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Convergence of an algorithm for the largest singular value of a nonnegative rectangular tensor

Guanglu Zhou [a,*], Louis Caccetta [a], Liqun Qi [b,1]

[a] Department of Mathematics and Statistics, Curtin University of Technology, Perth, Australia
[b] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong

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ABSTRACT

In this paper, we present an iterative algorithm for computing the largest singular value of a nonnegative rectangular tensor. We establish the convergence of this algorithm for any irreducible nonnegative rectangular tensor.

1. Introduction

Let R be the real field. An mth order n dimensional *square* tensor $\mathcal{B}$ consists of n^m entries in R, which is defined as follows:

$$\mathcal{B} = (B_{i_1 i_2 \cdots i_m}), \quad B_{i_1 i_2 \cdots i_m} \in R, \quad 1 \leq i_1, i_2, \ldots, i_m \leq n. \tag{1.1}$$

$\mathcal{B}$ is called nonnegative (or, respectively, positive) if $B_{i_1 i_2 \cdots i_m} \geq 0$ (or, respectively, $B_{i_1 i_2 \cdots i_m} > 0$). An mth order n dimensional square tensor $\mathcal{B}$ is called *reducible* if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$B_{i_1 i_2 \cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{B}$ is not reducible, then we call $\mathcal{B}$ *irreducible* [3,16].

* Corresponding author.
E-mail addresses: G.Zhou@curtin.edu.au (G. Zhou), caccetta@maths.curtin.edu.au (L. Caccetta), maqilq@polyu.edu.hk (L. Qi).

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Assume that p, q, m and n are positive integers, and $m, n \geq 2$. In this paper, we consider a nonnegative (p, q)th order $m \times n$ dimensional *rectangular* tensor

$$\mathcal{A} = (A_{i_1 \cdots i_p j_1 \cdots j_q}), \quad A_{i_1 \cdots i_p j_1 \cdots j_q} \in R, \ 1 \leq i_1, \ldots, i_p \leq m, \ 1 \leq j_1, \ldots, j_q \leq n. \tag{1.2}$$

Let $\mathcal{A}x^{p-1}y^q$ be a vector in R^m such that

$$\left(\mathcal{A}x^{p-1}y^q\right)_i = \sum_{i_2,\ldots,i_p=1}^{m} \sum_{j_1,\ldots,j_q=1}^{n} A_{ii_2\cdots i_p j_1\cdots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}, \quad i = 1, 2, \ldots, m.$$

Similarly, let $\mathcal{A}x^p y^{q-1}$ be a vector in R^n such that

$$\left(\mathcal{A}x^p y^{q-1}\right)_j = \sum_{i_1,\ldots,i_p=1}^{m} \sum_{j_2,\ldots,j_q=1}^{n} A_{i_1\cdots i_p j j_2\cdots j_q} x_{i_1} \cdots x_{i_p} y_{j_2} \cdots y_{j_q}, \quad j = 1, 2, \ldots, n.$$

Throughout this paper, we let $M = p + q$ and $N = m + n$. Consider

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[M-1]} \\ \mathcal{A}x^p y^{q-1} = \lambda y^{[M-1]}. \end{cases} \tag{1.3}$$

Here, $x^{[\alpha]} = [x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha]^T$. Let C be the set of all complex numbers. If $\lambda \in C$, $x \in C^m \setminus \{0\}$ and $y \in C^n \setminus \{0\}$ are solutions of (1.3), then we say that λ is a *singular value* of $\mathcal{A}$, x and y are a *left* and a *right eigenvectors* of $\mathcal{A}$, associated with the singular value λ.

A rectangular tensor $\mathcal{A}$ is called nonnegative (or positive) if $A_{i_1\cdots i_p j_1\cdots j_q} \geq 0$ (or $A_{i_1\cdots i_p j_1\cdots j_q} > 0$). For any $j = 1, 2, \ldots, n$, let $\mathcal{A}_{\bullet j} = (A_{i_1\cdots i_p j\cdots j})$ be a pth order m dimensional square tensor. For any $i = 1, 2, \ldots, m$, let $\mathcal{A}_{i\bullet} = (A_{i\cdots i j_1\cdots j_q})$ be a qth order n dimensional square tensor.

Definition 1.1 [5,16]**.** A nonnegative rectangular tensor $\mathcal{A}$ is called *irreducible* if all the square tensors $\mathcal{A}_{\bullet j}, j = 1, \ldots n$, and $\mathcal{A}_{i\bullet}, i = 1, \ldots m$, are irreducible.

For square tensors, the definition of eigenvalues has been recently introduced in [3,16,23]. Nice properties such as the Perron–Frobenius theorem for eigenvalues of nonnegative square tensors [3] have been established. The Perron–Frobenius Theorem for nonnegative tensors is related to measuring higher order connectivity in linked objects [17] and hyper-graphs [2,11]. Applications of eigenvalues of tensors include medical resonance imaging [1,28], higher-order Markov chains [19], positive definiteness of even-order multivariate forms in automatical control [20], and best-rank one approximation in data analysis [9,15,26,27], etc.

Recently, Ng et al. [19] proposed an iterative method for computing the largest eigenvalue of a nonnegative square tensor. This method is an extension of a method of Collatz [7,32,35] for calculating the spectral radius of an irreducible nonnegative matrix. In [21], Pearson introduced the notion of *essentially positive* tensors, and conjectured that the convergence of the Ng–Qi–Zhou method could be established for essentially positive tensors. In [22], Pearson established the convergence of the Ng–Qi–Zhou method for *primitive* nonnegative tensors. In [36], Zhang and Qi established linear convergence of the Ng–Qi–Zhou method for essentially positive tensors.

Real rectangular tensors arise from the strong ellipticity condition problem in solid mechanics [13,14,29,31,33] and the entanglement problem in quantum physics [8,10,30]. In [25], M-eigenvalues of such tensors are introduced. Algorithms for finding the largest M-eigenvalues are discussed in [12,18,34]. M-eigenvalues are parallel to Z-eigenvalues for square tensors [1,4,16,23,24,27]. Singular values of non-square tensors have been introduced in [16].

In [5,6,16], properties of singular values of non-square tensors have been discussed. In particular, the Perron–Frobenius theorem to singular values of non-square tensors was established in [16]. Chang et al. [5] established the Perron–Frobenius theorem to singular values of nonnegative rectangular tensors and proposed an iterative algorithm to find the largest singular value of a nonnegative rectangular tensor. However, they did not study the convergence of the proposed algorithm. In the next section, we propose a modified version of the algorithm given in [5] and show this modified algorithm is convergent for any irreducible nonnegative rectangular tensor.

2. Convergence of an iterative algorithm

In this section we propose an iterative algorithm to calculate the largest singular value of a nonnegative rectangular tensor. This algorithm is a modified version of the one given in [5], and we will show the convergence of the proposed algorithm for any irreducible nonnegative rectangular tensors. In this section, we always suppose that $\mathcal{A}$ is an irreducible nonnegative rectangular tensor of order (p, q) and dimension $m \times n$.

Let $P_n = \{x \in R^n : x_i \geq 0,\ 1 \leq i \leq n\}$ and $int(P_n) = \{x \in R^n : x_i > 0,\ 1 \leq i \leq n\}$. For any two vectors $x^1 \in R^n$ and $x^2 \in R^n$, $x^1 \geq x^2$ and $x^1 > x^2$ mean that $x^1 - x^2 \in P_n$ and $x^1 - x^2 \in int(P_n)$, respectively.

In the following, we state the Perron–Frobenius Theorem for nonnegative rectangular tensors proposed in [5,16] for reference. The Perron–Frobenius theorem to singular values of non-square tensors was first proposed in [16].

Theorem 2.1 [5,16]. *If $\mathcal{A}$ is an irreducible nonnegative rectangular tensor of order (p, q) and dimension $m \times n$, then there exist $\lambda_0 > 0$, $x_0 \in int(P_m)$ and $y_0 \in int(P_n)$ such that*

$$\begin{cases} \mathcal{A}x_0^{p-1}y_0^q = \lambda_0 x_0^{[M-1]} \\ \mathcal{A}x_0^p y_0^{q-1} = \lambda_0 y_0^{[M-1]}. \end{cases} \tag{2.4}$$

Moreover, if λ is a singular value with strongly positive left and right eigenvectors, then $\lambda = \lambda_0$. For all singular values λ of $\mathcal{A}$, $|\lambda| \leq \lambda_0$.

Clearly, from this result, λ_0 is the largest singular value of $\mathcal{A}$.

Theorem 2.2 [5]. *Assume that $\mathcal{A}$ is an irreducible nonnegative rectangular tensor of order (p, q) and dimension $m \times n$, then*

$$\begin{aligned} \lambda_0 &= \min_{(x,y)\in(P_m\setminus\{0\})\times(P_n\setminus\{0\})} \max_{i,j} \left(\frac{\left(\mathcal{A}x^{p-1}y^q\right)_i}{x_i^{M-1}}, \frac{\left(\mathcal{A}x^p y^{q-1}\right)_j}{y_j^{M-1}} \right) \\ &= \max_{(x,y)\in(P_m\setminus\{0\})\times(P_n\setminus\{0\})} \min_{i,j} \left(\frac{\left(\mathcal{A}x^{p-1}y^q\right)_i}{x_i^{M-1}}, \frac{\left(\mathcal{A}x^p y^{q-1}\right)_j}{y_j^{M-1}} \right), \end{aligned}$$

where λ_0 is the unique positive singular value corresponding to strongly positive left and right eigenvectors.

For a rectangular tensor $\mathcal{A}$, $\rho > 0$, $x \in P_m$ and $y \in P_n$, let

$$\mathcal{B}_x(x, y) = \mathcal{A}x^{p-1}y^q + \rho x^{[M-1]}, \tag{2.5}$$

$$\mathcal{B}_y(x, y) = \mathcal{A}x^p y^{q-1} + \rho y^{[M-1]}. \tag{2.6}$$

By Theorems 2.1 and 2.2, we have the following theorem.

Theorem 2.3. *If $\mathcal{A}$ is an irreducible nonnegative rectangular tensor of order (p, q) and dimension $m \times n$, then there exist $\mu_0 > 0$, $x_0 \in int(P_m)$ and $y_0 \in int(P_n)$ such that*

$$\begin{cases} \mathcal{B}_x(x_0, y_0) = \mu_0 x_0^{[M-1]} \\ \mathcal{B}_y(x_0, y_0) = \mu_0 y_0^{[M-1]}. \end{cases} \tag{2.7}$$

Moreover, μ_0 satisfies the following equalities:

$$\mu_0 = \min_{(x,y)\in(P_m\backslash\{0\})\times(P_n\backslash\{0\})} \max_{i,j} \left(\frac{\mathcal{B}_x(x,y)_i}{x_i^{M-1}}, \frac{\mathcal{B}_y(x,y)_j}{y_j^{M-1}} \right)$$

$$= \max_{(x,y)\in(P_m\backslash\{0\})\times(P_n\backslash\{0\})} \min_{i,j} \left(\frac{\mathcal{B}_x(x,y)_i}{x_i^{M-1}}, \frac{\mathcal{B}_y(x,y)_j}{y_j^{M-1}} \right),$$

and $\mu_0 - \rho$ is the largest singular value of $\mathcal{A}$.

By a direct computation, we obtain the following two lemmas.

Lemma 2.1. *For any $x, \bar{x} \in P_m$, $y, \bar{y} \in P_n$ and $t > 0$, we have the following results:*

(1) *If $x \geq \bar{x}$ and $y \geq \bar{y}$, then $\mathcal{B}_x(x,y) \geq \mathcal{B}_x(\bar{x},\bar{y})$ and $\mathcal{B}_y(x,y) \geq \mathcal{B}_y(\bar{x},\bar{y})$. Furthermore, if $x_i > \bar{x}_i$ for some $1 \leq i \leq m$, then $\mathcal{B}_x(x,y)_i > \mathcal{B}_x(\bar{x},\bar{y})_i$. Similarly, if $y_j > \bar{y}_j$ for some $1 \leq j \leq n$, then $\mathcal{B}_y(x,y)_j > \mathcal{B}_y(\bar{x},\bar{y})_j$.*
(2) *$\mathcal{B}_x(tx,ty) = t^{M-1}\mathcal{B}_x(x,y)$ and $\mathcal{B}_y(tx,ty) = t^{M-1}\mathcal{B}_y(x,y)$.*

Lemma 2.2. *For any $x \in int(P_m)$, $y \in int(P_n)$ and $\rho > 0$, $\mathcal{B}_x(x,y)$ and $\mathcal{B}_y(x,y)$ are strongly positive vectors.*

For any vectors $x \in P_m\backslash\{0\}$ and $y \in P_n\backslash\{0\}$, we define the following sequences $\{\mathcal{B}_x^{(k)}(x,y)\}$ and $\{\mathcal{B}_y^{(k)}(x,y)\}$:

$$\begin{aligned}
&\mathcal{B}_x^{(1)}(x,y) = \mathcal{B}_x(x,y), \quad \mathcal{B}_y^{(1)}(x,y) = \mathcal{B}_y(x,y),\\
&\qquad a^{(1)} = \left(\mathcal{B}_x^{(1)}(x,y)\right)^{\left[\frac{1}{M-1}\right]}, \quad b^{(1)} = \left(\mathcal{B}_y^{(1)}(x,y)\right)^{\left[\frac{1}{M-1}\right]},\\
&\mathcal{B}_x^{(2)}(x,y) = \mathcal{B}_x(a^{(1)},b^{(1)}), \quad \mathcal{B}_y^{(2)}(x,y) = \mathcal{B}_y(a^{(1)},b^{(1)}),\\
&\qquad \vdots\\
&\qquad a^{(k)} = \left(\mathcal{B}_x^{(k-1)}(x,y)\right)^{\left[\frac{1}{M-1}\right]}, \quad b^{(k)} = \left(\mathcal{B}_y^{(k-1)}(x,y)\right)^{\left[\frac{1}{M-1}\right]}, k \geq 1,\\
&\mathcal{B}_x^{(k+1)}(x,y) = \mathcal{B}_x\left(a^{(k)},b^{(k)}\right), \quad \mathcal{B}_y^{(k+1)}(x,y) = \mathcal{B}_y\left(a^{(k)},b^{(k)}\right), k \geq 1.
\end{aligned} \tag{2.8}$$

We have the following results for the sequences $\{\mathcal{B}_x^{(k)}(x,y)\}$ and $\{\mathcal{B}_y^{(k)}(x,y)\}$.

Theorem 2.4. *Suppose that $\mathcal{A}$ is an irreducible nonnegative (p,q)th order $m \times n$ dimensional rectangular tensor. Then there exists a positive integer s such that $\mathcal{B}_x^{(s)}(x,y) \in int(P_m)$ and $\mathcal{B}_y^{(s)}(x,y) \in int(P_n)$ for any $x \in P_m\backslash\{0\}$ and $y \in P_n\backslash\{0\}$.*

Proof. For any $x \in P_m\backslash\{0\}$ and $y \in P_n\backslash\{0\}$, let $I(x) = \{i : x_i > 0, i = 1, 2, \ldots, m\}$ and $J(y) = \{i : y_i > 0, i = 1, 2, \ldots, n\}$. For any integer $k \geq 1$, we let $\mathcal{B}_x^{(k)} = \mathcal{B}_x^{(k)}(x,y)$ and $\mathcal{B}_y^{(k)} = \mathcal{B}_y^{(k)}(x,y)$, where $\mathcal{B}_x^{(k)}(x,y)$ and $\mathcal{B}_y^{(k)}(x,y)$ are defined in (2.8). From (2.5) and (2.6), we obtain $I(x) \subseteq I(\mathcal{B}_x^{(1)})$, $J(y) \subseteq J(\mathcal{B}_y^{(1)})$, and for any positive integer $k \geq 2$, $I(\mathcal{B}_x^{(k-1)}) \subseteq I(\mathcal{B}_x^{(k)})$ and $J(\mathcal{B}_y^{(k-1)}) \subseteq J(\mathcal{B}_y^{(k)})$. Let

$$\lim_{k\to+\infty} I(\mathcal{B}_x^{(k)}) = \{i : \text{there exists } k_0 \text{ such that } i \in I(\mathcal{B}_x^{(k)}) \text{ for all } k \geq k_0, 1 \leq i \leq m\},$$

$$\lim_{k\to+\infty} J(\mathcal{B}_y^{(k)}) = \{j : \text{there exists } k_1 \text{ such that } j \in J(\mathcal{B}_y^{(k)}) \text{ for all } k \geq k_1, 1 \leq j \leq n\},$$

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$I = \lim_{k\to+\infty} I(\mathcal{B}_x^{(k)})$ and $J = \lim_{k\to+\infty} J(\mathcal{B}_y^{(k)})$. Clearly, for any sufficiently large k, $I = I(\mathcal{B}_x^{(k)})$ and $J = J(\mathcal{B}_y^{(k)})$. Suppose $I \neq \{1, 2, \ldots, m\}$. Then there exists a nonempty proper index subset $K \subset \{1, 2, \ldots, m\}$ such that $I \cup K = \{1, 2, \ldots, m\}$, and if $l \in K$ then $l \notin I(\mathcal{B}_x^{(k)}) = I$ for any sufficiently large k. Hence, by (2.5), for any $j \in J$, $l \in K$ and $i_2, \ldots, i_p \in I$, $A_{li_2\cdots i_p jj\cdots j}$ must be zero, which contradicts that $\mathcal{A}$ is irreducible. This implies that $I = \{1, 2, \ldots, m\}$. Similarly, we have $J = \{1, 2, \ldots, n\}$. Hence, there exists a positive integer s such that $\mathcal{B}_x^{(s)} \in int(P_m)$ and $\mathcal{B}_y^{(s)} \in int(P_n)$, which completes the proof. □

Theorem 2.5. *Let $\mathcal{A}$ be an irreducible nonnegative (p, q)th order $m \times n$ dimensional rectangular tensor. Suppose $x^1, x^2 \in P_m\backslash\{0\}$, $x^2 \geq x^1$, and $y^1, y^2 \in P_n\backslash\{0\}$, $y^2 \geq y^1$. If $x^1_{i_0} < x^2_{i_0}$ for some $1 \leq i_0 \leq m$, or $y^1_{j_0} < y^2_{j_0}$ for some $1 \leq j_0 \leq n$, then there exists a positive integer s such that $\mathcal{B}_x^{(s)}(x^1, y^1) < \mathcal{B}_x^{(s)}(x^2, y^2)$ and $\mathcal{B}_y^{(s)}(x^1, y^1) < \mathcal{B}_y^{(s)}(x^2, y^2)$.*

Proof. We assume $x^1_{i_0} < x^2_{i_0}$ for some $1 \leq i_0 \leq m$ and $y^2_{j_0} \geq y^1_{j_0} > 0$ for some $1 \leq j_0 \leq n$. Suppose for any integer $k \geq 1$, $\left(\mathcal{B}_x^{(k)}(x^1, y^1)\right)_i = \left(\mathcal{B}_x^{(k)}(x^2, y^2)\right)_i$ for some $1 \leq i \leq m$. Then, by (1) of Lemma 2.1, we have $i \neq i_0$. Since $\mathcal{A}$ is nonnegative, the term $(x^2_{i_0})^{p-1}(y^2_{j_0})^q$ must be missing from the ith coordinate of $\mathcal{B}_x^{(k)}(x^2, y^2)$. Let $e \in R^m$, $e_{i_0} = 1$ and zero elsewhere, and $f \in R^n$, $f_{j_0} = 1$ and zero elsewhere. Then $\left(\mathcal{B}_x^{(k)}(e, f)\right)_i = 0$ for any $k \geq 1$, which contradicts with Theorem 2.4. Hence, there exists a positive integer s_1 such that $0 < \mathcal{B}_x^{(k)}(x^1, y^1) < \mathcal{B}_x^{(k)}(x^2, y^2)$ for any $k \geq s_1$.

Suppose for any integer $k \geq 1$, $\left(\mathcal{B}_y^{(k)}(x^1, y^1)\right)_j = \left(\mathcal{B}^{(k)}(x^2, y^2)\right)_j$ for some $1 \leq j \leq n$. If $j = j_0$, then we must have $A_{i_1 i_2\cdots i_p j_0\cdots j_0} = 0$ for all $1 \leq i_1, i_2, \ldots, i_p \leq m$ because $0 < \mathcal{B}_x^{(k)}(x^1, y^1) < \mathcal{B}_x^{(k)}(x^2, y^2)$ for any $k \geq s_1$. This contradicts with that $\mathcal{A}$ is irreducible. Now we suppose $j \neq j_0$. Since $\mathcal{A}$ is nonnegative, the term $(x^2_{i_0})^p(y^2_{j_0})^{q-1}$ must be missing from the j-th coordinate of $\mathcal{B}^{(k)}(x^2, y^2)$. Then $\left(\mathcal{B}_y^{(k)}(e, f)\right)_j = 0$ for any $k \geq 1$, which contradicts with Theorem 2.4. Hence, there exists a positive integer s_2 such that $0 < \mathcal{B}_y^{(k)}(x^1, y^1) < \mathcal{B}_y^{(k)}(x^2, y^2)$ for any $k \geq s_2$. Let $s = \max\{s_1, s_2\}$. Then $\mathcal{B}_x^{(s)}(x^1, y^1) < \mathcal{B}_x^{(s)}(x^2, y^2)$ and $\mathcal{B}_y^{(s)}(x^1, y^1) < \mathcal{B}_y^{(s)}(x^2, y^2)$. Therefore, Theorem 2.5 holds. □

Now we state an iterative algorithm for calculating μ_0 in Theorem 2.3, which is a modified version of the algorithm proposed in [5].

Algorithm 2.1.
Step 0. Choose $\rho > 0$, $x^{(1)} > 0$, and $y^{(1)} > 0$. Set $k := 1$.
Step 1. Compute

$$\xi^{(k)} = \mathcal{B}_x(x^{(k)}, y^{(k)}), \tag{2.9}$$

$$\eta^{(k)} = \mathcal{B}_y(x^{(k)}, y^{(k)}). \tag{2.10}$$

Let

$$\underline{\mu}_k = \min_{x_i^{(k)}>0,\ y_j^{(k)}>0} \left\{ \frac{\xi_i^{(k)}}{\left(x_i^{(k)}\right)^{M-1}}, \quad \frac{\eta_j^{(k)}}{\left(y_j^{(k)}\right)^{M-1}} \right\}, \tag{2.11}$$

$$\bar{\mu}_k = \max_{x_i^{(k)}>0,\ y_j^{(k)}>0} \left\{ \frac{\xi_i^{(k)}}{\left(x_i^{(k)}\right)^{M-1}}, \quad \frac{\eta_j^{(k)}}{\left(y_j^{(k)}\right)^{M-1}} \right\}. \tag{2.12}$$

Step 2. If $\bar{\mu}_k = \underline{\mu}_k$, then stop. Otherwise, compute

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$$x^{(k+1)} = \frac{\left(\xi^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\left\| \left(\xi^{(k)},\ \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]} \right\|}, \tag{2.13}$$

$$y^{(k+1)} = \frac{\left(\eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\left\| \left(\xi^{(k)},\ \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]} \right\|}, \tag{2.14}$$

replace k by $k+1$ and go to Step 1.

In the following, we will give a convergence result for Algorithm 2.1. Note that Theorem 2.6 is a modification of the corresponding result in [5].

Lemma 2.3. *Suppose $\{x^{(k)}\}$, $\{y^{(k)}\}$, $\{\xi^{(k)}\}$ and $\{\eta^{(k)}\}$ are the sequences produced by Algorithm 2.1. Then,*

(1) *For any $k \geq 1$, $x^{(k)} > 0$, $y^{(k)} > 0$, $\xi^{(k)} > 0$, $\eta^{(k)} > 0$,*

$$\left(x^{(k+1)}\right)^{[M-1]} = \frac{\xi^{(k)}}{\left\| \left(\xi^{(k)},\ \eta^{(k)}\right) \right\|} \text{ and } \left(y^{(k+1)}\right)^{[M-1]} = \frac{\eta^{(k)}}{\left\| \left(\xi^{(k)},\ \eta^{(k)}\right) \right\|}.$$

(2) *For any positive integer s,*

$$\mathcal{B}_x^{(s)}(x^{(k)}, y^{(k)}) = \left\| \left(\xi^{(k)},\ \eta^{(k)}\right) \right\| \cdots \left\| \left(\xi^{(k+s-2)},\ \eta^{(k+s-2)}\right) \right\| \xi^{(k+s-1)},$$

$$\mathcal{B}_y^{(s)}(x^{(k)}, y^{(k)}) = \left\| \left(\xi^{(k)},\ \eta^{(k)}\right) \right\| \cdots \left\| \left(\xi^{(k+s-2)},\ \eta^{(k+s-2)}\right) \right\| \eta^{(k+s-1)},$$

$$\mathcal{B}_x^{(s)}(e^{(k)}, f^{(k)}) = \left\| \left(\xi^{(k)},\ \eta^{(k)}\right) \right\| \cdots \left\| \left(\xi^{(k+s-1)},\ \eta^{(k+s-1)}\right) \right\| \xi^{(k+s)},$$

$$\mathcal{B}_y^{(s)}(e^{(k)}, f^{(k)}) = \left\| \left(\xi^{(k)},\ \eta^{(k)}\right) \right\| \cdots \left\| \left(\xi^{(k+s-1)},\ \eta^{(k+s-1)}\right) \right\| \eta^{(k+s)},$$

where $e^{(k)} = \left(\xi^{(k)}\right)^{\left[\frac{1}{M-1}\right]}$, $f^{(k)} = \left(\eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}$, and $\mathcal{B}_x^{(s)}$ and $\mathcal{B}_y^{(s)}$ are defined in (2.8).

Proof. By (2.13), (2.14) and Lemma 2.2, the first statement holds. From (1) and (2.8), we have (2) holds. □

Theorem 2.6. *Assume that (μ_0, x_0, y_0) is a solution of (2.7). Then,*

$$\rho < \underline{\mu}_1 \leq \underline{\mu}_2 \leq \cdots \leq \mu_0 \leq \cdots \leq \bar{\mu}_2 \leq \bar{\mu}_1.$$

Proof. By (2.11), $\rho < \underline{\mu}_1$. From Theorem 2.3, for $k = 1, 2, \ldots$,

$$\underline{\mu}_k \leq \mu_0 \leq \bar{\mu}_k.$$

We now prove for any $k \geq 1$,

$$\underline{\mu}_k \leq \underline{\mu}_{k+1} \quad \text{and} \quad \bar{\mu}_{k+1} \leq \bar{\mu}_k.$$

For each $k = 1, 2, \ldots$, by the definition of $\underline{\mu}_k$ and Lemma 2.2, we have

$$\xi^{(k)} \geq \underline{\mu}_k \left(x^{(k)}\right)^{[M-1]} > 0, \quad \eta^{(k)} \geq \underline{\mu}_k \left(y^{(k)}\right)^{[M-1]} > 0.$$

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Then,

$$\left(\xi^{(k)}\right)^{\left[\frac{1}{M-1}\right]} \geq \left(\underline{\mu}_k\right)^{\frac{1}{M-1}} x^{(k)} > 0, \quad \left(\eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]} \geq \left(\underline{\mu}_k\right)^{\frac{1}{M-1}} y^{(k)} > 0.$$

So,

$$x^{(k+1)} = \frac{\left(\xi^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|} \geq \frac{\left(\underline{\mu}_k\right)^{\frac{1}{M-1}} x^{(k)}}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|} > 0,$$

$$y^{(k+1)} = \frac{\left(\eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|} \geq \frac{\left(\underline{\mu}_k\right)^{\frac{1}{M-1}} y^{(k)}}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|} > 0.$$

Hence, by Lemma 2.1, we get

$$\begin{aligned}\mathcal{B}_x\left(x^{(k+1)}, y^{(k+1)}\right) &\geq \frac{\underline{\mu}_k \mathcal{B}_x\left(x^{(k)}, y^{(k)}\right)}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|^{M-1}} \\ &= \frac{\underline{\mu}_k \xi^{(k)}}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|^{M-1}} \\ &= \underline{\mu}_k \left(x^{(k+1)}\right)^{[M-1]}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_y\left(x^{(k+1)}, y^{(k+1)}\right) &\geq \frac{\underline{\mu}_k \mathcal{B}_y\left(x^{(k)}, y^{(k)}\right)}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|^{M-1}} \\ &= \frac{\underline{\mu}_k \eta^{(k)}}{\left\|\left(\xi^{(k)}, \eta^{(k)}\right)^{\left[\frac{1}{M-1}\right]}\right\|^{M-1}} \\ &= \underline{\mu}_k \left(y^{(k+1)}\right)^{[M-1]},\end{aligned}$$

which means for each $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$,

$$\underline{\mu}_k \leq \frac{\left(\mathcal{B}_x\left(x^{(k+1)}, y^{(k+1)}\right)\right)_i}{\left(x_i^{(k+1)}\right)^{M-1}}, \quad \underline{\mu}_k \leq \frac{\left(\mathcal{B}_y\left(x^{(k+1)}, y^{(k+1)}\right)\right)_j}{\left(y_j^{(k+1)}\right)^{M-1}}.$$

Therefore, we obtain

$$\underline{\mu}_k \leq \underline{\mu}_{k+1}.$$

Similarly, we can prove that

$$\bar{\mu}_{k+1} \leq \bar{\mu}_k.$$

This completes our proof. □

From Theorem 2.6, $\{\underline{\mu}_k\}$ is a monotonic increasing sequence and it has an upper bound, so the limit exists. Since $\{\bar{\mu}_k\}$ is monotonic decreasing sequence and it has a lower bound, the limit exists as well. We suppose

$$\underline{\mu} = \lim_{k\to\infty} \underline{\mu}_k, \quad \text{and} \quad \bar{\mu} = \lim_{k\to\infty} \bar{\mu}_k.$$

By Theorem 2.6, we have

$$\rho < \underline{\mu} \leq \mu_0 \leq \bar{\mu}. \tag{2.15}$$

Theorem 2.7. *Let $\{x^{(k)}\}$, $\{y^{(k)}\}$, $\{\xi^{(k)}\}$ and $\{\eta^{(k)}\}$ be the sequences produced by Algorithm 2.1. Then,*

(a) $\{x^{(k)}\}$ *and* $\{y^{(k)}\}$ *have convergent subsequences which converge to* x^* *and* y^**, respectively. Moreover,* $x^* \in P_m\backslash\{0\}$ *and* $y^* \in P_n\backslash\{0\}$.
(b) $\mathcal{B}_x(x^*, y^*) \geq \underline{\mu}\,(x^*)^{[M-1]}$ *and* $\mathcal{B}_y(x^*, y^*) \geq \underline{\mu}\,(y^*)^{[M-1]}$.
(c) $\underline{\mu} = \bar{\mu}$.

Proof. As the sequences $\{x^{(k)}\}$ and $\{y^{(k)}\}$ are bounded, $\{x^{(k)}\}$ and $\{y^{(k)}\}$ have convergent subsequences, respectively. Without loss of generality, we suppose $x^* = \lim_{j\to\infty} x^{(k_j)}$ and $y^* = \lim_{j\to\infty} y^{(k_j)}$, where $\{x^{(k_j)}\}$ and $\{y^{(k_j)}\}$ are subsequences of $\{x^{(k)}\}$ and $\{y^{(k)}\}$, respectively. Since $x^{(k)} > 0$ and $y^{(k)} > 0$ for all $k \geq 1$, we have $x^* \geq 0$ and $y^* \geq 0$. As $\|(x^{(k)}, y^{(k)})\| = 1$ for all $k \geq 2$, (x^*, y^*) must not be a zero vector. We suppose $x^*_{i_0} \neq 0$ for some $1 \leq i_0 \leq m$. Then, $y^* \neq 0$. Otherwise, by continuity of $\mathcal{B}_x(x, y)$, we have $\xi^{(k_j)}_{i_0} = \mathcal{B}_x(x^{(k_j)}, y^{(k_j)})_{i_0} \to \rho(x^*_{i_0})^{M-1}$ as $k_j \to \infty$. By (2.11), $\underline{\mu}_{k_j} \leq \frac{\xi^{(k_j)}_{i_0}}{\left(x^{(k_j)}_{i_0}\right)^{M-1}} \to \rho$ as $j \to \infty$. Hence, $\underline{\mu} \leq \rho$, which contradicts with (2.15). Therefore, we obtain $x^* \neq 0$ and $y^* \neq 0$.

For the second statement, by continuity of $\mathcal{B}_x(x, y)$ and $\mathcal{B}_y(x, y)$, (2.11) and (2.12), the statement follows.

Now we prove (c). Suppose $\underline{\mu} < \bar{\mu}$. Then, by (b), (2.11) and (2.12), we have $\mathcal{B}_x(x^*, y^*) \neq \underline{\mu}\,(x^*)^{[M-1]}$ or $\mathcal{B}_y(x^*, y^*) \neq \underline{\mu}\,(y^*)^{[M-1]}$. Let $B^*_x = (\mathcal{B}_x(x^*, y^*))^{\left[\frac{1}{M-1}\right]}$ and $B^*_y = (\mathcal{B}_y(x^*, y^*))^{\left[\frac{1}{M-1}\right]}$. By Theorem 2.5, there exists a positive integer s such that $\underline{\mu}\mathcal{B}^{(s)}_x(x^*, y^*) < \mathcal{B}^{(s)}_x(B^*_x, B^*_y)$ and $\underline{\mu}\mathcal{B}^{(s)}_y(x^*, y^*) < \mathcal{B}^{(s)}_y(B^*_x, B^*_y)$. By (a) and the continuity of $\mathcal{B}_x(x, y)$ and $\mathcal{B}_y(x, y)$, for any sufficiently large k_j, we obtain

$$\underline{\mu}\mathcal{B}^{(s)}_x\left(x^{(k_j)}, y^{(k_j)}\right) < \mathcal{B}^{(s)}_x\left(B^{(k_j)}_x, B^{(k_j)}_y\right), \underline{\mu}\mathcal{B}^{(s)}_y\left(x^{(k_j)}, y^{(k_j)}\right) < \mathcal{B}^{(s)}_y\left(B^{(k_j)}_x, B^{(k_j)}_y\right), \tag{2.16}$$

where $B^{(k_j)}_x = \left(\mathcal{B}_x(x^{(k_j)}, y^{(k_j)})\right)^{\left[\frac{1}{M-1}\right]}$ and $B^{(k_j)}_y = \left(\mathcal{B}_y(x^{(k_j)}, y^{(k_j)})\right)^{\left[\frac{1}{M-1}\right]}$. It follows from (2.9) and (2.10) that we have $B^{(k_j)}_x = \left(\xi^{(k_j)}\right)^{\left[\frac{1}{M-1}\right]}$ and $B^{(k_j)}_y = \left(\eta^{(k_j)}\right)^{\left[\frac{1}{M-1}\right]}$. By Lemma 2.3 and (2.16), we have

$$\underline{\mu}\left(\xi^{(k_j+s-1)}, \eta^{(k_j+s-1)}\right) < \|\left(\xi^{(k_j+s-1)}, \eta^{(k_j+s-1)}\right)\|\left(\xi^{(k_j+s)}, \eta^{(k_j+s)}\right). \tag{2.17}$$

By Lemma 2.3, (2.11) and (2.17), we obtain $\underline{\mu}_{k_j+s} > \underline{\mu}$, which contradicts with Theorem 2.6. So (c) holds. □

By Theorem 2.7, we have the following convergence result.

Theorem 2.8. *Suppose that a nonnegative (p, q)th order $m \times n$ dimensional rectangular tensor $\mathcal{A}$ is irreducible. Assume that (μ_0, x_0, y_0) is a solution of (2.7). Then, Algorithm 2.1 produces the value of μ_0 in* *a finite number of steps, or generates two convergent sequences* $\{\underline{\mu}_k\}$ *and* $\{\bar{\mu}_k\}$*, both of which converge to* μ_0*. Furthermore,* $\mu_0 - \rho$ *is the largest singular value of* $\mathcal{A}$*.*

Remark 1. In the following example, we will show the sequence generated by the algorithm in [5] may not converge for some nonnegative rectangular tensors, but we can obtain the largest singular value by using our proposed algorithm in this paper. Consider the $(1, 1)$-th order 2×2 dimensional rectangular tensor $\mathcal{A}$ given by $A_{12} = 1, A_{21} = 5$ and zero elsewhere. We choose $x^{(1)} = (1, 1)^T$ and $y^{(1)} = (1, 1)^T$. By the algorithm in [5], we cannot obtain the largest singular value of $\mathcal{A}$ after 1000 iterations. Let $\rho = 1$. By Algorithm 2.1, we obtain the largest singular value of $\mathcal{A}$ is 2.24 after 20 iterations.

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