\right)^T$. Then, $f(x)$ can be converted into a homogeneous polynomial form of y, and we denote it by $g(y)$. Hence, Problem (3.10) can be formulated as the following optimization problem:

$$\begin{array}{ll} \max & g(y) \\ s.t & \sum_{i=1}^{n} y_i = 1 \\ & y \geqslant 0. \end{array} \tag{3.28}$$

Theorem 3.7
Suppose the nonnegative tensor $\mathcal{A}$ is symmetric. Then, $g(y)$ in Problem (3.28) is a concave function on $int(P_n)$.

Proof
By Theorem 5.2 [32], this theorem holds. □

By Theorem 3.7, Problem (3.28) can be converted into the following convex optimization problem:

$$\begin{array}{ll} \min & -g(y) \\ s.t & \sum_{i=1}^{n} y_i = 1 \\ & y \geqslant 0. \end{array} \tag{3.29}$$

Theorem 3.8
Suppose the nonnegative tensor $\mathcal{A}$ is symmetric. Then, Problem (3.29) is a convex optimization problem. Furthermore, if $\mathcal{A}$ is weakly irreducible, then Problem (3.29) has a unique solution y^* and $y^* \in int(P_n)$.

Proof
By Lemma 3.5 and Theorem 3.7, this theorem holds. □

4. ALGORITHMS FOR THE SPECTRAL RADIUS FOR SYMMETRIC NONNEGATIVE TENSORS

In this section, we discuss two algorithms for computing the spectral radius for symmetric nonnegative tensors. One is a power type algorithm based on Theorem 3.2, and the other is a geometric programming (GP) method due to Theorem 3.3.

4.1. Power type algorithm

Given a symmetric nonnegative tensor $\mathcal{A}$ of order m and dimension n, we first propose a procedure to compute a partition $\{I_1, I_2, \ldots, I_k\}$ of $\{1, 2, \ldots, n\}$ such that each induced tensor $\mathcal{A}_{I_i}, i = 1, 2, \ldots, k$ is either weakly irreducible or a zero tensor. This procedure is based on Definition 2.3 that the graph of a weakly irreducible tensor is strongly connected.

Procedure 1

1. Let $J_1 = \{1, 2, \ldots, n\}$ and set $j = 1$.
2. If $J_j = \emptyset$, then stop. Otherwise, choose an element i_1 from J_j, and let $I_j := \{i_1\}$.

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3. Check all elements $a_{i_1 i_2 \ldots i_m}$ such that $i_d \in J_j, d = 2, 3, \ldots, m$. If all these elements are zero, then let $J_{j+1} := J_j \backslash I_j$, set $j := j + 1$, and go to Step 2. Otherwise, for all nonzero elements $a_{i_1 i_2 \ldots i_m}$ and all $d = 2, 3, \ldots, m$, $I_j := I_j \cup \{i_d\}$ if $i_d \notin I_j$. Let $K := \{i_1\}$.
4. If $I_j \backslash K = \emptyset$, then let $J_{j+1} := J_j \backslash I_j$, set $j := j + 1$, and go to Step 2. Otherwise, go to Step 5.
5. Choose an $i_k \in I_j \backslash K$, and check all elements $a_{i_k i_2 \ldots i_m}$ such that $i_d \in J_j, d = 2, 3, \ldots, m$. For all nonzero elements $a_{i_k i_2 \ldots i_m}$ and all $d = 2, 3, \ldots, m$, $I_j := I_j \cup \{i_d\}$ if $i_d \notin I_j$. Let $K := K \cup \{i_k\}$, and go to Step 4.

Procedure 1 can be implemented easily in a computer programming such as MATLAB. To show how Procedure 1 works, we consider the three-order four-dimensional tensor $\mathcal{M}$ given by $m_{222} = 2, m_{333} = m_{444} = 1, m_{443} = m_{434} = m_{344} = 1$, and zero elsewhere. By Procedure 1, a partition of $\{1, 2, 3, 4\}$, can be obtained by the following steps:

Step 1: Let $J_1 := \{1, 2, 3, 4\}$ and $j := 1$.
Step 2: Choose 1 from J_1, and let $I_1 := \{1\}$.
Step 3: Check all elements $m_{1 i_2 i_3}, i_2 \in J_1, j_3 \in J_1$. Because they are all zero, let $J_2 := J_1 \backslash I_1 = \{2, 3, 4\}$ and $j := 2$.
Step 4: Choose 2 from J_2, and let $I_2 := \{2\}$.
Step 5: Check all elements $m_{2 i_2 i_3}, i_2 \in J_2, j_3 \in J_2$, and we have $m_{222} \neq 0$. So let $I_2 := I_2$ and $K := \{2\}$.
Step 6: Because $I_2 \backslash K = \emptyset$, let $J_3 := J_2 \backslash I_2 = \{3, 4\}$ and $j := 3$.
Step 7: Choose 3 from J_3, and let $I_3 := \{3\}$.
Step 8: Check all elements $m_{3 i_2 i_3}, i_2 \in J_3, j_3 \in J_3$, and we have $m_{344} \neq 0$. So let $I_3 := I_3 \cup \{4\} = \{3, 4\}$ and $K := \{3\}$.
Step 9: Because $I_3 \backslash K = \{4\}$, we choose 4 from $I_3 \backslash K$ and check all elements $m_{4 i_2 i_3}, i_2 \in J_3, j_3 \in J_3$. We have $m_{443} \neq 0$ and $m_{434} \neq 0$. Let $I_3 := I_3$ and $K := K \cup \{4\} = \{3, 4\}$.
Step 10: Because $I_3 \backslash K = \emptyset$, let $J_4 := J_3 \backslash I_3$. Because $J_4 = \emptyset$, we stop the procedure and obtain $I_1 = \{1\}, I_2 = \{2\}$ and $I_3 = \{3, 4\}$, which is a partition of $\{1, 2, 3, 4\}$.

We now state a power-type algorithm for the largest eigenvalue of $\mathcal{A}$ as follows.

Algorithm 4.1

Step 1. By using Procedure 1, compute a partition $\{I_1, I_2, \ldots, I_k\}$ of $\{1, 2, \ldots, n\}$ such that each induced tensor $\mathcal{A}_{I_i}, i = 1, 2, \ldots, k$ is either weakly irreducible or a zero tensor.
Step 2. **For** $i = 1, 2, \cdots, k$, **do**
If $\mathcal{A}_{I_i}$ is a zero tensor, then let $\lambda^{(i)} = 0$. Otherwise, compute the largest eigenvalue $\lambda^{(i)}$ of $\mathcal{A}_{I_i}$ and a corresponding eigenvector $\boldsymbol{u}^{(i)}$ by using Algorithm 2.1
End
Step 3. Output $\rho(\mathcal{A}) = \max_{1 \leqslant i \leqslant k} \lambda^{(i)}$. Suppose $\rho(\mathcal{A}) = \lambda^{(i)}$, and let $\boldsymbol{u}$ be defined by

$$\boldsymbol{u}_j = (\boldsymbol{u}^{(i)})_j \text{ if } j \in I_i. \text{ Otherwise, } \boldsymbol{u}_j = 0.$$

Then, $\boldsymbol{u}$ is an eigenvector associated with the largest eigenvalue $\rho(\mathcal{A})$.

By Theorems 2.4 and 2.5, Algorithm 2.1 can produce the largest eigenvalue and its associated eigenvector for weakly irreducible nonnegative tensors. Therefore, in Step 2 of Algorithm 4.1, the largest eigenvalue $\lambda^{(i)}$ of $\mathcal{A}_{I_i}$ and a corresponding eigenvector $\boldsymbol{u}^{(i)}$ can be obtained by using Algorithm 2.1 because $\mathcal{A}_{I_i}$ is weakly irreducible. By Theorem 3.2, any weakly reducible symmetric nonnegative tensor can be decomposed into some weakly irreducible tensors. Hence, Algorithm 4.1 can produce the largest eigenvalue and its associated eigenvector for any symmetric nonnegative tensor. To show this, we report our numerical results as follows.

Algorithms 2.1 and 4.1 are implemented in MATLAB (R2008b), and all the numerical computations are conducted using an Intel 3.20-GHz computer with 2 GB of RAM. All test tensors are

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Table I. Numerical results of Algorithms 4.1 and 2.1.

Problem		Algorithm 4.1				Algorithm 2.1			
n	np	k	err	ρ	cpu(s)	k	err	ρ	cpu(s)
30	2	9	1.57e-006	51.39	0.11	100	2.24e-001	51.39	0.41
60	2	8	1.11e-007	200.12	0.38	100	9.47e-002	200.07	2.17
90	2	6	9.84e-007	451.40	0.92	100	1.43e-001	450.67	4.77
120	2	6	7.11e-007	802.00	2.13	100	1.69e-001	800.57	9.30
150	2	6	3.14e-007	1251.52	4.08	100	1.17e-001	1250.29	18.73
180	2	6	1.64e-007	1799.09	7.25	100	1.33e-001	1797.24	59.20
210	2	6	1.36e-007	2448.70	11.72	100	6.88e-002	2447.93	114.67
240	2	6	9.42e-008	3198.34	17.50	100	4.11e-002	3198.15	171.98
270	2	6	6.49e-008	4050.56	25.08	100	5.77e-002	4049.77	248.80
300	2	6	7.51e-008	5001.54	34.33	100	8.02e-002	4999.91	341.02
30	3	12	2.53e-006	50.75	0.09	100	2.34e-001	50.62	0.42
60	3	12	1.53e-007	201.60	0.42	100	9.90e-002	201.35	2.17
90	3	9	1.80e-006	450.26	1.05	100	1.17e-001	449.42	4.75
120	3	9	1.13e-006	803.34	2.31	100	2.04e-001	801.02	9.38
150	3	9	3.32e-007	1253.10	4.52	100	1.70e-001	1250.41	18.36
180	3	9	1.97e-007	1797.85	7.95	100	4.30e-002	1797.52	63.23
210	3	9	1.42e-007	2449.32	12.72	100	6.34e-002	2448.21	114.52
240	3	9	1.26e-007	3205.06	19.06	100	1.72e-001	3200.43	171.83
270	3	9	7.07e-008	4057.06	27.23	100	1.15e-001	4053.68	250.78
300	3	9	4.66e-008	5005.00	37.36	100	7.68e-002	5002.75	341.73

randomly generated with order $d = 3$. Throughout the computational experiments, we use the vector of ones as a starting point for Algorithm 2.1, and we terminate it when one of the following conditions is satisfied:

$$(1)\ k \geqslant 100;\ \ (2)\ \bar{\lambda}_k - \underline{\lambda}_k \leqslant 10^{-6};\ \ \text{or}\ \ (3)\ \|x^{(k)} - H\left(x^{(k)}\right)\|_\infty \leqslant 10^{-6}.$$

Our numerical results are summarized in Table I. In this table, n is the dimension of the randomly generated tensor, and np denotes the number of the partition of $\{1, 2, \ldots, n\}$. k denotes the number of total iterations needed for Algorithm 2.1, and cpu(s) denotes the total computer time in seconds used to solve the problem. ρ is the largest eigenvalue obtained by these algorithms with a corresponding eigenvector $\boldsymbol{x}_f$, and $err = \|A\boldsymbol{x}_f{}^{m-1} - \rho \boldsymbol{x}_f^{[m-1]}\|_\infty$. The results reported in Table I clearly show that the performance of Algorithm 4.1 is better than Algorithm 2.1 for weakly reducible tensors. Algorithm 4.1 is able to produce the largest eigenvalue for all these randomly generated reducible tensors within 12 iterations. However, Algorithm 2.1 fails to give the largest eigenvalue within 100 iterations for most of test problems.

4.2. Geometric programming method

Let $\mathcal{A}$ be a symmetric nonnegative tensor of order m and dimension n. By Theorem 3.3, clearly, $\rho(\mathcal{A})$ can be obtained by solving the following optimization problem:

$$\mathrm{Min}_{\boldsymbol{x} \in int(\boldsymbol{P})} \mathrm{Max}_{1 \leqslant i \leqslant n} \frac{\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i}{\boldsymbol{x}_i^{m-1}}. \tag{4.30}$$

Let

$$\lambda = \mathrm{Max}_{1 \leqslant i \leqslant n} \frac{\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i}{\boldsymbol{x}_i^{m-1}}$$

Then, we have

$$\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i \leqslant \lambda \boldsymbol{x}_i^{m-1}, i = 1, 2, \ldots, n,$$

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which can also be written as

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{i\,i_2\ldots i_m}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_m}\lambda^{-1}\boldsymbol{x}_i^{1-m} \leqslant 1, i = 1,2,\ldots,n.$$

Hence, (4.30) can be reformulated into the following problem:

$$\begin{aligned}(GP)\quad &\text{Min}\quad \lambda\\ &\text{s.t.}\quad \sum_{i_2,\ldots,i_m=1}^{n} a_{i\,i_2\ldots i_m}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_m}\lambda^{-1}\boldsymbol{x}_i^{1-m} \leqslant 1, i = 1,2,\ldots,n\\ &\quad\ \boldsymbol{x} \in int(\boldsymbol{P}).\end{aligned} \tag{4.31}$$

Problem (GP) is a GP, which is extensively studied in [34,35]. The standard barrier-based interior-point method for convex optimization can be applied to (GP) in a straightforward way with a polynomial-time complexity; see [35].

We conclude this paper with a remark on these two algorithms. Recently, GP method has been discussed in [31] for computing the largest eigenvalue for nonnegative tensors, and it is shown in [31, 36] that the performance of power type method is better than that of the GP method. The advantage of the GP method is that it has a polynomial-time complexity.

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