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Journal of the Operations Research Society of China

ISSN 2194-668X

J. Oper. Res. Soc. China
DOI 10.1007/s40305-016-0141-3

Springer

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The Adjacency and Signless Laplacian Spectra of Cored Hypergraphs and Power Hypergraphs

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Received: 25 May 2016 / Revised: 26 August 2016 / Accepted: 31 October 2016

Abstract In this paper, we study the adjacency and signless Laplacian tensors of cored hypergraphs and power hypergraphs. We investigate the properties of their adjacency and signless Laplacian H-eigenvalues. Especially, we find out the largest H-eigenvalues of adjacency and signless Laplacian tensors for uniform squids. We also compute the H-spectra of sunflowers and some numerical results are reported for the H-spectra.

Keywords H-eigenvalue · Hypergraph · Adjacency tensor · Signless Laplacian tensor · Sunflower · Squid

Mathematics Subject Classification 74B99 · 15A18 · 15A69

This work was supported by the National Natural Science Foundation of China (No. 11271221) and the Specialized Research Fund for State Key Laboratories.

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Published online: 29 November 2016

1 Introduction

In recent years, the study of spectral hypergraph theory via tensors [1–8] has attracted extensive attention and interest since the work of [1,4,8,9].

As was in [9], a real tensor $\mathcal{T} = (t_{i_1,\cdots,i_k})$ of order k and dimension n refers to a multidimensional array (also called hypermatrix) with entries $t_{i_1,\cdots,i_k}$ such that $t_{i_1,\cdots,i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1, \cdots, n\}$ and $j \in [k]$. Given a vector $x \in \mathbb{R}^n$, $\mathcal{T}x^{k-1}$ is defined as an n-dimensional vector such that its ith element is $\sum_{i_2,\cdots,i_k \in [n]} t_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k}$ for $i \in [n]$. Let $\mathcal{I}$ be the identity tensor of appropriate dimension, e.g., $i_{i_1 \cdots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise when the dimension is n. The following definition was introduced by Qi [9].

Definition 1.1 Let $\mathcal{T}$ be a k-th order n-dimensional real tensor. For some $\lambda \in \mathbb{R}$, if polynomial system $(\lambda\mathcal{I} - \mathcal{T})x^{k-1} = 0$ has a solution $x \in \mathbb{R}^n \backslash \{0\}$, then λ is called an H-eigenvalue and x an H-eigenvector.

Obviously, H-eigenvalues are real number. By [9,10], we have the number of H-eigenvalues of a real tensor as finite. By [8], we have that all the tensors considered in this paper have at least one H-eigenvalue. Hence, we can denote $\lambda(\mathcal{T})$ as the largest H-eigenvalue of a real tensor $\mathcal{T}$.

As was in [8], a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V, which is labeled as $[n]$, and edge set E. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. Throughout this paper, $k \geqslant 3$ and $n \geqslant k$. Moreover, since the trivial hypergraph (i.e., $E = \emptyset$) is of less interest, we consider only hypergraphs to have at least one edge (i.e., nontrival) in this paper. The following definition was introduced by Qi [8].

Definition 1.2 Let $G = (V, E)$ be a k-uniform hypergraph. The adjacency tensor of G is defined as the k-th order n-dimensional tensor $\mathcal{A}$ whose $(i_1, \cdots, i_k)$-entry is

$$a_{i_1,\cdots,i_k} := \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, \cdots, i_k\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}$ be a k-th order n-dimensional diagonal tensor with its diagonal element $d_{i,\cdots,i}$ being d_i, the degree of vertex i, for all $i \in [n]$. Then $\mathcal{L} := \mathcal{D} - \mathcal{A}$ is the Laplacian tensor of the hypergraph G, and $\mathcal{Q} := \mathcal{D} + \mathcal{A}$ is the signless Laplacian tensor of the hypergraph G.

Zero is always the smallest H-eigenvalue of $\mathcal{L}$ [8] and we have $d \leqslant \lambda(\mathcal{L}) \leqslant \lambda(\mathcal{Q}) \leqslant 2d$, where d is the maximum degree of G. By [1, Theorem 3.8], we have $\bar{d} \leqslant \lambda(A) \leqslant d$, where $\bar{d}$ is the average degree of G.

Recently, Hu et al. [2] introduced the class of cored hypergraphs and power hypergraphs, and investigated the properties of their Laplacian H-eigenvalues. Power hypergraphs are cored hypergraphs, but not vice versa. Sunflowers are power hypergraphs, while squids are cored hypergraphs, but not power hypergraphs in general. They showed that the largest Laplacian H-eigenvalue of an even-uniform cored hypergraph is equal to its largest signless Laplacian H-eigenvalue and also computed the

Laplacian H-spectra of sunflowers. Indeed, the results of [1,2,4,5,7,8] raised the interests to study the Laplacian H-eigenvalues of a k-uniform hypergraph. Actually, there are still some changing problems which are worthy of being investigated. First, can we describe the properties of adjacency and signless Laplacian H-eigenvalues of cored hypergraphs and power hypergraphs in the similar way of [2]? Second, can we calculate all adjacency and signless Laplacian H-eigenvalues for some special k-uniform hypergraphs, such as sunflowers and squids? Motivated by these questions, we study adjacency and signless Laplacian H-eigenvalues of the class of cored hypergraphs and power hypergraphs in this paper.

Using the similar methods as in [2], we first investigate the properties of H-eigenvectors of adjacency tensor and signless Laplacian tensors for cored hypergraphs. Especially, we show that the largest adjacency H-eigenvalue of k-uniform squid is in the interval $(1, 2)$ and we can find it out. We also show that its largest signless Laplacian H-eigenvalue is in the interval $(2, 4)$. By [2, Proposition 3.2], it is clear that the result in [2, Proposition 3.4] is a corollary of the above conclusion. We next investigate the H-spectra of adjacency and signless Laplacian tensors of power hypergraphs. Especially, we compute all H-spectra of adjacency and signless Laplacian tensors for sunflowers.

The rest of this paper is organized as follows. We list some known results of cored hypergraphs and power hypergraphs in the next section. In Sect. 3, we discuss some properties of adjacency and signless Laplacian H-eigenvalues of cored hypergraphs. Especially, we compute the largest adjacency H-eigenvalues and the largest signless Laplacian H-eigenvalues of squids. In Sect. 4, we investigate some properties of adjacency and signless Laplacian H-eigenvalues of odd-uniform power hypergraphs and even-uniform power hypergraphs. We especially investigate the H-spectra of sunflowers. In Sect. 5, we compute all adjacency and signless Laplacian H-eigenvalues of sunflowers. Moreover, some numerical results are reported to verify our conclusion.

2 Preliminaries

In this section, we list some essential notions of uniform hypergraphs which will be used in the sequel. Please refer to [8,11–14] for comprehensive references. In this paper, unless stated otherwise, a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V and edge set E. For a subset $S \subset [n]$, we denote by E_S the set of edges $\{e \in E | S \cap e \neq \varnothing\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. It is the set of edges containing the vertex i, i.e., $E_i := \{e \in E | i \in e\}$. The cardinality $|E_i|$ of the set E_i is defined as the degree of the vertex i, which is denoted by d_i. Two different vertices i and j are connected to each other (or the pair i and j is connected), if there is a sequence of edges $(e_1, \cdots, e_m)$ such that $i \in e_1$, $j \in e_m$, and $e_r \cap e_{r+1} \neq \varnothing$ for all $r \in [m-1]$. A hypergraph is called connected, if every pair of different vertices of G is connected. In the sequel, unless stated otherwise, all the notations introduced above are reserved for the specific meanings. For the sake of simplicity, we mainly consider connected hypergraphs in the subsequent analysis. By the techniques in [8,14], the conclusion on connected hypergraphs can be easily generalized to general hypergraphs.

In the following, we recall the definitions of cored hypergraphs and power hypergraphs introduced in [2]. We also list the definitions of sunflowers and squids respectively introduced in [2, 15].

Definition 2.1 Let $G = (V, E)$ be a k-uniform hypergraph. If for every edge $e \in E$, there is a vertex $i_e \in e$ such that the degree of the vertex i_e is one, then G is a cored hypergraph. A vertex with degree one is a cored vertex, and a vertex with degree larger than one is an intersectional vertex.

Definition 2.2 Let $G = (V, E)$ be a k-uniform nontrivial hypergraph. If there is a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \cdots \cup V_d$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_d| = k - 1$, and $E = \{V_0 \cup V_i | i \in [d]\}$, then G is called a sunflower. The degree d of the vertex in V_0, which is called the heart, is the size of the sunflower. The edges of G are leaves, and the vertices other than the heart are vertices of leaves.

Definition 2.3 Let $G = (V, E)$ be a 2-uniform graph. For any $k \geqslant 3$, the kth power of G, $G^k := (V^k, E^k)$ is defined as the k-uniform hypergraph with the set of edges $E^k := \{e \cup \{i_{e,1}, \cdots, i_{e,k-2}\} \mid e \in E\}$, and the set of vertices $V^k := V \cup \{i_{e,1}, \cdots, i_{e,k-2}, e \in E\}$.

Definition 2.4 Let $G = (V, E)$ be a k-uniform hypergraph. If we can number the vertex set V as $V := \{i_{1,1}, \cdots, i_{1,k}, \cdots, i_{k-1,1}, \cdots, i_{k-1,k}, i_k\}$ such that the set of edges $E = \{\{i_{1,1}, \cdots, i_{1,k}\}, \cdots, \{i_{k-1,1}, \cdots, i_{k-1,k}\}, \{i_{1,1}, \cdots, i_{k-1,1}, i_k\}\}$, then G is a squid.

It is easy to see that the class of power hypergraphs is a subclass of cored hypergraphs and not all cored hypergraphs are power hypergraphs. It can be seen that the class of sunflowers is a subclass of power hypergraphs. It is also easy to see that the class of squids is a subclass of cored hypergraphs but is not contained in the class of power hypergraphs. Recently, the properties of their Laplacian H-spectra were investigated by Hu et al. [2]. For completeness, we list here some results given in [2], which will be used in the sequel.

The following result was given in [2, Proposition 3.4].

Proposition 2.5 *Let k be even and $G = (V, E)$ be a k-uniform squid. Let $\mathcal{L}$ be the Laplacian tensor of G. Then $\lambda(\mathcal{L}) = \lambda(\mathcal{Q})$ is the unique root of $(\mu - 2) - (\frac{1}{\mu-1})^{\frac{1}{k-1}} - (\frac{1}{\mu-1})^{k-1} = 0$ in the interval $(2, 4)$.*

The following proposition was proposed in [2, Proposition 3.5], but its proof is not completed. Here, we complete the proof.

Proposition 2.6 *Let k be odd and $G = (V, E)$ be a k-uniform squid. Let $\mathcal{L}$ be the Laplacian tensor of G. Then $\lambda(\mathcal{L}) = 2$.*

Proof Suppose that $V := \{i_{1,1}, \cdots, i_{1,k}, \cdots, i_{k-1,1}, \cdots, i_{k-1,k}, i_k\}$ such that the set of edges is $E = \{\{i_{1,1}, \cdots, i_{1,k}\}, \cdots, \{i_{k-1,1}, \cdots, i_{k-1,k}\}, \{i_{1,1}, \cdots, i_{k-1,1}, i_k\}\}$. Let $\omega \in \mathbb{R}^n$ be an H-eigenvector of $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$. Then we have

$$(\lambda(\mathcal{L}) - 1)\, \omega_{i_{j,s}}^{k-1} = -\omega_{i_j,1} \prod_{t \in \{2,\cdots,k\} \setminus \{s\}} \omega_{i_j,t}, \quad \forall j \in [k-1], s \in \{2, \cdots, k\}.$$

Thus,

$$(\lambda(\mathcal{L})-1)\omega_{i_j,s}^{k}=-\omega_{i_j,1}\prod_{t\in\{2,\cdots,k\}}\omega_{i_j,t}.$$

Hence, $\omega_{i_j,2}=\cdots=\omega_{i_j,k}=: z_j$ for all $j\in[k-1]$, since $\lambda(\mathcal{L})\geqslant 2$.

Let $x:=\omega_{i_k}$ and $y_j:=\omega_{i_j,1}$ for all $j\in[k-1]$.
(1) If $z_j\neq 0$ for all $j\in[k-1]$, then we have $y_j=(1-\lambda(\mathcal{L}))z_j$. Moreover,

$$(\lambda(\mathcal{L})-2){y_j}^{k-1}=-x\prod_{s\in[k-1]\backslash\{j\}}y_s-{z_j}^{k-1},\quad \forall j\in[k-1].$$

Consequently,

$$(\lambda(\mathcal{L})-2)1-\lambda(\mathcal{L})^{k-1}{z_j}^{k-1}+{z_j}^{k-1}=-x\prod_{s\in[k-1]\backslash\{j\}}y_s,\quad \forall j\in[k-1].$$

Then,

$$[(\lambda(\mathcal{L})-2)1-\lambda(\mathcal{L})^{k-1}+1]{z_j}^{k}=-z_jx\prod_{s\in[k-1]\backslash\{j\}}y_s=-\frac{1}{1-\lambda(\mathcal{L})}x\prod_{s\in[k-1]}y_s.$$

So,

$$[(\lambda(\mathcal{L})-2)(1-\lambda(\mathcal{L}))^{k}+(1-\lambda(\mathcal{L}))]{z_j}^{k}=-x\prod_{s\in[k-1]}y_s,\quad \forall j\in[k-1].$$

Now, it follows from the fact $\lambda(\mathcal{L})\geqslant 2$ that $(\lambda(\mathcal{L})-2)(1-\lambda(\mathcal{L}))^k+(1-\lambda(\mathcal{L}))<0$. So $z_1=z_2=\cdots=z_{k-1}=: z$. If $x\prod_{s\in[k-1]}y_s=0$, then $z_1=z_2=\cdots=z_{k-1}=0$, which is a contradiction. So $z_1=z_2=\cdots=z_{k-1}\neq 0$. And $y_1=y_2=\cdots=y_{k-1}=: y\neq 0$. By Definition 1.1, we have $0\leqslant(\lambda(\mathcal{L})-1)x^{k-1}=-y^{k-1}\leqslant 0$, then $x=y=0$. Thus, this situation can never happen.

(2) If $z_1=\cdots=z_p=0$ and $z_{p+1}\neq 0,\cdots,z_{k-1}\neq 0$, $1\leqslant p\leqslant k-2$

$$(\lambda(\mathcal{L})-1){z_j}^{k-1}=-y_j{z_j}^{k-2},\quad \forall j\in[k-1]\backslash[p]. \tag{2.1}$$

$$(\lambda(\mathcal{L})-2){y_j}^{k-1}=-x\prod_{s\in[k-1]\backslash\{j\}}y_s,\quad \forall j\in[p]. \tag{2.2}$$

$$(\lambda(\mathcal{L})-2){y_j}^{k-1}=-x\prod_{s\in[k-1]\backslash\{j\}}y_s-{z_j}^{k-1},\quad \forall j\in[k-1]\backslash[p]. \tag{2.3}$$

$$(\lambda(\mathcal{L})-1)x^{k-1}=-\prod_{s\in[k-1]}y_s. \tag{2.4}$$

Case 1 If $x=0$ and $\lambda>2$, by (2.2) we know $y_j=0, \forall j\in[p]$, and by (2.3), we have

$$(\lambda(\mathcal{L})-2){y_j}^{k-1}=-{z_j}^{k-1},\quad \forall j\in[k-1]\backslash[p].$$

Here $(\lambda(\mathcal{L})-2){y_j}^{k-1} \geqslant 0$ and $-{z_j}^{k-1} \leqslant 0$. Thus, this situation can never happen.
Case 2 If $x \neq 0$, p is even and $\lambda > 2$, then we have $y_1 = \cdots = y_p =: y_{\star 1}$ by (2.2). By (2.3), we have

$$[(\lambda(\mathcal{L})-2)(1-\lambda(\mathcal{L}))^k + (1-\lambda(\mathcal{L}))]{z_j}^k = -x \prod_{s\in[k-1]} y_s, \quad \forall j \in [k-1]\backslash[p].$$

So, $z_{p+1} = \cdots = z_{k-1}$ and $y_{p+1} = \cdots = y_{k-1} =: y_{\star 2}$. By (2.4), we have

$$(\lambda(\mathcal{L})-1)x^{k-1} = -{y_{\star 1}}^p {y_{\star 2}}^{k-1-p}.$$

Since p is even, ${y_{\star 1}}^p \leqslant 0$ and ${y_{\star 2}}^{k-1-p} < 0$. This implies $-{y_{\star 1}}^p {y_{\star 2}}^{k-1-p} \leqslant 0$, which is a construction with the fact $(\lambda(\mathcal{L})-1)x^{k-1} > 0$.
Case 3 If $x \neq 0$, p is odd and $\lambda > 2$, then by (2.2) and (2.4), we have

$$(\lambda(\mathcal{L})-2){y_j}^k = (\lambda(L)-1)x^k, \quad \forall j \in [p].$$

Thus, x and $y_j, \quad \forall j \in [p]$ are the same sign. By (2.3), we have $(\lambda(\mathcal{L})-2){y_j}^{k-1} > 0$ and $-{z_j}^{k-1} < 0$. So,

$$x\Pi_{s\in[k-1]\backslash\{j\}} y_s = x{y_1}^p {y_{k-1}}^{k-p-2} < 0.$$

Because p is odd, $k-p-2$ is even and ${y_{k-1}}^{k-p-2} > 0$, x and y_1 are not the same sign. This is a contradiction. Consequently, $\lambda(\mathcal{L}) = 2$.

In the sequel, we investigate the adjacency tensor and signless Laplacian tensor of sunflower and squid in the similar way of [2].

3 Spectral Structures of Cored Hypergraphs and Squids

Some facts about the H-eigenvalues and H-eigenvectors of the adjacency tensors and signless Laplacian tensors of cored hypergraphs are established in this section. Especially, we calculate the largest adjacency H-eigenvalue and the largest signless Laplacian H-eigenvalue of a k-uniform squid.

The following lemma shows some structure exhibited by the H-eigenvectors of cored hypergraphs.

Lemma 3.1 *Let $G = (V, E)$ be a k-uniform cored hypergraph and $x \in \mathbb{R}^n$ be an H-eigenvector of its adjacency tensor $\mathcal{A}$ corresponding to an H-eigenvalue $\lambda \neq 0$. If there are two core vertices i, j in an edge $e \in E$, then $|x_i| = |x_j|$. Moreover, $x_i = x_j$ when k is an odd number.*

Proof By the definition of H-eigenvalues and the fact that i and j are core vertices, we have

$$\lambda x_i^{k-1} = (\mathcal{A}x^{k-1})_i = x_j \prod_{s\in e\backslash\{i,j\}} x_s, \quad \lambda x_j^{k-1} = (\mathcal{A}x^{k-1})_j = x_i \prod_{s\in e\backslash\{i,j\}} x_s.$$

Hence,

$$\lambda x_i^k = \lambda x_j^k.$$

Since $\lambda \neq 0$, we have that $|x_i| = |x_j|$. Moreover, when k is odd, we see that $x_i = x_j$.

Using the similar proof of the above lemma, we have the following lemma.

Lemma 3.2 *Let $G = (V, E)$ be a k-uniform cored hypergraph and $x \in \mathbb{R}^n$ be an H-eigenvector of its signless Laplacian tensor $\mathcal{Q}$ corresponding to an H-eigenvalue $\lambda \neq 1$. If there are two cored vertices i, j in an edge $e \in E$, then $|x_i| = |x_j|$. Moreover, $x_i = x_j$ when k is an odd number.*

By the above lemmas, we calculate the largest H-eigenvalues of the signless Laplacian tensor and the adjacency tensor of a k-uniform squid.

Proposition 3.3 *Let $G = (V, E)$ be a k-uniform squid. Let $\mathcal{A}$ be the adjacency tensor of G. Then $\lambda(\mathcal{A})$ is the unique root of $\mu - (\frac{1}{\mu})^{\frac{1}{k-1}} - (\frac{1}{\mu})^{k-1} = 0$ in the interval* $(1, 2)$.

Proof Suppose that $V := \{i_{1,1}, \cdots, i_{1,k}, \cdots, i_{k-1,1}, \cdots, i_{k-1,k}, i_k\}$ such that the set of edges is $E = \{\{i_{1,1}, \cdots, i_{1,k}\}, \cdots, \{i_{k-1,1}, \cdots, i_{k-1,k}\}, \{i_{1,1}, \cdots, i_{k-1,1}, i_k\}\}$.

By [7, Lemma 3.1], [16, Theorem 4], and [17, Theorem 3.20], if we can find a positive H-eigenvector $x \in \mathbb{R}^n$ of $\mathcal{A}$ corresponding to an H-eigenvalue μ, then $\mu = \lambda(\mathcal{A})$.

Let $x_{i_k} = \alpha$, $x_{i_{j,1}} = 1$, and $x_{i_{j,2}} = \cdots = x_{i_{j,k}} = \gamma > 0$ for all $j \in [k-1]$. Suppose that x is an H-eigenvector of $\mathcal{A}$ corresponding to the H-eigenvalue $\mu = \lambda(\mathcal{A})$. By Definition 1.1, we have

$$\mu\alpha^{k-1} = 1, \quad \mu = \alpha + \gamma^{k-1}, \quad \text{and} \quad \mu\gamma^{k-1} = \gamma^{k-2}.$$

By [1, Theorem 3.8], we have $\mu \geqslant 1$. Thus, the first and the third equalities imply that $\alpha^{k-1} = \gamma$. Hence,

$$\mu = \left(\frac{1}{\mu}\right)^{\frac{1}{k-1}} + \left(\frac{1}{\mu}\right)^{k-1}.$$

Let $f(\mu) = \mu - (\frac{1}{\mu})^{\frac{1}{k-1}} - (\frac{1}{\mu})^{k-1}$. We have $f(1) = -1 < 0$ and $f(2) = 2 - (\frac{1}{2})^{\frac{1}{k-1}} - \frac{1}{2^{(k-1)}} > 0$. So, $f(\mu) = 0$ does have a root in the interval $(1, 2)$. Since $\mathcal{A}$ has a unique positive H-eigenvector, the equation $\mu - (\frac{1}{\mu})^{\frac{1}{k-1}} - (\frac{1}{\mu})^{k-1} = 0$ has a unique positive solution which is in interval $(1, 2)$. Hence, the result follows.

Similarly, we can show the following proposition.

Proposition 3.4 *Let $G = (V, E)$ be a k-uniform squid. Let $\mathcal{Q}$ be the signless Laplacian tensor of G. Then $\lambda(\mathcal{Q})$ is the unique root of $(\mu - 2) - (\frac{1}{\mu-1})^{\frac{1}{k-1}} - (\frac{1}{\mu-1})^{k-1} = 0$ in the interval* $(2, 4)$.

Clearly, by Hu et al. [2, Proposition 3.2] and the above proposition, Proposition 2.5, i.e., [2, Proposition 3.4] is a direct corollary of Proposition 3.4.

4 Spectral Structures of Power Hypergraphs and Sunflowers

Some facts about the H-eigenvalues and H-eigenvectors of the adjacency tensors and signless Laplacian tensors of power hypergraphs are given in this section. Note that the Laplacian H-spectra of even-uniform power hypergraphs are not given in [2]. Here, we establish detailed H-spectra of such class. Moreover, we investigate the largest adjacency H-eigenvalue and the largest signless Laplacian H-eigenvalues of a k-uniform sunflower.

The following two lemmas are given for odd-uniform power hypergraphs.

Lemma 4.1 *Let k be odd and $G = (V, E)$ be a k-uniform power hypergraph and $x \in \mathbb{R}^n$ be an H-eigenvector of its adjacency tensor $\mathcal{A}$ corresponding to an H-eigenvalue $\lambda \neq 0$. Let $e \in E$ be an arbitrary but fixed edge.*

- (i) *If e has only one intersectional vertex i, and $x_s \neq 0$ for some cored vertex $s \in e$, then $\lambda x_s = x_i$.*
- (ii) *If e has two intersectional vertices i and j, and $x_s \neq 0$ for some cored vertex $s \in e$, then $\lambda {x_s}^2 = x_i x_j$.*

Proof For (i), by Definition 1.1 and Lemma 3.1, we have

$$\lambda x_s^{k-1} = x_s^{k-2} x_i.$$

Thus, $x_i = \lambda x_s$. For (ii), by Definition 1.1 and Lemma 3.1, we have

$$\lambda x_s^{k-1} = x_s^{k-3} x_i x_j.$$

Thus, $x_i x_j = \lambda x_s^2$.

Similarly, we show the following lemma.

Lemma 4.2 *Let k be odd and $G = (V, E)$ be a k-uniform power hypergraph and $x \in \mathbb{R}^n$ be an H-eigenvector of its signless Laplacian tensor $\mathcal{Q}$ corresponding to an H-eigenvalue $\lambda \neq 1$. Let $e \in E$ be an arbitrary but fixed edge.*

- (i) *If e has only one intersectional vertex i, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(\lambda - 1)x_s = x_i$.*
- (ii) *If e has two intersectional vertices i and j, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(\lambda - 1){x_s}^2 = x_i x_j$.*

The following results are given for even-uniform power hypergraphs.

Lemma 4.3 *Let k be even and $G = (V, E)$ be a k-uniform power hypergraph and $x \in \mathbb{R}^n$ be an H-eigenvector of its adjacency tensor $\mathcal{A}$ corresponding to an H-eigenvalue $\lambda \neq 0$. Let $e \in E$ be an arbitrary but fixed edge and e' be the set of its intersectional vertices. Let α be the cardinality of the set $\{i \in e \backslash e' \mid x_i < 0\}$.*

- (i) *If e has only one intersectional vertex i, and $x_s \neq 0$ for some cored vertex $s \in e$, then $\lambda x_i > 0$ when α is even and $\lambda x_i < 0$ when α is odd. Here, $x_s = \frac{x_i}{\lambda}$ or $-\frac{x_i}{\lambda}$.*

(ii) *If e has two intersectional vertices i and j, and $x_s \neq 0$ for some cored vertex $s \in e$, then $\lambda x_i x_j > 0$ when α is even and $\lambda x_i x_j < 0$ when α is odd. Here, $x_s = \pm\sqrt{\frac{x_i x_j}{\lambda}}$ or $\pm\sqrt{-\frac{x_i x_j}{\lambda}}$.*

Proof Let $x_+ = |x_s|$. By Definition 1.1, we have

$$\lambda x_s^k = \prod_{t \in e} x_t. \tag{4.1}$$

For (i), if α is even, then we have (4.1)$\Leftrightarrow$ $\lambda x_+^k = x_+^{k-1} x_i \Leftrightarrow \lambda x_+ = x_i$. If α is odd, we have (4.1)$\Leftrightarrow$ $\lambda x_+^k = -x_+^{k-1} x_i \Leftrightarrow \lambda x_+ = -x_i$.

For (ii), if α is even, then (4.1)$\Leftrightarrow$ $\lambda x_+^k = x_+^{k-2} x_i x_j \Leftrightarrow \lambda x_+^2 = x_i x_j$. If α is odd, we have (4.1)$\Leftrightarrow$ $\lambda x_+^k = -x_+^{k-2} x_i x_j \Leftrightarrow \lambda x_+^2 = -x_i x_j$. Thus, we obtain the desired results.

Lemma 4.4 *Let k be even and $G = (V, E)$ be a k-uniform power hypergraph and $x \in \mathbb{R}^n$ be an H-eigenvector of its Laplacian tensor $\mathcal{L}$ corresponding to an H-eigenvalue $\lambda \neq 1$. Let $e \in E$ be an arbitrary but fixed edge and e' be the set of its intersectional vertices. Let α be the cardinality of the set $\{i \in e \backslash e' | \, x_i < 0\}$.*

(i) *If e has only one intersectional vertex i, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(1 - \lambda) x_i > 0$ when α is even and $(1 - \lambda) x_i < 0$ when α is odd. Here, $x_s = \frac{x_i}{1-\lambda}$ or $\frac{x_i}{\lambda - 1}$.*

(ii) *If e has two intersectional vertices i and j, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(1 - \lambda) x_i x_j > 0$ when α is even and $(1 - \lambda) x_i x_j < 0$ when α is odd. Here, $x_s = \pm\sqrt{\frac{x_i x_j}{1-\lambda}}$ or $\pm\sqrt{\frac{x_i x_j}{\lambda - 1}}$.*

Proof Let $x_+ = |x_s|$. By Definition 1.1, we have

$$(\lambda - 1) x_s^k = -\prod_{t \in e} x_t. \tag{4.2}$$

For (i), if α is even, then we have (4.2)$\Leftrightarrow$ $(1 - \lambda) x_+^k = x_+^{k-1} x_i \Leftrightarrow (1 - \lambda) x_+ = x_i$. If α is odd, we have (4.2)$\Leftrightarrow$ $(1 - \lambda) x_+^k = -x_+^{k-1} x_i \Leftrightarrow (1 - \lambda) x_+ = -x_i$.

For (ii), if α is even, then (4.2)$\Leftrightarrow$ $(1 - \lambda) x_+^k = x_+^{k-2} x_i x_j \Leftrightarrow (1 - \lambda) x_+^2 = x_i x_j$. If α is odd, we have (4.2)$\Leftrightarrow$ $(1 - \lambda) x_+^k = -x_+^{k-2} x_i x_j \Leftrightarrow (1 - \lambda) x_+^2 = -x_i x_j$. Thus, we obtain the desired results.

The proof for the following lemma is similar.

Lemma 4.5 *Let k be even and $G = (V, E)$ be a k-uniform power hypergraph and $x \in \mathbb{R}^n$ be an H-eigenvector of its signless Laplacian tensor $\mathcal{Q}$ corresponding to an H-eigenvalue $\lambda \neq 1$. Let $e \in E$ be an arbitrary but fixed edge and e' be the set of its intersectional vertices. Let α be the cardinality of the set $\{i \in e \backslash e' | \, x_i < 0\}$.*

(i) *If e has only one intersectional vertex i, and* $x_s \neq 0$ *for some cored vertex* $s \in e$, *then* $(\lambda - 1)x_i > 0$ *when* α *is even and* $(\lambda - 1)x_i < 0$ *when* α *is odd. Here,* $x_s = \frac{x_i}{\lambda-1}$ *or* $\frac{x_i}{1-\lambda}$.

(ii) *If e has two intersectional vertices i and j, and* $x_s \neq 0$ *for some cored vertex* $s \in e$, *then* $(\lambda - 1)x_i x_j > 0$ *when* α *is even and* $(\lambda - 1)x_i x_j < 0$ *when* α *is odd. Here,* $x_s = \pm\sqrt{\frac{x_i x_j}{\lambda-1}}$ *or* $\pm\sqrt{\frac{x_i x_j}{1-\lambda}}$.

Sunflower is a special class of power hypergraphs. Now, we pay attention to establish H-spectra for sunflowers. Let $G = (V, E)$ be a k-uniform sunflower with $k \geqslant 3$ and the size $d \geqslant 2$, where $V = [n]$, $E = \{e_1, \cdots, e_d\}$, and $d_1 = d$ (i.e., the vertex 1 is the heart). Let $e(j)$ denote the unique edge containing the vertex j for $j \geqslant 2$.

Lemma 4.6 *Let G, k, and d be as above. Suppose that* (λ, x) *is an H-eigenpair of its adjacency tensor* $\mathcal{A}$ *with* $\lambda \neq 0$. *Then,* $x_1 \neq 0$. *Moreover, if* $i, j \geqslant 2$ *and* x_i, x_j *are both nonzero, then* $x_i = x_j$ *when k is odd, and* $|x_i| = |x_j|$ *when k is even.*

Proof If $x_1 = 0$, we have

$$\lambda x_j^{k-1} = 0, \quad \forall j \in \{2, 3, \cdots, n\},$$

which, together with $\lambda \neq 0$, implies $x_j = 0, \quad \forall j \in \{2, 3, \cdots, n\}$. This is a contradiction with the fact that x is an H-eigenvector. So, $x_1 \neq 0$.

We next show the second conclusion in the following two cases.

Case 1 k is odd. If $j \geqslant 2$ and $x_j \neq 0$, by Lemma 4.1, we have $x_1 = \lambda x_j$. Similarly, for $i \geqslant 2$ and $x_i \neq 0$, we also have $x_1 = \lambda x_i$. Thus, $x_i = x_j$.

Case 2 k is even. If $j \geqslant 2$ and $x_j \neq 0$, by Lemma 4.3, we have $|x_1| = |\lambda x_j|$. Similarly, for $i \geqslant 2$ and $x_i \neq 0$, we also have $|x_1| = |\lambda x_i|$. Thus, $|x_i| = |x_j|$.

Lemma 4.7 *Let G, k, d, and* $\mathcal{A}$ *be as above. Then we have*:

(1) *If* $x_1 \neq 0$, (λ, x) *is an H-eigenpair of* $\mathcal{A}$ *with* $\lambda = 0$ *if and only if* $\prod_{s\in e(j)\setminus\{j\}} x_s = 0, \forall j \in \{2, \cdots, n\}$.

(2) *If* $x_1 = 0$, (λ, x) *is an H-eigenpair of* $\mathcal{A}$ *with* $\lambda = 0$ *if and only if* $\sum_{i=1}^{d} \prod_{s\in e_i\setminus\{1\}} x_s = 0$.

Proof Necessity: It is easy to see that the eigenvalue equations $(\lambda\mathcal{I} - \mathcal{A})x^{k-1} = 0$ are equivalent to the following two relations:

$$0 = \sum_{i=1}^{d} \prod_{s\in e_i\setminus\{1\}} x_s \tag{4.3}$$

and

$$\prod_{s\in e(j)\setminus\{j\}} x_s = 0, \quad \forall j \in \{2, \cdots, n\}. \tag{4.4}$$

Case 1 If $x_1 \neq 0$, by (4.4) we know that at least two vertices except for 1-vertex in every edge is equal to zero. So we must have $\sum_{i=1}^{d} \prod_{s \in e_i \setminus \{1\}} x_s = 0$.

Case 2 If $x_1 = 0$, we must have $\prod_{s \in e(j) \setminus \{j\}} x_s = 0, \quad \forall j \in \{2, \cdots, n\}$.

Sufficiency: It is easy to verify these results by above conclusions and $x \neq 0$.

Similarly, we have the following lemma for the signless Laplacian tensor of a k-uniform sunflower.

Lemma 4.8 *Let G, k, and d be as above. Suppose that (λ, x) is an H-eigenpair of the signless Laplacian tensor $\mathcal{Q}$ with $\lambda \neq 1$. Then, $x_1 \neq 0$. Moreover, if $i, j \geqslant 2$ and x_i, x_j are both nonzero, then $x_i = x_j$ when k is odd, and $|x_i| = |x_j|$ when k is even.*

Lemma 4.9 *Let G, k, d, and $\mathcal{Q}$ be as above. Then (λ, x) is an H-eigenpair of $\mathcal{Q}$ with $\lambda = 1$ if and only if we have $x_1 = 0$ and $\sum_{i=1}^{d} \prod_{s \in e_i \setminus \{1\}} x_s = 0$.*

The following theorem gives the largest adjacency H-eigenvalue of a k-uniform sunflower.

Theorem 4.10 *Let $G = (V, E)$ be a k-uniform sunflower of size $d \geqslant 2$. Let $\mathcal{A}$ be the adjacency tensor of G. Then $\lambda(\mathcal{A}) = d^{\frac{1}{k}}$.*

Proof By [7, Lemma 3.1], [16, Theorem 4], and [17, Theorem 3.20], if we can find a positive H-eigenvector x of $\mathcal{A}$ corresponding to an H-eigenvalue μ, then $\mu = \lambda(\mathcal{A})$. Hence, we may assume that x with $x_1 = \alpha > 0$ and $x_j = 1$ for $j \in \{2, \cdots, n\}$ is an H-eigenvector of $\mathcal{A}$ corresponding to $\lambda(\mathcal{A})$. By Definition 1.1, we have

$$\lambda(\mathcal{A})\, \alpha^{k-1} = d, \quad \lambda(\mathcal{A}) = \alpha.$$

Hence,

$$\lambda(\mathcal{A})^k = d.$$

From the proof of the above theorem, we immediately obtain the following result.

Corollary 4.11 *Let $G = (V, E)$ be a sunflower of size $d \geqslant 2$. If $x \in \mathbb{R}^n$ is an H-eigenvector of its adjacency tensor $\mathcal{A}$ corresponding to $\lambda(\mathcal{A})$, then $sup(x) = [n]$. Hence, there is an H-eigenvector $z \in \mathbb{R}^n$ of $\mathcal{A}$ corresponding to $\lambda(\mathcal{A})$ satisfying that z_i is a constant for all vertices other than the heart.*

The following result shows the largest signless Laplacian H-eigenvalue of sunflowers.

Theorem 4.12 *Let $G = (V, E)$ be a k-uniform hyperstar of size $d \geqslant 2$ and $\mathcal{Q}$ be its signless Laplacian tensor. Then $\lambda(\mathcal{Q})$ is the unique real root of the equations $(\lambda - 1)^{k-1}(\lambda - d) - d = 0$ in the interval $(d, d + 1)$.*

Proof By [7, Lemma 3.1], [16, Theorem 4], [17, Theorem 3.20] and (see also [14, Lemmas 2.2 and 2.3]), if we can find a positive H-eigenvector $x \in \mathbb{R}^n$ of $\mathcal{Q}$ corresponding to an H-eigenvalue μ, then $\mu = \lambda(\mathcal{Q})$.

Let $x_1 = \alpha > 0, x_j = 1, j \in \{2, \cdots, n\}$. Suppose that x is an H-eigenvector of $\mathcal{Q}$ corresponding to the H-eigenvalue $\mu = \lambda(\mathcal{Q})$. By Definition 1.1, we have

$$(\mu - d)\ \alpha^{k-1} = d, \quad (\mu - 1) = \alpha.$$

Hence,

$$(\mu - d)(\mu - 1)^{k-1} - d = 0.$$

Let $f(\mu) = (\mu - d)(\mu - 1)^{k-1} - d$. We have $f^{'}(\mu) = (\mu - 1)^{k-2}(\mu k - dk + d - 1)$. So if $\mu > d$, $f(\mu)$ is monotone increasing. Clearly, $f(d) = -d < 0$ and $f(d + 1) = d^{k-1} - d > 0$. Thus, $f(\mu) = 0$ does have a unique root in the interval $(d, d + 1)$. The desired result follows.

We immediately get the following result.

Corollary 4.13 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two sunflowers of size d_1 and $d_2 \geqslant 2$, respectively. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the signless Laplacian tensors of G_1 and G_2, respectively. If $d_1 > d_2$, then $\lambda(\mathcal{Q}_1) > \lambda(\mathcal{Q}_2)$.*

By [2, Proposition 3.2], we also have the following corollary.

Corollary 4.14 *Let k be even and $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two sunflowers of size d_1 and $d_2 \geqslant 2$, respectively. Let $\mathcal{L}_1, \mathcal{Q}_1$ and $\mathcal{L}_2, \mathcal{Q}_2$ be the Laplacian and signless Laplacian tensors of G_1 and G_2, respectively. If $d_1 > d_2$, then $\lambda(\mathcal{L}_1) = \lambda(\mathcal{Q}_1) > \lambda(\mathcal{L}_2) = \lambda(\mathcal{Q}_2)$.*

5 H-spectra of Sunflowers

In this section, we compute all the adjacency H-eigenvalues and all the signless Laplacian H-eigenvalues of sunflowers. From Lemma 4.6 we can obtain the set of all distinct H-eigenvalues and all corresponding H-eigenvectors of the adjacency tensor $\mathcal{A}$ of the sunflower G (except for the eigenvalue 0) in the following Proposition 5.1 (for the case when k is odd) and Proposition 5.2 (for the case when k is even). The set of all eigenvectors corresponding to the eigenvalue 0 will be given in Proposition 5.3. From Lemma 4.8 we can construct the set of all distinct H-eigenvalues and all corresponding H-eigenvectors of the signless Laplacian tensor $\mathcal{Q}$ of the hyperstar G (except for the eigenvalue 1) in the following Proposition 5.4 (for the case when k is odd) and Proposition 5.5 (for the case when k is even). The set of all eigenvectors corresponding to the eigenvalue 1 will be given in Proposition 5.6. Their proofs are similar to those of Propositions 5.1, 5.2, and 5.3 in [2], so we omit them here.

Proposition 5.1 *Let $G = (V, E)$ be a k-uniform sunflower with odd $k \geqslant 3$ and the size $d \geqslant 2$, where $V = [n]$, $E = \{e_1, e_2, \cdots, e_d\}$, and $d_1 = d$ (i.e., the vertex 1 is the heart). Let $\mathcal{A}$ be the adjacency tensor of G. Let*

$$f_r(\lambda) = \lambda^k - r, \quad r = 0, 1, \cdots, d.$$

Then we have:

(i) $\lambda \neq 0$ *is an H-eigenvalue of* $\mathcal{A}$ *if and only if it is a real root of the polynomial* $f_r(\lambda)$ *for some* $r \in \{0, 1, \cdots, d\}$.
(ii) *If* $\lambda \neq 0$ *is a root of the polynomial* $f_r(\lambda)$, *then we can construct all the H-eigenvectors of* $\mathcal{A}$ *corresponding to* λ *(up to a constant multiple) by going through the following procedure*:

Step 1. *Take* $x_1 = \lambda$.
Step 2. *Choose any r edges of G. Take the x-values of all the pendant vertices of these r edges to be* 1.
Step 3. *Take the x-values of all the other vertices of G to be zero.*

Proposition 5.2 *Let* $G = (V, E)$ *be a k-uniform sunflower with even* $k \geqslant 4$ *and the size* $d \geqslant 2$, *where* $V = [n]$, $E = \{e_1, e_2, \cdots, e_d\}$, *and* $d_1 = d$ *(i.e., the vertex* 1 *is the heart). Let* $\mathcal{A}$ *be the adjacency tensor of G. Let*

$$f_r(\lambda) = \lambda^k - r, \quad r = 0, 1, \cdots, d.$$

Then we have

(i) $\lambda \neq 0$ *is an H-eigenvalue of* $\mathcal{A}$ *if and only if it is a real root of the polynomial* $f_r(\lambda)$ *for some* $r \in \{0, 1, \cdots, d\}$.
(ii) *If* $\lambda \neq 0$ *is a root of the polynomial* $f_r(\lambda)$, *then we can construct all the H-eigenvectors of* $\mathcal{A}$ *corresponding to* λ *(up to a constant multiple) by going through the following procedure*:

Step 1. *Take* $x_1 = \lambda$.
Step 2. *Choose any r edges of G. Take the x-values of all the pendant vertices of these r edges to be* ± 1, *where the number of* -1 *value in each edge is even.*
Step 3. *Take the x-values of all the other vertices of G to be zero.*

Proposition 5.3 *Let* $G = (V, E)$ *be a k-uniform sunflower with* $k \geqslant 4$ *and the size* $d \geqslant 2$, *where* $V = [n]$, $E = \{e_1, e_2, \cdots, e_d\}$, *and* $d_1 = d$ *(i.e., the vertex* 1 *is the heart). Let* $\mathcal{A}$ *be the adjacency tensor of G. Then we have*

(i) *If* $x_1 \neq 0$, *a nonzero vector x is an eigenvector corresponding to eigenvalue* 0 *if and only if the x-values of all the pendant vertices of G satisfy the following relation*:
$$\prod_{s \in e(j) \setminus \{j\}} x_s = 0, \quad \forall j \in \{2, \cdots, n\}.$$
(ii) *If* $x_1 = 0$, *a nonzero vector x is an eigenvector corresponding to eigenvalue* 0 *if and only if the x-values of all the pendant vertices of G satisfy the following relation*:
$$\sum_{i=1}^{d} \prod_{s \in e_i \setminus \{1\}} x_s = 0.$$

Proposition 5.4 *Let* $G = (V, E)$ *be a k-uniform sunflower with odd* $k \geqslant 3$ *and the size* $d \geqslant 2$, *where* $V = [n]$, $E = \{e_1, e_2, \cdots, e_d\}$, *and* $d_1 = d$*(i.e., the vertex* 1 *is the heart). Let* $\mathcal{Q}$ *be the signless Laplacian tensor of G. Let*

$$f_r(\lambda) = (\lambda - d)(\lambda - 1)^{k-1} - r \quad (r = 0, 1, \cdots, d).$$

Then we have

(i) $\lambda \neq 1$ *is an H-eigenvalue of* $\mathcal{Q}$ *if and only if it is a real root of the polynomial* $f_r(\lambda)$ *for some* $r \in \{0, 1, \cdots, d\}$.

(ii) *If* $\lambda \neq 1$ *is a root of the polynomial* $f_r(\lambda)$, *then we can construct all the H-eigenvectors of* $\mathcal{Q}$ *corresponding to* λ (*up to a constant multiple*) *by going through the following procedure*:

Step1. *Take* $x_1 = \lambda - 1$.
Step2. *Choose any r edges of G. Take the x-values of all the pendant vertices of these r edges to be* 1.
Step3. *Take the x-values of all the other vertices of G to be zero.*

Proposition 5.5 *Let* $G = (V, E)$ *be a k-uniform sunflower with even* $k \geqslant 4$ *and the size* $d \geqslant 2$, *where* $V = [n]$, $E = \{e_1, e_2, \cdots, e_d\}$, *and* $d_1 = d$(*i.e., the vertex* 1 *is the heart*). *Let* $\mathcal{Q}$ *be the signless Laplacian tensor of G. Let*

$$f_r(\lambda) = (\lambda - d)(\lambda - 1)^{k-1} - r, \quad r = 0, 1, \cdots, d.$$

Then we have

(i) $\lambda \neq 1$ *is an H-eigenvalue of* $\mathcal{Q}$ *if and only if it is a real root of the polynomial* $f_r(\lambda)$ *for some* $r \in \{0, 1, \cdots, d\}$.

(ii) *If* $\lambda \neq 1$ *is a root of the polynomial* $f_r(\lambda)$, *then we can construct all the H-eigenvectors of* $\mathcal{Q}$ *corresponding to* λ (*up to a constant multiple*) *by going through the following procedure*:

Step1. *Take* $x_1 = \lambda - 1$.
Step2. *Choose any r edges of G. Take the x-values of all the pendant vertices of these r edges to be* ± 1, *where the number of* -1 *value in each edge is even.*
Step3. *Take the x-values of all the other vertices of G to be zero.*

Proposition 5.6 *Let* $G = (V, E)$ *be a k-uniform sunflower with* $k \geqslant 4$ *and the size* $d \geqslant 2$, *where* $V = [n]$, $E = \{e_1, e_2, \cdots, e_d\}$, *and* $d_1 = d$ (*i.e., the vertex* 1 *is the heart*). *Let* $\mathcal{Q}$ *be the signless Laplacian tensor of G. Then a nonzero vector x is an eigenvector corresponding to eigenvalue* 1 *if and only if* $x_1 = 0$ *and the x-values of all the pendant vertices of G satisfy the following relation*:

$$\sum_{i=1}^{d} \prod_{s \in e_i \setminus \{1\}} x_s = 0.$$

From the above propositions, we know that it is essential to solve the equation $f_r(\lambda) = 0$ in order to compute all H-eigenvalues. The following theorems fix the roots of $f_r(\lambda)$.

Theorem 5.7 *Let k be odd and $G = (V, E)$ be a k-uniform sunflower of size $d \geqslant 2$. Let $\mathcal{L}$ be its Laplacian tensor. Consider the real roots of $f_r(\lambda) := (\lambda - d)(1-\lambda)^{k-1} + r$ ($r \in \{0, 1, \cdots, d\}$), we have the following statements.*
Case 1 *If $r = 0$, then $f_r(\lambda)$ has two real roots. 1 is a $k-1$ multiples root and d is a single root.*
Case 2 *If $0 < r \leqslant d$, then there are three cases.*

- (i) *If $(1-d)(1-\frac{(k-1)d+1}{k})^{k-1} + kr > 0$, $f_r(\lambda)$ has only one real root and it falls in $[0, 1)$. In this case, 0 is the root if and only if $r = d$.*
- (ii) *If $(1-d)(1-\frac{(k-1)d+1}{k})^{k-1} + kr = 0$, $f_r(\lambda)$ has two real roots, one falls in $[0, 1)$ and the other is $\lambda = \frac{(k-1)d+1}{k}$. In this case, 0 is the root if and only if $r = d$.*
- (iii) *If $(1-d)(1-\frac{(k-1)d+1}{k})^{k-1} + kr < 0$, $f_r(\lambda)$ has three real roots. The first falls in $[0, 1)$, the second falls in $(1, \frac{(k-1)d+1}{k})$ and the third falls in $(\frac{(k-1)d+1}{k}, d)$. In this case, 0 is the root if and only if $r = d$.*

Proof We have

$$f_r'(\lambda) = (1-\lambda)^{k-2}[-k\lambda + (k-1)d + 1].$$

Let $f_r'(\lambda) = 0$, then

$$\lambda = \frac{(k-1)d+1}{k} \geqslant \frac{(k-1)2+1}{k} = 2 - \frac{1}{k} > 1.$$

Hence, $f_r'(\lambda) > 0$ when $\lambda < 1$, $f_r'(\lambda) < 0$ when $1 < \lambda < \frac{(k-1)d+1}{k}$ and $f_r'(\lambda) > 0$ when $\lambda > \frac{(k-1)d+1}{k}$. We also have

$$f_r(0) = -d + r, \quad f_r(1) = r,$$
$$f_r\left(\frac{(k-1)d+1}{k}\right) = \frac{1}{k}(1-d)\left(1 - \frac{(k-1)d+1}{k}\right)^{k-1} + kr.$$

Hence, the desired results hold.

Theorem 5.8 *Let k be even and $G = (V, E)$ be a k-uniform sunflower of size $d \geqslant 2$. Let $\mathcal{L}$ be the Laplacian tensor of G. Consider the real roots of $f_r(\lambda) := (\lambda - d)(1-\lambda)^{k-1} + r$ ($r \in \{0, 1, \cdots, d\}$), we obtain the following conclusions:*
Case 1 *If $r = 0$, then $f_r(\lambda)$ has two real roots. 1 is a $k-1$ multiples root and d is a single root.*
Case 2 *If $0 < r \leqslant d$, then $f_r(\lambda)$ has two real roots. One falls in $(0, 1)$ and the other is in $(d, d+1)$. In this case, 0 is its root if and only if $r = d$.*

Proof By straightforward computation, we have

$$f_r'(\lambda) = (1-\lambda)^{k-2}[-k\lambda + (k-1)d + 1].$$

Let $f_r'(\lambda) = 0$, then we get

$$\lambda = \frac{(k-1)d+1}{k} \geqslant \frac{(k-1)2+1}{k} = 2 - \frac{1}{k} > 1.$$

So, $f_r'(\lambda) > 0$ when $\lambda < \frac{(k-1)d+1}{k}$ and $f_r'(\lambda) < 0$ when $\lambda > \frac{(k-1)d+1}{k}$. We have $f_r(0) = -d + r$, $f_r(1) = r$, $f_r(d) = r$, and $f_r(d+1) = (-d)^{k-1} + r < 0$. Hence, the proof is complete.

Theorem 5.9 *Let k be odd and $G = (V, E)$ be a k-uniform sunflower of size $d \geqslant 2$. Let $\mathcal{Q}$ be its signless Laplacian tensor. Consider the real roots of $f_r(\lambda) := (\lambda - d)(\lambda - 1)^{k-1} - r$ $(r \in \{0, 1, \cdots, d\})$, we obtain the following results.*
Case 1 If $r = 0$, then $f_r(\lambda)$ has two real roots. 1 is a $k-1$ multiples root and d is a single root.
Case 2 If $0 < r \leqslant d$, then $f_r(\lambda)$ has only one real root and it falls in $(d, d+1)$.

Proof By straightforward computation, we have

$$f_r'(\lambda) = (\lambda - 1)^{k-2}[k\lambda - (k-1)d - 1].$$

Let $f_r'(\lambda) = 0$, then we get

$$\lambda = \frac{(k-1)d+1}{k} \geqslant \frac{(k-1)2+1}{k} = 2 - \frac{1}{k} > 1.$$

So, $f_r'(\lambda) > 0$ when $\lambda < 1$, $f_r'(\lambda) < 0$ when $1 < \lambda < \frac{(k-1)d+1}{k}$, and $f_r'(\lambda) > 0$ when $\lambda > \frac{(k-1)d+1}{k}$. We have $f_r(0) = -d - r < 0$, $f_r(1) = -r$, $f_r(d) = -r$, and $f_r(d+1) = d^{k-1} - r > 0$. These facts imply that the desired results hold.

By Theorem 5.9 and [2, Proposition 3.2], we immediately get the following theorem:

Theorem 5.10 *Let k be even and $G = (V, E)$ be a k-uniform sunflower of size $d \geqslant 2$. Let $\mathcal{Q}$ be the signless Laplacian tensor of G. Consider the real roots of $f_r(\lambda) := (\lambda - d)(\lambda - 1)^{k-1} - r$ $(r \in \{0, 1, \cdots, d\})$, we have the following results*:

Case 1 *If $r = 0$, then $f_r(\lambda)$ has two real roots. 1 is a $k-1$ multiples root and d is a single root.*

Case 2 *If $0 < r \leqslant d$, then $f_r(\lambda)$ has two real roots. One falls in $(0, 1)$ and the other is in $(d, d+1)$. In this case, 0 is its root if and only if $r = d$.*

We now give some numerical experiments to show the conclusions given in the above theorems. We apply the bisection method to solve the real root of above polynomials. When d is fixed, we investigate the changing trend of Laplacian and signless Laplacian H-spectra of sunflower as k increasing.

Through a lot of experiments, we find some good rules for above problems. The numerical results are reported in Fig. 1, which show that H-Eigenvalues of sunflowers will tend to 0, 2, and d when k is increasing and d is fixed.

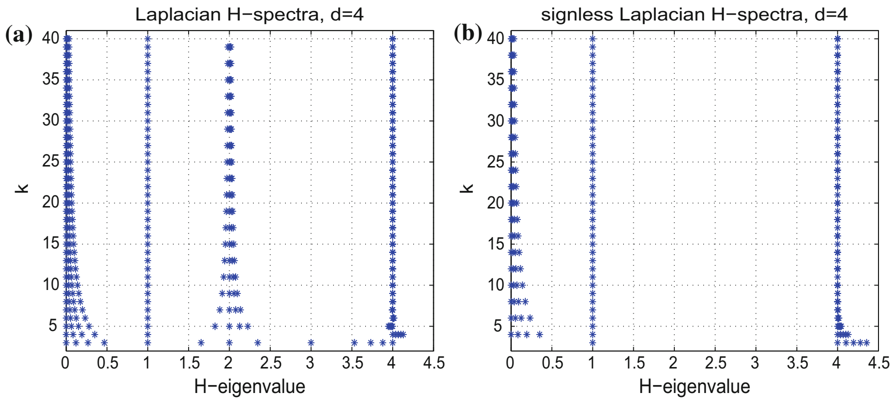

Fig. 1 Distribution of H-spectra of sunflower when d is fixed and k is changed

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