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Eigenvalue analysis of constrained minimization problem for homogeneous polynomial

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Abstract In this paper, the concepts of Pareto H-eigenvalue and Pareto Z-eigenvalue are introduced for studying constrained minimization problem and the necessary and sufficient conditions of such eigenvalues are given. It is proved that a symmetric tensor has at least one Pareto H-eigenvalue (Pareto Z-eigenvalue). Furthermore, the minimum Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a symmetric tensor is exactly equal to the minimum value of constrained minimization problem of homogeneous polynomial deduced by such a tensor, which gives an alternative methods for solving the minimum value of constrained minimization problem. In particular, a symmetric tensor $\mathcal{A}$ is strictly copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of $\mathcal{A}$ is positive, and $\mathcal{A}$ is copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of $\mathcal{A}$ is non-negative.

Keywords Constrained minimization · Principal sub-tensor · Pareto H-eigenvalue · Pareto Z-eigenvalue

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1 Introduction and basic facts

Throughout this paper, let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x \geq 0\}$, and $\mathbb{R}^n_- = \{x \in \mathbb{R}^n; x \leq 0\}$, and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n; x > 0\}$, and $e = (1, 1, \dots, 1)^T$, and $x^{[m]} = (x_1^m, x_2^m, \dots, x_n^m)^T$ for $x = (x_1, x_2, \dots, x_n)^T$, where x^T is the transposition of a vector x and $x \geq 0$ ($x > 0$) means $x_i \geq 0 (x_i > 0)$ for all $i \in \{1, 2, \dots, n\}$.

As a natural extension of the concept of matrices, a m-order n-dimensional tensor $\mathcal{A}$ consists of n^m entries in the real field $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 \dots i_m}), \quad a_{i_1 \dots i_m} \in \mathbb{R}, \ i_1, i_2, \dots, i_m = 1, 2, \dots, n.$$

For a vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ or $\mathbb{C}^n$, $\mathcal{A}x^m$ is defined by

$$\mathcal{A}x^m = \sum_{i_1, i_2, \dots, i_m = 1}^{n} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}; \tag{1.1}$$

$\mathcal{A}x^{m-1}$ is a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) with its ith component defined by

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2, \dots, i_m = 1}^{n} a_{i i_2 \dots i_m} x_{i_2} \cdots x_{i_m} \text{ for } i = 1, 2, \dots, n. \tag{1.2}$$

A m-order n-dimensional tensor $\mathcal{A}$ is said to be *symmetric* if its entries $a_{i_1 \dots i_m}$ are invariant for any permutation of the indices. Clearly, each m-order n-dimensional symmetric tensor $\mathcal{A}$ defines a homogeneous polynomial $\mathcal{A}x^m$ of degree m with n variables and vice versa.

For a given m-order n-dimensional symmetric tensor $\mathcal{A}$, we consider a constrained optimization problem of the form:

$$\begin{aligned} &\min \frac{1}{m} \mathcal{A}x^m \\ &s.t. \ x^T x^{[m-1]} = 1 \\ &\quad x \in \mathbb{R}^n_+. \end{aligned} \tag{1.3}$$

Then the Lagrange function of the problem (1.3) is given clearly by

$$L(x, \lambda, y) = \frac{1}{m} \mathcal{A}x^m + \frac{1}{m} \lambda \left(1 - x^T x^{[m-1]}\right) - x^T y \tag{1.4}$$

where $x, y \in \mathbb{R}^n_+$, $\frac{\lambda}{m} \in \mathbb{R}$ is the Lagrange multiplier of the equality constraint and y is the Lagrange multiplier of non-negative constraint. So the solution x of the problem (1.3) satisfies the following KKT conditions ([4,6]):

$$\mathcal{A}x^{m-1} - \lambda x^{[m-1]} - y = 0 \tag{1.5}$$

$$1 - x^T x^{[m-1]} = 0 \tag{1.6}$$

$$x^T y = 0 \tag{1.7}$$

$$x, y \in \mathbb{R}^n_+. \tag{1.8}$$

The Eq. (1.6) means that $\sum_{i=1}^{n} x_i^m = 1$. It follows from the Eqs. (1.5), (1.7) and (1.8) that

$$\begin{aligned} x^T y = x^T \mathcal{A}x^{m-1} - \lambda x^T x^{[m-1]} = 0 \\ x \geq 0, \mathcal{A}x^{m-1} - \lambda x^{[m-1]} = y \geq 0, \end{aligned}$$

and hence,

$$\begin{cases} \mathcal{A}x^m = \lambda x^T x^{[m-1]} \\ \mathcal{A}x^{m-1} - \lambda x^{[m-1]} \geq 0 \\ x \geq 0. \end{cases} \tag{1.9}$$

Following Qi [11] (H-eigenvalue of the tensor $\mathcal{A}$) and Seeger [23] (Pareto eigenvalue of the matrix A), for a m-order n-dimensional tensor $\mathcal{A}$, a real number λ is called *Pareto H-eigenvalue* of the tensor $\mathcal{A}$ if there exists a non-zero vector $x \in \mathbb{R}^n$ satisfying the system (1.9). The non-zero vector x is called a *Pareto H-eigenvector* of $\mathcal{A}$ associated to λ.

Similarly, for a given m-order n-dimensional symmetric tensor $\mathcal{A}$, we consider another constrained optimization problem of the form ($m \geq 2$):

$$\begin{aligned} \min\ & \frac{1}{m}\mathcal{A}x^m \\ s.t.\ & x^T x = 1 \\ & x \in \mathbb{R}^n_+. \end{aligned} \tag{1.10}$$

Obviously, when $x \in \mathbb{R}^n$, $x^T x = 1$ if and only if $(x^T x)^{\frac{m}{2}} = 1$. The corresponding Lagrange function may be written in the form

$$L(x, \mu, y) = \frac{1}{m}\mathcal{A}x^m + \frac{1}{m}\mu\left(1 - (x^T x)^{\frac{m}{2}}\right) - x^T y.$$

So the solution x of the problem (1.10) satisfies KKT conditions ([4,6]):

$$\mathcal{A}x^{m-1} - \mu(x^T x)^{\frac{m}{2}-1}x - y = 0,\ 1 - (x^T x)^{\frac{m}{2}} = 0,\ x^T y = 0,\ x, y \in \mathbb{R}^n_+.$$

Then we have $\sum_{i=1}^n x_i^2 = 1$ and

$$\begin{cases} \mathcal{A}x^m = \mu(x^T x)^{\frac{m}{2}} \\ \mathcal{A}x^{m-1} - \mu(x^T x)^{\frac{m}{2}-1}x \geq 0 \\ x \geq 0. \end{cases} \tag{1.11}$$

Following Qi [11] (Z-eigenvalue of the tensor $\mathcal{A}$) and Seeger [23] (Pareto eigenvalue of the matrix A), for an m-order n-dimensional tensor $\mathcal{A}$, a real number μ is said to be *Pareto Z-eigenvalue* of the tensor $\mathcal{A}$ if there is a non-zero vector $x \in \mathbb{R}^n$ satisfying the system (1.11). The non-zero vector x is called a *Pareto Z-eigenvector* of $\mathcal{A}$ associated to μ.

So the constrained optimization problem (1.3) and (1.10) of homogeneous polynomial may be respectively solved by means of the Pareto H-eigenvalue (1.9) and Pareto Z-eigenvalue (1.11) of the corresponding tensor. It will be an interesting work to compute the Pareto H-eigenvalue (Z-eigenvalue) of a higher order tensor.

When $m = 2$, both Pareto H-eigenvalue and Pareto Z-eigenvalue of the m-order n-dimensional tensor obviously changes into Pareto eigenvalue of the matrix. The concept of Pareto eigenvalue is first introduced and used by Seeger [23] for studying the equilibrium processes defined by linear complementarity conditions. For more details, also see Hiriart-Urruty and Seeger [5].

Let $\mathcal{A}$ be a m-order n-dimensional symmetric tensor. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue of* $\mathcal{A}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1.12}$$

where $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$, and call x an *eigenvector* of $\mathcal{A}$ associated with the eigenvalue λ. We call such an eigenvalue *H-eigenvalue* if it is real and has a real eigenvector x, and call such a real eigenvector x an *H-eigenvector*.

These concepts were first introduced by Qi [11] to the higher order symmetric tensor, and the existence of the eigenvalues and its some application were studied also. Lim [9] independently introduced these concept and obtained the existence results using the variational approach. Subsequently, this topics are attracted attention of many mathematicians from different disciplines. For various studies and applications, see Chang [1], Chang et al. [2,3], Chang et al. [7], Li et al. [8], Qi and Song [16], Song and Qi [18–22], Yang and Yang [24,25] and references cited therein.

A number $\mu \in \mathbb{C}$ is said to be an *E-eigenvalue of $\mathcal{A}$* if there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \mu x(x^T x)^{\frac{m-2}{2}}. \tag{1.13}$$

Such a nonzero vector $x \in \mathbb{C}^n$ is called an *E-eigenvector* of $\mathcal{A}$ associated with μ, If x is real, then μ is also real. In this case, μ and x are called a *Z-eigenvalue* of $\mathcal{A}$ and a *Z-eigenvector* of $\mathcal{A}$ (associated with μ), respectively. Qi [11–13] first introduced and used these concepts and showed that a symmetric and real tensor has always Z-eigenvalue.

In homogeneous polynomial $\mathcal{A}x^m$ defined by (1.1), if we let some (but not all) x_i be zero, then we have a homogeneous polynomial with fewer variables, which defines a lower dimensional tensor. We call such a lower dimensional tensor a *principal sub-tensor* of $\mathcal{A}$. The concept were first introduced and used by Qi [11] to the higher order symmetric tensor.

Recently, Qi [14] introduced and used the following concepts for studying the properties of hypergraph. An H-eigenvalue λ of $\mathcal{A}$ is said to be (i) *H^+-eigenvalue of $\mathcal{A}$*, if its H-eigenvector $x \in \mathbb{R}^n_+$; (ii) *H^{++}-eigenvalue of $\mathcal{A}$*, if its H-eigenvector $x \in \mathbb{R}^n_{++}$. Similarly, we introduce the concepts of Z^+-eigenvalue and Z^{++}-eigenvalue. An Z-eigenvalue μ of $\mathcal{A}$ is said to be (i) *Z^+-eigenvalue of $\mathcal{A}$*, if its Z-eigenvector $x \in \mathbb{R}^n_+$; (ii) *Z^{++}-eigenvalue of $\mathcal{A}$*, if its Z-eigenvector $x \in \mathbb{R}^n_{++}$.

Obviously, the definition of H^+-eigenvalue (Z^+-eigenvalue) λ of $\mathcal{A}$ means that

$$\mathcal{A}x^{m-1} - \lambda x^{[m-1]} = 0 \ \ (\mathcal{A}x^{m-1} - \lambda (x^T x)^{\frac{m}{2}-1} x = 0, \text{ respectively})$$

for some non-zero vector $x \geq 0$. However, Pareto H-eigenvalue (Z-eigenvalue) λ of $\mathcal{A}$ means that

$$\mathcal{A}x^{m-1} - \lambda x^{[m-1]} \geq 0 \ \ (\mathcal{A}x^{m-1} - \lambda (x^T x)^{\frac{m}{2}-1} x \geq 0, \text{ respectively})$$

for some non-zero vector $x \geq 0$. So, the following conclusions are trivial.

Proposition 1.1 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. Then each H^+-eigenvalue (Z^+-eigenvalue) of $\mathcal{A}$ is its Pareto H-eigenvalue (Z-eigenvalue, respectively).*

Remark 1 (1) A Pareto H-eigenvalue (Z-eigenvalue) of a tensor $\mathcal{A}$ may not be its H^+ -eigenvalue (Z^+-eigenvalue). Such an example may see Example 2.

(2) Pareto H-eigenvalue (Z-eigenvalue) of a tensor $\mathcal{A}$ must be H^{++}-eigenvalue (Z^{++} -eigenvalue, respectively) of some principal sub-tensor of $\mathcal{A}$. For detailed proof, see Theorems 2.1 and 2.2.

In this paper, we mainly study the properties of the Pareto H-eigenvalue (Z-eigenvalue) of a higher order tensor $\mathcal{A}$.

In Sect. 2, it will be proved that a real number λ is Pareto H-eigenvalue (Z-eigenvalue) of $\mathcal{A}$ if and only if λ is H^{++}-eigenvalue (Z^{++}-eigenvalue) of some principal sub-tensor of $\mathcal{A}$ with corresponding H-eigenvector (Z-eigenvector) w and

$$\sum_{i_2,\ldots,i_m \in N} a_{ii_2\ldots i_m} w_{i_2} w_{i_3} \ldots w_{i_m} \geq 0 \text{ for } i \in \{1, 2, \ldots, n\} \setminus N.$$

So we may calculate some Pareto H-eigenvalue (Z-eigenvalue) of a higher order tensor by means of H^{++}-eigenvalue (Z^{++}-eigenvalue) of some lower dimensional tensors.

In Sect. 3, we will show that

$$\min_{\substack{x \geq 0 \\ \|x\|_m = 1}} \mathcal{A}x^m = \min\{\mu;\ \mu \text{ is Pareto } H\text{-eigenvalue of } \mathcal{A}\} \tag{1.14}$$

$$\min_{\substack{x \geq 0 \\ \|x\|_2 = 1}} \mathcal{A}x^m = \min\{\mu;\ \mu \text{ is Pareto } Z\text{-eigenvalue of } \mathcal{A}\}. \tag{1.15}$$

Therefore, we may solve the constrained minimization problem for homogeneous polynomial and test the (strict) copositivity of a symmetric tensor $\mathcal{A}$ with the help of computing the Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a symmetric tensor. As a corollary, a symmetric tensor $\mathcal{A}$ is copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of $\mathcal{A}$ is non-negative and $\mathcal{A}$ is strictly copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of $\mathcal{A}$ is positive.

2 Pareto H-eigenvalue and Pareto Z-eigenvalue

Let N be a subset of the index set $\{1, 2, \ldots, n\}$ and $\mathcal{A}$ be a tensor of order m and dimension n. We denote the principal sub-tensor of $\mathcal{A}$ by $\mathcal{A}^N$ which is obtained by homogeneous polynomial $\mathcal{A}x^m$ for all $x = (x_1, x_2, \ldots, x_n)^T$ with $x_i = 0$ for $i \in \{1, 2, \ldots, n\} \setminus N$. The symbol $|N|$ denotes the cardinality of N. So, $\mathcal{A}^N$ is a tensor of order m and dimension $|N|$ and the principal sub-tensor $\mathcal{A}^N$ is just $\mathcal{A}$ itself when $N = \{1, 2, \ldots, n\}$.

Theorem 2.1 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. A real number λ is Pareto H-eigenvalue of $\mathcal{A}$ if and only if there exists a nonempty subset $N \subseteq \{1, 2, \ldots, n\}$ and a vector $w \in \mathbb{R}^{|N|}$ such that*

$$\mathcal{A}^N w^{m-1} = \lambda w^{[m-1]}, \quad w \in \mathbb{R}^{|N|}_{++} \tag{2.1}$$

$$\sum_{i_2,\ldots,i_m \in N} a_{ii_2\ldots i_m} w_{i_2} w_{i_3} \ldots w_{i_m} \geq 0 \textit{ for } i \in \{1, 2, \ldots, n\} \setminus N \tag{2.2}$$

In such a case, the vector $y \in \mathbb{R}^n_+$ defined by

$$y_i = \begin{cases} w_i, & i \in N \\ 0, & i \in \{1, 2, \ldots, n\} \setminus N \end{cases} \tag{2.3}$$

is a Pareto H-eigenvector of $\mathcal{A}$ associated to the real number λ.

Proof First we show the necessity. Let the real number λ be a Pareto H-eigenvalue of $\mathcal{A}$ with a corresponding Pareto H-eigenvector y. Then by the definition (1.9) of the Pareto H-eigenvalue, the Pareto H-eigenpairs (λ, y) may be rewritten in the form

$$\begin{aligned} y^T\left(\mathcal{A}y^{m-1}-\lambda y^{[m-1]}\right) &= 0 \\ \mathcal{A}y^{m-1}-\lambda y^{[m-1]} &\geq 0 \\ y &\geq 0 \end{aligned} \tag{2.4}$$

and hence

$$\sum_{i=1}^{n} y_i\left(\mathcal{A}y^{m-1}-\lambda y^{[m-1]}\right)_i = 0 \tag{2.5}$$

$$\left(\mathcal{A}y^{m-1}-\lambda y^{[m-1]}\right)_i \geq 0, \quad \text{for } i=1,2,\ldots,n \tag{2.6}$$

$$y_i \geq 0, \quad \text{for } i=1,2,\ldots,n. \tag{2.7}$$

Combining the Eqs. (2.5) with (2.6) and (2.7), we have

$$y_i\left(\mathcal{A}y^{m-1}-\lambda y^{[m-1]}\right)_i = 0, \quad \text{for all } i \in \{1,2,\ldots,n\}. \tag{2.8}$$

Take $N=\{i \in \{1,2,\ldots,n\};\, y_i>0\}$. Let the vector $w \in \mathbb{R}^{|N|}$ be defined by

$$w_i = y_i \text{ for all } i \in N.$$

Clearly, $w \in \mathbb{R}^{|N|}_{++}$. Combining the Eq. (2.8) with the fact that $y_i > 0$ for all $i \in N$, we have

$$\left(\mathcal{A}y^{m-1}-\lambda y^{[m-1]}\right)_i = 0, \quad \text{for all } i \in N,$$

and so

$$\mathcal{A}^N w^{m-1} = \lambda w^{[m-1]}, \quad w \in \mathbb{R}^{|N|}_{++}.$$

It follows from the Eq. (2.6) and the fact that $y_i = 0$ for all $i \in \{1,2,\ldots,n\}\setminus N$ that

$$(\mathcal{A}y^{m-1})_i \geq 0, \quad \text{for all } i \in \{1,2,\ldots,n\}\setminus N.$$

By the definition (1.2) of $\mathcal{A}y^{m-1}$, the conclusion (2.2) holds.

Now we show the sufficiency. Suppose that there exists a nonempty subset $N \subseteq \{1,2,\ldots,n\}$ and a vector $w \in \mathbb{R}^{|N|}$ satisfying (2.1) and (2.2). Then the vector y defined by (2.3) is a non-zero vector in $\mathbb{R}^n_+$ such that (λ, y) satisfying (2.4). The desired conclusion follows. □

Using the same proof techniques as that of Theorem 2.1 with appropriate changes in the inequalities or equalities ($y^{[m-1]}$ is replaced by $(y^Ty)^{\frac{m-2}{2}}y$ and so on). We can obtain the following conclusions about the Pareto Z-eigenvalue of $\mathcal{A}$.

Theorem 2.2 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. A real number μ is Pareto Z-eigenvalue of $\mathcal{A}$ if and only if there exists a nonempty subset $N \subseteq \{1,2,\ldots,n\}$ and a vector $w \in \mathbb{R}^{|N|}$ such that*

$$\mathcal{A}^N w^{m-1} = \mu(w^Tw)^{\frac{m-2}{2}}w, \quad w \in \mathbb{R}^{|N|}_{++} \tag{2.9}$$

$$\sum_{i_2,\ldots,i_m \in N} a_{ii_2\ldots i_m} w_{i_2}w_{i_3}\ldots w_{i_m} \geq 0 \, for \, i \in \{1,2,\ldots,n\}\setminus N \tag{2.10}$$

In such a case, the vector $y \in \mathbb{R}^n_+$ defined by

$$y_i = \begin{cases} w_i, & i \in N \\ 0, & i \in \{1,2,\ldots,n\}\setminus N \end{cases} \tag{2.11}$$

is a Pareto Z-eigenvector of $\mathcal{A}$ associated to the real number μ.

Following Theroems 2.1 and 2.2, the following results are obvious.

Corollary 2.3 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. If a real number λ is Pareto H-eigenvalue (Z-eigenvalue) of $\mathcal{A}$, then λ is H^{++}-eigenvalue (Z^{++}-eigenvalue, respectively) of some principal sub-tensor of $\mathcal{A}$.*

Corollary 2.4 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. Then the Pareto H-eigenvalues (Z-eigenvalues) of a diagonal tensor $\mathcal{A}$ coincide with its diagonal entries. In particular, a n-dimensional and diagonal tensor may have at most n distinct Pareto H-eigenvalues (Z-eigenvalues).*

It follows from the above results that some Pareto H-eigenvalue (Z-eigenvalue) of a higher order tensor may be calculated by means of H^{++}-eigenvalue (Z^{++}-eigenvalue, respectively) of the lower dimensional tensors.

Example 1 Let $\mathcal{A}$ be a 4-order and 2-dimensional tensor. Suppose that $a_{1111} = 1, a_{2222} = 2, a_{1122} + a_{1212} + a_{1221} = -1, a_{2121} + a_{2112} + a_{2211} = -2$, and other $a_{i_1 i_2 i_3 i_4} = 0$. Then

$$\mathcal{A}x^4 = x_1^4 + 2x_2^4 - 3x_1^2 x_2^2$$
$$\mathcal{A}x^3 = \begin{pmatrix} x_1^3 - x_1 x_2^2 \\ 2x_2^3 - 2x_1^2 x_2 \end{pmatrix}$$

When $N = \{1, 2\}$, the principal sub-tensor $\mathcal{A}^N$ is just $\mathcal{A}$ itself. $\lambda_1 = 0$ is a H^{++}-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(1)} = (\frac{\sqrt[4]{8}}{2}, \frac{\sqrt[4]{8}}{2})^T$, and so it follows from Theorem 2.1 that $\lambda_1 = 0$ is a Pareto H-eigenvalue with Pareto H-eigenvector $x^{(1)} = (\frac{\sqrt[4]{8}}{2}, \frac{\sqrt[4]{8}}{2})^T$.

$\lambda_2 = 0$ is a Z^{++}-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(2)} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T$, and so it follows from Theorem 2.2 that $\lambda_2 = 0$ is a Pareto Z-eigenvalue of $\mathcal{A}$ with Pareto Z-eigenvector $x^{(2)} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T$.

When $N = \{1\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^N = 1$. Obviously, $\lambda_3 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^N$ with a corresponding eigenvector $w = 1$ and $a_{2111} w^3 = 0$, and hence it follows from Theorems 2.1 and 2.2 that $\lambda_3 = 1$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(3)} = (1, 0)^T$.

Similarly, when $N = \{2\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^N = 2$. Clearly, $\lambda_4 = 2$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^N$ with a corresponding eigenvector $w = 1$ and $a_{1222} w^3 = 0$, and so $\lambda_4 = 2$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(4)} = (0, 1)^T$.

Example 2 Let $\mathcal{A}$ be a 3-order and 2-dimensional tensor. Suppose that $a_{111} = 1, a_{222} = 2, a_{122} = a_{212} = a_{221} = \frac{1}{3}$, and $a_{112} = a_{121} = a_{211} = -\frac{2}{3}$. Then

$$\mathcal{A}x^3 = x_1^3 + x_1 x_2^2 - 2x_1^2 x_2 + 2x_2^3$$
$$\mathcal{A}x^2 = \begin{pmatrix} x_1^2 + \frac{1}{3}x_2^2 - \frac{4}{3}x_1 x_2 \\ 2x_2^2 + \frac{2}{3}x_1 x_2 - \frac{2}{3}x_1^2 \end{pmatrix}$$

When $N = \{1\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^N = 1$. Obviously, $\lambda_1 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^N$ with a corresponding eigenvector $w = 1$ and

$a_{211}w^2 = -\frac{2}{3} < 0$, and so $\lambda_1 = 1$ is neither Pareto H-eigenvalue nor Pareto Z-eigenvalue of $\mathcal{A}$.

When $N = \{2\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^N = 2$. Clearly, $\lambda_2 = 2$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^N$ with a corresponding eigenvector $w = 1$ and $a_{122}w^2 = \frac{1}{3} > 0$, and so $\lambda_2 = 2$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(2)} = (0, 1)^T$. But $\lambda = 2$ is neither H^+-eigenvalue nor Z^+-eigenvalue of $\mathcal{A}$.

Remark 2 The Example 2 reveals that a Pareto H-eigenvalue (Z-eigenvalue) of a tensor $\mathcal{A}$ may not be its H^+-eigenvalue (Z^+-eigenvalue) even when $\mathcal{A}$ is symmetric.

3 Constrained minimization and Pareto eigenvalue

Let $\mathcal{A}$ be a symmetric tensor of order m and dimension n and $\|x\|_k = (|x_1|^k + |x_2|^k + \cdots + |x_n|^k)^{\frac{1}{k}}$ for $k \geq 1$. Denote by $e^{(i)} = (e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)})^T$ the ith unit vector in $\mathbb{R}^n$, i.e.,

$$e_j^{(i)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \text{ for } i, j \in \{1, 2, \ldots, n\}.$$

We consider the constrained minimization problem

$$\gamma(\mathcal{A}) = \min\{\mathcal{A}x^m;\ x \geq 0 \text{ and } \|x\|_m = 1\}, \tag{3.1}$$

Theorem 3.1 *Let $\mathcal{A}$ be a m-order and n-dimensional symmetric tensor. If*

$$\lambda(\mathcal{A}) = \min\{\lambda; \lambda \textit{ is Pareto } H\textit{-eigenvalue of } \mathcal{A}\},$$

then $\gamma(\mathcal{A}) = \lambda(\mathcal{A})$.

Proof Let λ be a Pareto H-eigenvalue of $\mathcal{A}$. Then there exists a non-zero vector $y \in \mathbb{R}^n$ such that

$$\mathcal{A}y^m = \lambda y^T y^{[m-1]},\ y \geq 0,$$

and so

$$\mathcal{A}y^m = \lambda \sum_{i=1}^n y_i^m = \lambda \|y\|_m^m \text{ and } \|y\|_m > 0. \tag{3.2}$$

Then we have

$$\lambda = \mathcal{A}\left(\frac{y}{\|y\|_m}\right)^m \text{ and } \left\|\frac{y}{\|y\|_m}\right\|_m = 1.$$

From (3.1), it follows that $\gamma(\mathcal{A}) \leq \lambda$. Since λ is arbitrary, we have

$$\gamma(\mathcal{A}) \leq \lambda(\mathcal{A}).$$

Now we show $\gamma(\mathcal{A}) \geq \lambda(\mathcal{A})$. Let $S = \{x \in \mathbb{R}^n; x \geq 0 \text{ and } \|x\|_m = 1\}$. It follows from the continuity of the homogeneous polynomial $\mathcal{A}x^m$ and the compactness of the set S that there exists a $v \in S$ such that

$$\gamma(\mathcal{A}) = \mathcal{A}v^m,\ v \geq 0,\ \|v\|_m = 1. \tag{3.3}$$

Let $g(x) = \mathcal{A}x^m - \gamma(\mathcal{A})x^T x^{[m-1]}$ for all $x \in \mathbb{R}^n$. We claim that for all $x \geq 0$, $g(x) \geq 0$. Suppose not, then there exists non-zero vector $y \geq 0$ such that

$$g(y) = \mathcal{A}y^m - \gamma(\mathcal{A}) \sum_{i=1}^{n} y_i^m < 0,$$

and hence $\gamma(\mathcal{A}) \leq \mathcal{A}(\frac{y}{\|y\|_m})^m < \gamma(\mathcal{A})$, a contradiction. Thus we have

$$g(x) = \mathcal{A}x^m - \gamma(\mathcal{A})x^T x^{[m-1]} \geq 0 \text{ for all } x \in \mathbb{R}^n_+. \tag{3.4}$$

For each $i \in \{1, 2, \ldots, n\}$, we define a one-variable function

$$f(t) = g(v + te^{(i)}) \text{ for all } t \in \mathbb{R}^1.$$

Clearly, $f(t)$ is continuous and $v + te^{(i)} \in \mathbb{R}^n_+$ for all $t \geq 0$. It follows from (3.3) and (3.4) that

$$f(0) = g(v) = 0 \text{ and } f(t) \geq 0 \text{ for all } t \geq 0.$$

From the necessary conditions of extremum of one-variable function, it follows that the right-hand derivative $f'_+(0) \geq 0$, and hence

$$\begin{aligned} f'_+(0) = (e^{(i)})^T \nabla g(v) &= m(e^{(i)})^T \left(\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]}\right) \\ &= m\left(\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]}\right)_i \geq 0. \end{aligned}$$

So we have

$$\left(\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]}\right)_i \geq 0, \text{ for } i \in \{1, 2, \ldots, n\}.$$

Therefore, we obtain

$$f(0) = g(v) = \mathcal{A}v^m - \gamma(\mathcal{A})v^T v^{[m-1]} = 0 \tag{3.5}$$
$$\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]} \geq 0$$
$$v \geq 0 \tag{3.6}$$

Namely, $\gamma(\mathcal{A})$ is a Pareto H-eigenvalue of $\mathcal{A}$, and hence $\gamma(\mathcal{A}) \geq \lambda(\mathcal{A})$, as required. □

It follows from the proof of the inquality $\gamma(\mathcal{A}) \geq \lambda(\mathcal{A})$ in Theorem 3.1 that $\gamma(\mathcal{A})$ is a Pareto H-eigenvalue of $\mathcal{A}$, which implies the existence of Pareto H-eigenvalue of a symmetric tensor $\mathcal{A}$.

Theorem 3.2 *If a m-order and n-dimensional tensor $\mathcal{A}$ is symmetric, then $\mathcal{A}$ has at least one Pareto H-eigenvalue $\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_m = 1}} \mathcal{A}x^m$.*

Since $(x^T x)^{\frac{m}{2}} = \|x\|_2^m$ when $x \in \mathbb{R}^n$, using the same proof techniques as that of Theorem 3.1 with appropriate changes in the inequalities or equalities ($x^T x^{[m-1]}$ and $y^{[m-1]}$ are respectively replaced by $(x^T x)^{\frac{m}{2}}$ and $(y^T y)^{\frac{m-2}{2}} y$). We can obtain the following conclusions about the Pareto Z-eigenvalue of a symmetric tensor $\mathcal{A}$.

Theorem 3.3 *Let $\mathcal{A}$ be a m-order and n-dimensional symmetric tensor. Then $\mathcal{A}$ has at least one Pareto Z-eigenvalue $\mu(\mathcal{A}) = \min_{\substack{x\geq 0\\ \|x\|_2=1}} \mathcal{A}x^m$. What's more,*

$$\mu(\mathcal{A}) = \min\{\mu; \mu \text{ is Pareto Z-eigenvalue of } \mathcal{A}\}. \tag{3.7}$$

In 1952, Motzkin [10] introduced the concept of copositive matrices, which is an important in applied mathematics and graph theory. A real symmetric matrix A is said to be (i) *copositive* if $x \geq 0$ implies $x^T Ax \geq 0$; (ii) *strictly copositive* if $x \geq 0$ and $x \neq 0$ implies $x^T Ax > 0$. Recently, Qi [15] extended this concept to the higher order symmetric tensors and obtained its some nice properties as ones of copositive matrices. Let $\mathcal{A}$ be a real symmetric tensor of order m and dimension n.$\mathcal{A}$ is said to be

(i) *copositive* if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$;
(ii) *strictly copositive* if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$.

Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$. Now we give the equivalent definition of (strict) copositivity of a symmetric tensor in the sense of any norm on $\mathbb{R}^n$ (also see the refereence [17]).

Lemma 3.4 *Let $\mathcal{A}$ be a symmetric tensor of order m and dimension n. Then we have*

(i) $\mathcal{A}$ is copositive if and only if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$ with $\|x\| = 1$;
(ii) $\mathcal{A}$ is strictly copositive if and only if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n_+$ with $\|x\| = 1$;

Proof (i) When $\mathcal{A}$ is copositive, the conclusion is obvious. Conversely, take $x \in \mathbb{R}^n_+$. If $\|x\| = 0$, then it follows that $x = 0$, and hence $\mathcal{A}x^m = 0$. If $\|x\| > 0$, then let $y = \frac{x}{\|x\|}$. We have $\|y\| = 1$ and $x = \|x\|y$, and so

$$\mathcal{A}x^m = \mathcal{A}(\|x\|y)^m = \|x\|^m \mathcal{A}y^m \geq 0.$$

Therefore, $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$, as required.

Similarly, (ii) is easily proved. □

As the immediate conclusions of the above consequences, it is easy to obtain the following results about the copositive (strictly copositive) tensor $\mathcal{A}$.

Corollary 3.5 *Let $\mathcal{A}$ be a m-order and n-dimensional symmetric tensor. Then*

(a) $\mathcal{A}$ always has Pareto H-eigenvalue. $\mathcal{A}$ is copositive (strictly copositive) if and only if all of its Pareto H-eigenvalues are nonnegative (positive, respectively).
(b) $\mathcal{A}$ always has Pareto Z-eigenvalue. $\mathcal{A}$ is copositive (strictly copositive) if and only if all of its Pareto Z-eigenvalues are nonnegative (positive, respectively).

Now we give an example for solving the constrained minimization problem for homogeneous polynomial and testing the (strict) copositivity of a symmetric tensor $\mathcal{A}$ with the help of the above results.

Example 3 Let $\mathcal{A}$ be a 4-order and 2-dimensional tensor. Suppose that $a_{1111} = a_{2222} = 1$, $a_{1112} = a_{1211} = a_{1121} = a_{2111} = t$, and other $a_{i_1i_2i_3i_4} = 0$. Then

$$\mathcal{A}x^4 = x_1^4 + x_2^4 + 4tx_1^3x_2$$

$$\mathcal{A}x^3 = \begin{pmatrix} x_1^3+3tx_1^2x_2 \\ x_2^3+tx_1^3 \end{pmatrix}$$

When $N = \{1, 2\}$, the principal sub-tensor $\mathcal{A}^N$ is just $\mathcal{A}$ itself. $\lambda_1 = 1 + \sqrt[4]{27}t$ is H^{++}-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$. Then it follows from Theorem 2.1 and Proposition 2.4 that $\lambda_1 = 1 + \sqrt[4]{27}t$ is Pareto H-eigenvalues with Pareto H-eigenvector $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$.

When $N = \{1\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^N = 1$. Obviously, $\lambda_2 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^N$ with a corresponding eigenvector $w = 1$ and $a_{2111}w^3 = t$. Then when $t > 0$, it follows from Theorems 2.1 and 2.2 that $\lambda_2 = 1$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(2)} = (1, 0)^T$; when $t < 0$, $\lambda_2 = 1$ is neither Pareto H-eigenvalue nor Pareto Z-eigenvalue of $\mathcal{A}$.

Similarly, when $N = \{2\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^N = 1$. Clearly, $\lambda_3 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^N$ with a corresponding eigenvector $w = 1$ and $a_{1222}w^3 = 0$, and so $\lambda_3 = 1$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(3)} = (0, 1)^T$.

So the following conclusions are easily obtained:

(i) Let $t < -\frac{1}{\sqrt[4]{27}}$. Then $\lambda_1 = 1 + \sqrt[4]{27}t < 0$ and $\lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$ and $x^{(3)} = (0, 1)^T$, respectively. It follows from Theorems 3.1 and 3.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_3\} = 1 + \sqrt[4]{27}t < 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = \left(\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}}\right)^T$. It follows from Corollary 3.5 that $\mathcal{A}$ is not copositive.

(ii) Let $t = -\frac{1}{\sqrt[4]{27}}$. Then $\lambda_1 = 1 + \sqrt[4]{27}t = 0$ and $\lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = \left(\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}}\right)^T$ and $x^{(3)} = (0, 1)^T$, respectively. It follows from Theorems 3.1 and 3.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_3\} = 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$. It follows from Corollary 3.5 that $\mathcal{A}$ is copositive.

(iii) Let $0 > t > -\frac{1}{\sqrt[4]{27}}$. Clearly, $0 < 1 + \sqrt[4]{27}t < 1$. Then $\lambda_1 = 1 + \sqrt[4]{27}t$ and $\lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$. It follows from Theorems 3.1 and 3.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_3\} = 1 + \sqrt[4]{27}t > 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$. It follows from Corollary 3.5 that $\mathcal{A}$ is strictly copositive.

(iv) Let $t = 0$. Then $\lambda_1 = \lambda_2 = \lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = \left(\sqrt[4]{\tau}, \sqrt[4]{1-\tau}\right)^T$ for all $\tau \in (0, 1)$ and $x^{(2)} = (1, 0)^T$ and

$x^{(3)} = (0, 1)^T$, respectively. It follows from Theorems 3.1 and 3.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_2, \lambda_3\} = 1 > 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = (\sqrt[4]{\tau}, \sqrt[4]{1-\tau})^T$ or $x^{(2)} = (1, 0)^T$ or $x^{(3)} = (0, 1)^T$. It follows from Corollary 3.5 that $\mathcal{A}$ is strictly copositive.

(v) Let $t > 0$. Then $\lambda_1 = 1 + \sqrt[4]{27}t$ and $\lambda_2 = \lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$ and $x^{(2)} = (1, 0)^T$ and $x^{(3)} = (0, 1)^T$, respectively. It follows from Theorems 3.1 and 3.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_2, \lambda_3\} = 1 > 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(2)} = (1, 0)^T$ or $x^{(3)} = (0, 1)^T$. It follows from Corollary 3.5 that $\mathcal{A}$ is strictly copositive.

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