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NONNEGATIVE TENSOR FACTORIZATION, COMPLETELY POSITIVE TENSORS, AND A HIERARCHICAL ELIMINATION ALGORITHM*

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Abstract. Nonnegative tensor factorization has applications in statistics, computer vision, exploratory multiway data analysis, and blind source separation. A symmetric nonnegative tensor, which has an exact symmetric nonnegative factorization, is called a completely positive tensor. This concept extends the concept of completely positive matrices. A classical result in the theory of completely positive matrices is that a symmetric, diagonally dominated nonnegative matrix is a completely positive matrix. In this paper, we introduce strongly symmetric tensors and show that a symmetric tensor has a symmetric binary decomposition if and only if it is strongly symmetric. Then we show that a strongly symmetric, hierarchically dominated nonnegative tensor is a completely positive tensor, and present a hierarchical elimination algorithm for checking this. Numerical examples are given to illustrate this. Some other properties of completely positive tensors are discussed. In particular, we show that the completely positive tensor cone and the co-positive tensor cone of the same order are dual to each other.

Key words. nonnegative tensor factorization, completely positive tensor, eigenvalues, dominance properties, copositive tensor, strongly symmetric tensor, hierarchical dominance, hierarchical elimination algorithm

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1. Introduction.

Nonnegative tensor factorization has applications in statistics, computer vision, exploratory multiway data analysis, and blind source separation [3, 13]. In the literature, nonnegative matrix factorization refers to an approximation factorization [2, 5, 9]. A symmetric nonnegative matrix, which has an exact symmetric nonnegative factorization, is called a completely positive matrix [1, 6, 7, 14]. In this paper, as an extension of completely positive matrices, we introduce completely positive tensors and study their properties.

Let $\mathcal{A} = (a_{i_1 \ldots i_m})$ be a real mth order n-dimensional tensor. Denote the set of all nonnegative vectors in $\Re^n$ by $\Re^n_+$. For any vector $u \in \Re^n$, u^m is a rank-one mth order symmetric n-dimensional tensor $u^m = (u_{i_1} \ldots u_{i_m})$. If

$$\mathcal{A} = \sum_{k=1}^{r} \left(u^{(k)} \right)^m , \tag{1}$$

where $u^{(k)} \in \Re^n_+$ for $k = 1, \cdots, r$, then $\mathcal{A}$ is called a *completely positive* tensor. The minimum value of r is called the *CPrank* of $\mathcal{A}$. The concepts of completely positive

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tensors and their CPranks extend the concepts of completely positive matrices and their CPranks [1, 6, 7, 14].

A classical result in the theory of completely positive matrices is that a symmetric, diagonally dominated nonnegative matrix is a completely positive matrix [1, Theorem 2.5], [7]. This forms a checkable condition for a completely positive matrix. The main result of this paper is that a strongly symmetric, hierarchically dominated nonnegative tensor is a completely positive tensor. This extends the above classical result. We present a hierarchical elimination algorithm for checking this. Some other properties of completely positive tensors are also discussed.

The rest of this paper is distributed as follows. We discuss some properties of completely positive tensors in the next section. In section 3, we show that a strongly symmetric, hierarchically dominated nonnegative tensor is a completely positive tensor, and present a hierarchical elimination algorithm for checking this. Some numerical examples are given in section 4. We make some final remarks in section 5.

For a vector $x \in \Re^n$, denote $\mathrm{supp}(x) = \{i : 1 \leq i \leq n, x_i \neq 0\}$. For a finite set S, $|S|$ denotes its cardinality.

2. Properties of a completely positive tensor.

This section is divided to four subsections.

The Hadamard product of completely positive matrices is completely positive [1, Corollary 2.2]. The principal submatrices of a completely positive matrix are also completely positive [1, Proposition 2.4]. We extend these results to completely positive tensors in subsection 2.1.

The eigenvalues of a completely positive matrix are always nonnegative, as a completely positive matrix is positive semidefinite. In subsection 2.2, after summarizing some necessary knowledge about eigenvalues of tensors, we prove that the H-eigenvalues of a completely positive tensor are always nonnegative. We further show that when the order m is even, the Z-eigenvalues of a completely positive tensor are all nonnegative, while when the order m is odd, a Z-eigenvector associated with a positive (negative) Z-eigenvalue of a completely positive tensor is always nonnegative (nonpositive).

In subsection 2.3, we prove some dominance properties which the entries of a completely positive tensor must obey. These properties form some checkable necessary conditions for a completely positive tensor.

It is well-known that the completely positive matrix cone and the co-positive matrix cone are dual to each other [1, Theorem 2.3]. Recently, motivated by the study of spectral hypergraph theory, Qi [11] introduced co-positive tensors. In subsection 2.4, we show that the completely positive tensor cone and the co-positive tensor cone of the same order are dual to each other.

2.1. Hadamard product and principal subtensors.

Let $x = (x_1, \ldots, x_n)^\top, y = (y_1, \ldots, y_n)^\top \in \Re^n$. Then the Hadarmard product of x and y is $x \circ y = (x_1 y_1, \ldots, x_n y_n)^\top \in \Re^n$. Let $\mathcal{A} = (a_{i_1 \ldots i_m})$ and $\mathcal{B} = (b_{i_1 \ldots i_m})$ be two real mth order n-dimensional tensors. Then their Hadamard product is a real mth order n-dimensional tensor $\mathcal{A} \circ \mathcal{B} = (a_{i_1 \ldots i_m} b_{i_1 \ldots i_m})$ [12]. We have the following proposition.

PROPOSITION 1. *The Hadamard product of two completely positive tensors is completely positive.*

Proof. Assume that $\mathcal{A}$ and $\mathcal{B}$ are two mth order n-dimensional tensors. Then we may assume that

$$\mathcal{A} = \sum_{k=1}^{r} \left(u^{(k)}\right)^m \quad \text{and} \quad \mathcal{B} = \sum_{j=1}^{p} \left(v^{(j)}\right)^m,$$

where $u^{(k)}, v^{(j)} \in \Re^n_+$ for $k = 1, \dots, r$ and $j = 1, \dots, p$. Then for $i_1, \dots, i_m = 1, \dots, n$, we have $a_{i_1 \dots i_m} = \sum_{k=1}^r u^{(k)}_{i_1} \dots u^{(k)}_{i_m}$ and $b_{i_1 \dots i_m} = \sum_{j=1}^p v^{(j)}_{i_1} \dots v^{(j)}_{i_m}$. Thus, for $i_1, \dots, i_m = 1, \dots, n$,

$$a_{i_1 \dots i_m} b_{i_1 \dots i_m} = \sum_{k=1}^{r} \sum_{j=1}^{p} \left(u^{(k)}_{i_1} v^{(j)}_{i_1}\right) \dots \left(u^{(k)}_{i_m} v^{(j)}_{i_m}\right).$$

This implies that

$$\mathcal{A} \circ \mathcal{B} = \sum_{k=1}^{r} \sum_{j=1}^{p} \left(u^{(k)} \circ v^{(j)}\right)^m,$$

where $u^{(k)} \circ v^{(j)} \in \Re^n_+$ for $k = 1, \dots, r$ and $j = 1, \dots, p$. Hence, $\mathcal{A} \circ \mathcal{B}$ is also completely positive. □

Let S be a nonempty subset of $\{1, 2, \dots, n\}$ and $s = |S|$. Let $x \in \Re^n$. Then we use $x(S)$ to denote the subvector of x, such that $x(S) \in \Re^s$ and its components are $x_i, i \in S$. Let $\mathcal{A} = (a_{i_1 \dots i_m},)$ be a real mth order n-dimensional tensor. We use $\mathcal{A}(S)$ to denote a real mth order s-dimensional tensor such that its entries are $a_{i_1 \dots i_m}, i_1, \dots, i_m \in S$, and call $\mathcal{A}(S)$ a principal subtensor of $\mathcal{A}$ [10].

PROPOSITION 2. *All the principal subtensors of a completely positive tensor are completely positive.*

Proof. Let $\mathcal{A}$ be a completely positive tensor expressed by (1). Let S be a nonempty subset of $\{1, 2, \dots, n\}$ and $s = |S|$. Then we have

$$\mathcal{A}(S) = \sum_{k=1}^{r} \left(u^{(k)}(S)\right)^m,$$

where $u^{(k)}(S) \in \Re^n_+$ for $k = 1, \dots, r$. Thus, $\mathcal{A}(S)$ is also completely positive. □

2.2. Eigenvalues. Let $\mathcal{A} = (a_{i_1 \dots i_m})$ be a real mth order n-dimensional tensor, and $x \in C^n$. Then

$$\mathcal{A}x^m = \sum_{i_1, \dots, i_m = 1}^{n} a_{i_1 \dots i_m} x_{i_1} \dots x_{i_m},$$

and $\mathcal{A}x^{m-1}$ is a vector in C^n, with its ith component defined by

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2, \dots, i_m = 1}^{n} a_{i i_1 \dots i_m} x_{i_2} \dots x_{i_m}.$$

Let s be a positive integer. Then $x^{[s]}$ is a vector in C^n, with its ith component defined by x_i^s. We say that $\mathcal{A}$ is symmetric if its entries $a_{i_1, \dots, i_m}$ are invariant for any permutation of the indices. If $\mathcal{A}x^m \geq 0$ for all $x \in \Re^n$, then we say that $\mathcal{A}$ is positive semidefinite. Clearly, only when m is even, a nonzero tensor A can be positive semidefinite.

The following definitions of eigenvalues, H-eigenvalues, E-eigenvalues, and Z-eigenvalues were introduced in [10].

If $x \in C^n$, $x \neq 0$, $\lambda \in C$, x, and λ satisfy

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{2}$$

then we call λ an *eigenvalue* of $\mathcal{A}$, and x its corresponding *eigenvector*. By (2), if λ is an eigenvalue of $\mathcal{A}$ and x is its corresponding eigenvector, then

$$\lambda = \frac{(\mathcal{A}x^{m-1})_j}{x_j^{m-1}}$$

for some j with $x_j \neq 0$. In particular, if x is real, then λ is also real. In this case, we say that λ is an *H-eigenvalue* of $\mathcal{A}$ and x is its corresponding *H-eigenvector*.

We say a complex number λ is an *E-eigenvalue* of A if there exists a complex vector x such that

$$\begin{cases} Ax^{m-1} = \lambda x, \\ x^T x = 1. \end{cases} \tag{3}$$

In this case, we say that x is an E-eigenvector of the tensor A associated with the E-eigenvalue λ. By (3), if λ is an E-eigenvalue of $\mathcal{A}$ and x is its E-corresponding eigenvector, then

$$\lambda = \mathcal{A}x^m.$$

Thus, if x is real, then λ is also real. In this case, we say that λ is an *Z-eigenvalue* of $\mathcal{A}$ and x is its corresponding *Z-eigenvector*.

By [10, Theorem 5], we have the following proposition.

PROPOSITION 3. *A real mth order n-dimensional symmetric tensor A always has Z-eigenvalues. If m is even, then $\mathcal{A}$ always has at least one H-eigenvalue. When m is even, the following three statements are equivalent:*

(a) *$\mathcal{A}$ is positive semidefinite;*

(b) *all of the H-eigenvalues of $\mathcal{A}$ are nonnegative;*

(c) *all of the Z-eigenvalues of $\mathcal{A}$ are nonnegative.*

If all the entries of $\mathcal{A}$ are nonnegative, then we say that $\mathcal{A}$ is a nonnegative tensor. By (1), a completely positive tensor is a symmetric nonnegative tensor. By [15], a nonnegative tensor has at least one H-eigenvalue, which is the largest modulus of its eigenvalues.

We now have the following theorem on H-eigenvalues of a completely positive tensor.

THEOREM 1. *Suppose that $\mathcal{A} = (a_{i_1 \dots i_m})$ is an mth order n-dimensional completely positive tensor, expressed by* (1), *with $m \geq 2$. Then the H-eigenvalues of $\mathcal{A}$ are always nonnegative.*

Proof. First, assume that m is even. For any $x \in \Re^n$, we have

$$\mathcal{A}x^m = \sum_{k=1}^{r} \left(u^{(k)}\right)^m x^m = \sum_{k=1}^{r} \left[\left(u^{(k)}\right)^\top x\right]^m \geq 0.$$

Thus, $\mathcal{A}$ is positive semidefinite. By Proposition 3, all of the H-eigenvalues are nonnegative.

Now assume that m is odd. By the discussion before this theorem, $\mathcal{A}$ has at least one H-eigenvalue. Suppose that λ is an H-eigenvalue of $\mathcal{A}$, with an H-eigenvector x. Then $x \in \Re^n, x \neq 0$. By the definition of an H-eigenvalue and H-eigenvector, we have

$$\lambda x^{[m-1]} = \mathcal{A}x^{m-1} = \sum_{k=1}^{r} \left(u^{(k)}\right)^m x^{m-1} = \sum_{k=1}^{r} \left[\left(u^{(k)}\right)^\top x\right]^{m-1} u^{(k)} \geq 0.$$

Thus, $\lambda \geq 0$. This completes the proof. □

By (3), when m is odd, if λ is a Z-eigenvalue of a tensor $\mathcal{A}$ with a Z-eigenvector x, then $-\lambda$ is a Z-eigenvalue of a tensor $\mathcal{A}$ with a Z-eigenvector $-x$. Hence, when m is odd, we cannot expect that the Z-eigenvalues of a completely positive tensor are always nonnegative. However, in this case, we may get strong properties of Z-eigenvectors.

THEOREM 2. *Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is an mth order n-dimensional completely positive tensor, expressed by* (1)*, with $m \geq 2$. When the order m is even, the Z-eigenvalue of a completely positive tensor are all nonnegative. When the order m is odd, a Z-eigenvector associated with a positive (negative) Z-eigenvalue of a completely positive tensor is always nonnegative (nonpositive).*

Proof. The proof of the case that m is even is similar to the first part of the proof of Theorem 1.

Now assume that m is odd. Suppose that λ is a Z-eigenvalue of $\mathcal{A}$, with an Z-eigenvector x. By the definition of a Z-eigenvalue and Z-eigenvector, we have

$$\lambda x = \mathcal{A}x^{m-1} = \sum_{k=1}^{r} \left(u^{(k)}\right)^m x^{m-1} = \sum_{k=1}^{r} \left[\left(u^{(k)}\right)^\top x\right]^{m-1} u^{(k)} \geq 0.$$

Thus, if $\lambda > 0$, then $x \geq 0$, and if $\lambda < 0$, then $x \leq 0$. This completes the proof. □

2.3. Dominance properties. Denote $\mathcal{I} = \{(i_1, \ldots, i_m) : 1 \leq i_k \leq n, k = 1, \ldots, m\}$. For $(i_1, \ldots, i_m) \in \mathcal{I}$, let $[(i_1, \ldots, i_m)]$ be the set of all the distinct members in $\{i_1, \cdots, i_m\}$. For example, $[(1, 1, 4, 5)] = \{1, 4, 5\}$.

Let $(i_1, \ldots, i_m), (j_1, \ldots, j_m) \in \mathcal{I}$. We say that $(i_1, \ldots, i_m)$ is dominated by $(j_1, \ldots, j_m)$, and denote $(i_1, \ldots, i_m) \preceq (j_1, \ldots, j_m)$ if $[(i_1, \ldots, i_m)] \subseteq [(j_1, \ldots, j_m)]$. We say that $(i_1, \ldots, i_m)$ is similar to $(j_1, \ldots, j_m)$, and denote $(i_1, \ldots, i_m) \sim (j_1, \ldots, j_m)$ if $[(i_1, \ldots, i_m)] = [(j_1, \ldots, j_m)]$.

We have the following dominance property for a completely positive tensor.

THEOREM 3. *Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is an mth order n-dimensional completely positive tensor, expressed by* (1)*, with $m \geq 2$. If $(i_1, \ldots, i_m) \preceq (j_1, \ldots, j_m)$ and $a_{j_1 \ldots j_m} \neq 0$, then $a_{i_1 \ldots i_m} > 0$.*

Proof. We have that

$$0 \neq a_{j_1, \ldots, j_m} = \sum_{k=1}^{r} u_{j_1}^{(k)} \ldots u_{j_m}^{(k)}.$$

Since $u^{(k)} \in \Re^n_+$ for $k = 1, \ldots, r$, at least for one $k = \bar{k}$, $u_{j_1}^{(\bar{k})} > 0, \ldots, u_{j_m}^{(\bar{k})} > 0$. Since $\{i_1, \ldots, i_m\} \subseteq \{j_1, \ldots, j_m\}$, this implies that $u_{i_1}^{(\bar{k})} > 0, \ldots, u_{i_m}^{(\bar{k})} > 0$. Therefore,

$$a_{i_1, \ldots, i_m} = \sum_{k=1}^{r} u_{i_1}^{(k)} \ldots u_{i_m}^{(k)} > 0.$$

This completes the proof. ∎

Corollary 1. *Suppose that* $\mathcal{A} = (a_{i_1\dots i_m})$ *is an mth order n-dimensional completely positive tensor, expressed by* (1), *with* $m \ge 2$. *If* $(i_1,\dots,i_m) \sim (j_1,\dots,j_m)$, *then* $a_{j_1\dots j_m} = 0$ *if and only if* $a_{i_1\dots i_m} = 0$.

When $m = 2$, this property can be derived from the symmetric property of the matrix $\mathcal{A}$. When $m > 2$, this property cannot be derived from the symmetric property of the tensor $\mathcal{A}$. For example, for a third order completely positive tensor $\mathcal{A} = (a_{ijk})$, we have $a_{iij} = a_{ijj}$ for all i and j, satisfying $1 \le i, j \le n$. But this is not true for a general third order symmetric tensor. This motivates us to introduce strongly symmetric tensors in section 3.

Suppose that $(j_1,\dots,j_m) \in \mathcal{I}$ and $I = \{(i_1^{(1)},\dots,i_m^{(1)}),\dots,(i_1^{(s)},\dots,i_m^{(s)})\} \subseteq \mathcal{I}$. Assume that $(i_1^{(p)},\dots,i_m^{(p)}) \preceq (j_1,\dots,j_m)$ for $p = 1,\dots,s$, and for any index $i \in \{j_1,\dots,j_m\}$, if it appears t times in $\{j_1,\dots,j_m\}$, then it appears in I st times. Then we call I an *s-duplicate* of $(j_1,\dots,j_m)$.

We have the following strong dominance property for a completely positive tensor.

Theorem 4. *Suppose that* $\mathcal{A} = (a_{i_1\dots i_m})$ *is an mth order n-dimensional completely positive tensor, expressed by* (1), *with* $m \ge 2$. *Assume that*

$$I = \left\{\left(i_1^{(1)},\dots,i_m^{(1)}\right),\dots,\left(i_1^{(s)},\dots,i_m^{(s)}\right)\right\}$$

is an s-duplicate of $(j_1,\dots,j_m) \in \mathcal{I}$. *Then*

$$\frac{1}{s}\sum_{p=1}^{s} a_{i_1^{(p)}\dots i_m^{(p)}} \ge a_{j_1\dots j_m}.$$

Proof. We have that

$$\begin{aligned}\frac{1}{s}\sum_{p=1}^{s} a_{i_1^{(p)}\dots i_m^{(p)}} &= \sum_{k=1}^{r}\frac{1}{s}\sum_{p=1}^{s} u^{(k)}_{i_1^{(p)}}\dots u^{(k)}_{i_m^{(p)}} \ge \sum_{k=1}^{r}\left(\prod_{p=1}^{s} u^{(k)}_{i_1^{(p)}}\dots u^{(k)}_{i_m^{(p)}}\right)^{\frac{1}{s}}\\ &= \sum_{k=1}^{r} u^{(k)}_{j_1}\dots u^{(k)}_{j_m} = a_{j_1\dots j_m},\end{aligned}$$

where the inequality is due to the fact that the geometric mean of some positive numbers is never greater than their arithmetic mean. This completes the proof. ∎

Corollary 2. *Suppose that* $\mathcal{A} = (a_{i_1\dots i_m})$ *is an mth order n-dimensional completely positive tensor, expressed by* (1), *with* $m \ge 2$. *Assume that* $(j_1,\dots,j_m) \in \mathcal{I}$. *Then*

$$\frac{1}{m}\sum_{p=1}^{m} a_{j_p\dots j_p} \ge a_{j_1\dots j_m}.$$

2.4. The completely positive tensor cone and the co-positive tensor cone. Denote the set of all mth order n-dimensional completely positive tensors by $CP_{m,n}$. By (1), it is easy to see that $CP_{m,n}$ is a closed convex cone. Suppose that $\mathcal{B}$ is a real mth order nth dimensional symmetric tensor. If for all $x \in \Re^n_+$, we have $\mathcal{B}x^m \ge 0$, then $\mathcal{B}$ is called a *co-positive tensor* [11]. Denote the set of all mth order n-dimensional co-positive tensors by $COP_{m,n}$. Then, it is also easy to see that $COP_{m,n}$ is a closed convex cone. When $m = 2$, a classical result is that the completely positive matrix cone and the co-positive matrix cone are dual to each other [1, Theorem 2.3]. We now extend this result to the completely positive tensor cone and the co-positive tensor cone.

Let $\mathcal{A} = (a_{i_1 \dots i_m})$ and $\mathcal{B} = (b_{i_1 \dots i_m})$ be two real mth order n-dimensional symmetric tensors. Their inner product is defined as

$$\mathcal{A} \bullet \mathcal{B} = \sum_{i_1 \dots i_m = 1}^{n} a_{i_1 \dots i_m} b_{i_1, \cdots, i_m}.$$

THEOREM 5. *Let $m \geq 2$ and $n \geq 1$. Then $CP_{m,n}$ and $COP_{m,n}$ are dual to each other.*

Proof. Suppose that $\mathcal{B}$ is an mth order n-dimensional co-positive tensor. For any $\mathcal{A} \in CP_{m,n}$, by definition, we may assume that $\mathcal{A}$ can be expressed by (1). Since $\mathcal{B}$ is a co-positive tensor, by definition, $\mathcal{B}\left(u^{(k)}\right)^m \geq 0$, for $k = 1, \dots, r$. Thus,

$$\mathcal{A} \bullet \mathcal{B} = \sum_{k=1}^{r} \mathcal{B}\left(u^{(k)}\right)^m \geq 0.$$

Thus, $\mathcal{B}$ is in the dual cone of $CP_{m,n}$.

On the other hand, assume that $\mathcal{B}$ is in the dual cone of $CP_{m,n}$. Let $x \in \Re^n_+$. Then x^m is an m order n-dimensional completely positive tensor, i.e., $x \in CP_{m,n}$. We have $\mathcal{B}x^m = \mathcal{B} \bullet x^m \geq 0$. This shows that $\mathcal{B}$ is a co-positive tensor.

Together, we see that $CP_{m,n}$ and $COP_{m,n}$ are dual to each other. ∎

3. A checkable sufficient condition for completely positive tensors.

This section is divided into two subsections.

In subsection 3.1, we introduce strongly symmetric tensors and show that a symmetric tensor is strongly symmetric if and only if it has a symmetric binary decomposition. We present a hierarchical elimination algorithm for checking this.

In subsection 3.2, we further define strongly symmetric, hierarchically dominated nonnegative tensors and show that a strongly symmetric, hierarchically dominated nonnegative tensor is a completely positive tensor. We show that the hierarchical elimination algorithm given in subsection 3.1 can be used to check this condition, too.

3.1. Strongly symmetric tensors.

Suppose that $\mathcal{A} = (a_{i_1 \dots i_m})$ is a real mth order n-dimensional tensor. If for any $(i_1, \dots, i_m) \sim (j_1, \dots, j_m)$, $(i_1, \dots, i_m), (j_1, \dots, j_m) \in \mathcal{I}$, we have $a_{i_1 \dots i_m} = a_{j_1 \dots j_m}$, then we say that $\mathcal{A}$ is a *strongly symmetric* tensor. Clearly, a strongly symmetric tensor is a symmetric tensor. It is also clear that a linear combination of strongly symmetric tensors is still a strongly symmetric tensor. Thus, the set of all real mth order n-dimensional strongly symmetric tensors is a linear space.

Let $\mathcal{A} = (a_{i_1 \dots i_m})$ be a real mth order n-dimensional symmetric tensor. If

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k \left(v^{(k)}\right)^m, \tag{4}$$

where α_k are real numbers and $v^{(k)}$ are binary vectors in $\{0, 1\}^n$ for $k = 1, \dots, r$, then we say that $\mathcal{A}$ has a *symmetric binary decomposition*, which is not a nonnegative tensor factorization, but a general symmetric tensor decomposition [4, 8].

It is easy to show the following proposition.

PROPOSITION 4. *Suppose that $\mathcal{A} = (a_{i_1\ldots i_m})$ is a real mth order n-dimensional tensor with a symmetric binary decomposition. Then $\mathcal{A}$ is strongly symmetric.*

Proof. Suppose that $\mathcal{A} = (a_{i_1\ldots i_m})$ is expressed by (4). Assume that $(i_1 \ldots i_m) \sim (j_1 \ldots j_m)$. Then

$$\begin{aligned} a_{i_1\ldots i_m} &= \sum \left\{\alpha_k : (i_1, \ldots, i_m) \preceq \mathrm{supp}\left(v^{(k)}\right)\right\} \\ &= \sum \left\{\alpha_k : (j_1, \ldots, j_m) \preceq \mathrm{supp}\left(v^{(k)}\right)\right\} = a_{j_1\ldots j_m}. \end{aligned}$$

This completes the proof. ∎

For $k = 1, \ldots, m$, let

$$\mathcal{I}_k = \{(i_1, \ldots, i_m) \in \mathcal{I} : |[(i_1, \ldots, i_m)]| = k\}.$$

Then $\mathcal{I}_1, \ldots, \mathcal{I}_m$ are disjoint to each other and form a partition of $\mathcal{I}$.

In $\mathcal{I}_k$, there are some members similar to each other. For each class of similar members in I_k, we wish to pick only one representative member $(i_1, \ldots, i_m)$ such that only the last index is repeated, i.e., $i_1 < i_2 < \cdots < i_k = \cdots = i_m$. Hence, for $k = 1, \ldots, m$, let

$$\mathcal{I}_{k+} = \{(i_1, \ldots, i_{k-1}, i_k, i_k, \ldots, i_k) \in \mathcal{I}_k : 1 \le i_1 < i_2 \cdots < i_k \le n\}.$$

Then $\mathcal{I}_{k+}$ is the "representative" set of $\mathcal{I}_k$ in the sense that any member in $\mathcal{I}_k$ is similar to a member of $\mathcal{I}_{k+}$ and no two members in $\mathcal{I}_{k+}$ are similar.

Suppose that $\mathcal{A} = (a_{i_1\ldots i_m})$ is a real mth order n-dimensional tensor. For $k = 1, \ldots, m$, let

$$\mathcal{I}_{k+}(\mathcal{A}) = \{(i_1, \ldots, i_k, i_k, \ldots, i_k) \in \mathcal{I}_{k+} : a_{i_1\ldots i_k i_k \ldots i_k} \neq 0\}.$$

We now construct a hierarchical elimination algorithm to obtain symmetric binary decomposition of a strongly symmetric tensor. Suppose that $\mathcal{A} = (a_{i_1\ldots i_m})$ is a real mth order n-dimensional strongly symmetric tensor.

ALGORITHM 1.

Step 0. *Let $k = 0$ and $\mathcal{A}^{(0)} = \left(a^{(0)}_{i_1\ldots i_m}\right)$ be defined by $\mathcal{A}^{(0)} = \mathcal{A}$.*

Step 1. *For any $e = (i_1, \ldots, i_{m-k}, \ldots, i_{m-k}) \in \mathcal{I}_{(m-k)+}\left(\mathcal{A}^{(k)}\right)$, let $v^e \in \Re^n_+$ be a binary vector such that $v^e_{i_1} = \cdots = v^e_{i_{m-k}} = 1$ and $v^e_i = 0$ if $i \notin \{i_1, \ldots, i_{m-k}\}$.*

Let $\mathcal{A}^{(k+1)} = \left(a^{(k+1)}_{i_1\ldots i_m}\right)$ be defined by

$$\begin{aligned} &\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} \\ &-\sum \left\{a^{(k)}_{i_1\ldots i_{m-k}\ldots i_{m-k}} \left(v^e\right)^m : e = (i_1, \ldots, i_{m-k}, \ldots, i_{m-k}) \in \mathcal{I}_{(m-k)+}\left(\mathcal{A}^{(k)}\right)\right\}. \end{aligned} \tag{5}$$

Step 2. *Let $k = k + 1$. If $k = m$, stop. Otherwise, go to Step* 1.

THEOREM 6. *Suppose that $\mathcal{A} = (a_{i_1\ldots i_m})$ is a real mth order n-dimensional strongly symmetric tensor. Then we have $\mathcal{A}^{(m)} = 0$ in Algorithm* 1, *i.e., we have*

$$\mathcal{A} = \sum_{k=0}^{m-1} \sum \left\{a^{(k)}_{i_1\ldots i_{m-k}\ldots i_{m-k}} \left(v^e\right)^m : e = (i_1, \ldots, i_{m-k}, \ldots, i_{m-k}) \in \mathcal{I}_{(m-k)+}\left(\mathcal{A}^{(k)}\right)\right\}. \tag{6}$$

Thus, a symmetric tensor has a symmetric binary decomposition if and only if it is strongly symmetric.

Proof. For $k = 1, \ldots, m$, we now show by induction that $\mathcal{A}^{(k)}$ is strongly symmetric, and $\mathcal{I}_{(m-p)+}(\mathcal{A}^{(k)}) = \emptyset$ for $p = 0, \ldots, k-1$.

By Step 0 and the assumption, $\mathcal{A}^{(0)}$ is strongly symmetric.

For $k = 0, \ldots, m-1$, assume that $\mathcal{A}^{(k)}$ is strongly symmetric, and $\mathcal{I}_{(m-p)+}(\mathcal{A}^{(k)}) = \emptyset$ for $p = 0, \ldots, k-1$ if $k \geq 1$. By (5) and Proposition 4, $\mathcal{A}^{(k+1)}$ is a linear combination of strongly symmetric tensors, thus also a strongly symmetric tensor. As in the iteration $|\mathrm{supp}(v^e)| = m - k$ for all v^e in (5), $a^{(k+1)}_{i_1 \ldots i_m} = a^{(k)}_{i_1 \ldots i_m} = 0$ if $|[(i_1, \ldots, i_m)]| > m - k$. Thus, $\mathcal{I}_{(m-p)+}(\mathcal{A}^{(k+1)}) = \emptyset$ for $p = 0, \ldots, k-1$. By (5), we also have $\mathcal{I}_{(m-k)+}(\mathcal{A}^{(k+1)}) = \emptyset$. The induction proof is completed.

This shows that $\mathcal{A}^{(m)} = 0$. By this and (5), we have (6). Thus, a strongly symmetric tensor has a symmetric binary decomposition. By this and Proposition 4, the last conclusion also holds. This completes the proof. ∎

3.2. Strongly symmetric, hierarchically dominated tensors. In (6), if all the coefficients $a^{(k)}_{i_1 \ldots i_{m-k} \ldots i_{m-k}}$ are nonnegative, then $\mathcal{A}$ is a completely positive tensor. In this section, we explore a sufficient condition for this.

For $p = 1, \ldots, m-1$, and $q = 1, \ldots, m-p$, for any $(i_1, \ldots, i_p, i_p, \ldots, i_p) \in \mathcal{I}_{p+}$, define

$$\begin{aligned} &\mathcal{J}_q(i_1, \ldots, i_p) \\ &= \left\{ (j_1, \ldots, j_{p+q}, \ldots, j_{p+q}) \in \mathcal{I}_{(p+q)+} : (i_1, \ldots, i_p, \ldots, i_p) \preceq (j_1, \ldots, j_{p+q}, \ldots, j_{p+q}) \right\}. \end{aligned}$$

An mth order n-dimensional strongly symmetric nonnegative tensor $\mathcal{A} = (a_{i_1 \ldots i_m})$ is said to be *hierarchically dominated* if for $p = 1, \ldots, m-1$, and any $(i_1, \ldots, i_p, i_p, \ldots, i_p) \in \mathcal{I}_{p+}$, we have

$$a_{i_1 \ldots i_p i_p \ldots i_p} \geq \sum \left\{ a_{j_1 \ldots j_{p+1} \ldots j_{p+1}} : (j_1, \ldots, j_{p+1}, j_{p+1}, \ldots, j_{p+1}) \in \mathcal{J}_1(i_1, \ldots, i_p) \right\}. \tag{7}$$

Suppose that $\mathcal{A}$ is an mth order n-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. By (7), for $p = 1, \ldots, m-2$, and any $(i_1, \ldots, i_p, i_p, \ldots, i_p) \in \mathcal{I}_{p+}$, we have

$$\begin{aligned} a_{i_1 \ldots i_p i_p \ldots i_p} &\geq \sum \left\{ a_{j_1 \ldots j_{p+1} \ldots j_{p+1}} : (j_1, \ldots, j_{p+1}, j_{p+1}, \ldots, j_{p+1}) \in \mathcal{J}_1(i_1, \ldots, i_p) \right\} \\ &\geq \sum \left\{ \sum \left\{ a_{l_1 \ldots l_{p+2} \ldots l_{p+2}} : (l_1, \ldots, l_{p+2}, \ldots, l_{p+2}) \in \mathcal{J}_1(j_1, \ldots, j_{p+1}) \right\} \right. \\ &\qquad \left. : (j_1, \ldots, j_{p+1}, \ldots, j_{p+1}) \in \mathcal{J}_1(i_1, \ldots, i_p) \right\} \\ &\geq \sum \left\{ a_{l_1 \ldots l_{p+2} \ldots l_{p+2}} : (l_1, \ldots, l_{p+2}, \ldots, l_{p+2}) \in \mathcal{J}_2(i_1, \ldots, i_p) \right\}. \end{aligned}$$

Thus, by induction, we can prove the following proposition.

PROPOSITION 5. *Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is an mth order n-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. Then for $p = 1, \ldots, m-1$, and $q = 1, \ldots, m-p$, for any $(i_1, \ldots, i_p, i_p, \ldots, i_p) \in \mathcal{I}_{p+}$, we have*

$$a_{i_1 \ldots i_p i_p \ldots i_p} \geq \sum \left\{ a_{j_1 \ldots j_{p+q} \ldots j_{p+q}} : (j_1, \ldots, j_{p+q}, \ldots, j_{p+q}) \in \mathcal{J}_q(i_1, \ldots, i_p) \right\}. \tag{8}$$

With this proposition, we can prove the following main theorem of this section.

THEOREM 7. *Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is an mth order n-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. Then $\mathcal{A}^{(k)}$ are nonnegative for $k = 0, \ldots, m-1$, in Algorithm 1. Thus, $\mathcal{A}$ is a completely positive tensor. Thus, a strongly symmetric, hierarchically dominated nonnegative tensor is a completely positive tensor.*

Proof. For $k = 1, \ldots, m-1$, we now show by induction that $\mathcal{A}^{(k)}$ is a strongly symmetric, hierarchically dominated nonnegative tensor.

By Step 0 and the assumption, $\mathcal{A}^{(0)}$ is a strongly symmetric, hierarchically dominated nonnegative tensor.

For $k = 0, \ldots, m-1$, assume that $\mathcal{A}^{(k)}$ is a strongly symmetric, hierarchically dominated nonnegative tensor. We now consider $\mathcal{A}^{(k+1)}$.

By the proof of Theorem 6, $\mathcal{A}^{(k+1)}$ is also strongly symmetric and for $p = 0, \ldots, k$, and any $(i_1, \ldots, i_m) \in \mathcal{I}_{(m-p)+}$, $a^{(k+1)}_{i_1 \ldots i_m} = 0$. By strong symmetry of $\mathcal{A}^{(k+1)}$, for $p = 0, \ldots, k$ and any $(i_1, \ldots, i_m) \in \mathcal{I}_{m-p}$, $a^{(k+1)}_{i_1 \ldots i_m} = 0$.

Now for $p = k+1, \ldots, m-1$, and any $(i_1, \ldots, i_m) \in \mathcal{I}_{(m-p)+}$, by (5),

$$\begin{aligned} & a^{(k+1)}_{i_1 \ldots i_{m-p} \ldots i_{m-p}} \\ &= a^{(k)}_{i_1 \ldots i_{m-p} \ldots i_{m-p}} \\ &- \sum \left\{ a^{(k)}_{l_1 \ldots l_{m-k} \ldots l_{m-k}} : (l_1, \ldots, l_{m-k}, \ldots, l_{m-k}) \in \mathcal{J}_{p-k}(i_1, \ldots, i_{m-p}) \right\}. \end{aligned} \tag{9}$$

By Proposition 5, the right-hand side of (9) is nonnegative. Thus, for $p = k + 1, \ldots, m-1$, and any $(i_1, \ldots, i_m) \in \mathcal{I}_{(m-p)+}$, $a^{(k+1)}_{i_1 \ldots i_{m-p} \ldots i_{m-p}} \geq 0$. By strong symmetry of $\mathcal{A}^{(k+1)}$, for $p = k+1, \ldots, m-1$, and any $(i_1, \ldots, i_m) \in \mathcal{I}_{m-p}$, $a^{(k+1)}_{i_1 \ldots i_{m-p} \ldots i_{m-p}} \geq 0$. This shows that $\mathcal{A}^{(k+1)}$ is nonnegative.

Since $\mathcal{A}^{(k)} = (a^{(k)}_{i_1 \ldots i_m})$ is hierarchically dominated, for $p = 1, \ldots, m-1$, and any $(i_1, \ldots, i_p, \ldots, i_p) \in \mathcal{I}_{p+}$, we have

$$a^{(k)}_{i_1 \ldots i_p \ldots i_p} \geq \sum \left\{ a^{(k)}_{j_1 \ldots j_{p+1} \ldots j_{p+1}} : (j_1, \ldots, j_{p+1}, \ldots, j_{p+1}) \in \mathcal{J}_1(i_1, \ldots, i_p) \right\}. \tag{10}$$

By (9), we have

$$\begin{aligned} & a^{(k+1)}_{i_1 \ldots i_p \ldots i_p} \\ &= a^{(k)}_{i_1 \ldots i_p \ldots i_p} - \sum \left\{ a^{(k)}_{l_1 \ldots l_{m-k} \ldots l_{m-k}} : (l_1, \ldots, l_{m-k}, \ldots, l_{m-k}) \in \mathcal{J}_{m-p-k}(i_1, \ldots, i_p) \right\} \end{aligned} \tag{11}$$

and

$$\begin{aligned} & a^{(k+1)}_{j_1 \ldots j_{p+1} \ldots j_{p+1}} \\ &= a^{(k)}_{j_1 \ldots j_{p+1} \ldots j_{p+1}} \\ &- \sum \left\{ a^{(k)}_{l_1 \ldots l_{m-k} \ldots l_{m-k}} : (l_1, \ldots, l_{m-k}, \ldots, l_{m-k}) \in \mathcal{J}_{m-p-1-k}(j_1, \ldots, j_{p+1}) \right\}. \end{aligned} \tag{12}$$

Comparing (10), (11), and (12), for $p = 1, \ldots, m-1$, and any $(i_1, \ldots, i_p, \ldots, i_p) \in \mathcal{I}_{p+}$, we have

$$a^{(k+1)}_{i_1 \ldots i_p \ldots i_p} \geq \sum \left\{ a^{(k+1)}_{j_1 \ldots j_{p+1} \ldots j_{p+1}} : (j_1, \ldots, j_{p+1}, \ldots, j_{p+1}) \in \mathcal{J}_1(i_1, \ldots, i_p) \right\}.$$

Thus, $\mathcal{A}^{(k+1)}$ is also hierarchically dominated. The induction proof is completed.

Hence, $\mathcal{A}$ is a completely positive tensor. Therefore, a strongly symmetric, hierarchically dominated nonnegative tensor is a completely positive tensor. This completes the proof. ∎

When $m = 2$, Theorem 7 implies Kaykobad's result [7], [1, Theorem 2.5].

COROLLARY 3. *Suppose that $\mathcal{A} = (a_{i_1\ldots i_m})$ is a real mth order n-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. Then the CPrank of $\mathcal{A}$ is not bigger than $\sum_{k=0}^{m-1}\binom{n}{m-k}$.*

Proof. By (6), the CPrank of $\mathcal{A}$ is not bigger than

$$\sum_{k=0}^{m-1}\left|\mathcal{I}_{(m-k)+}\left(\mathcal{A}^{(k)}\right)\right| = \sum_{k=0}^{m-1}\binom{n}{m-k}.$$

This completes the proof. □

4. Numerical examples. In this section, we present some strongly symmetric, hierarchically dominated nonnegative tensors with $m = 3, n = 2$, $m = 3, n = 10$ and $m = 4, n = 10$, and use Algorithm 1 to decompose them.

First, we present a simple example with $m = 3, n = 2$ to show the steps of Algorithm 1.

Example 1. Let $m = 3, n = 2$, $\mathcal{A} \equiv (a_{ijk})$, $a_{111} = 2, a_{121} = 1, a_{211} = 1, a_{221} = 1, a_{112} = 1, a_{122} = 1, a_{212} = 1, a_{222} = 5$. It is clear that $\mathcal{A}$ is a strongly symmetric, hierarchically dominated nonnegative tensor. Thus, only the values of $a_{111}, a_{121}, a_{222}$ are independent.

From Algorithm 1, we have $\mathcal{A}^{(0)} \equiv \left(a_{ijk}^{(0)}\right) = \mathcal{A}$.

When $k = 0$, each member of $\mathcal{I}_{(m-k)+}(\mathcal{A}^{(k)}) \equiv \mathcal{I}_{3+}(\mathcal{A}^{(0)})$ is an index set (i, j, k), such that $i \neq j \neq k \neq i$ and $i, j, k = 1$ or 2. This implies that $\mathcal{I}_{3+}(\mathcal{A}^{(0)}) = \emptyset$, and $\mathcal{A}^{(1)} \equiv \left(a_{ijk}^{(1)}\right) = \mathcal{A}^{(0)}$.

When $k = 1$, from the definition of $\mathcal{I}_{(m-k)+}(\mathcal{A}^{(k)}) = \mathcal{I}_{2+}(\mathcal{A}^{(1)})$, it is easy to see that $\mathcal{I}_{2+}(\mathcal{A}^{(1)}) = \{(1, 2, 2)\}$. Hence, $v^{(1,2,2)} = \binom{1}{1}$. Then $\mathcal{A}^{(2)} \equiv \left(a_{ijk}^{(2)}\right) = \mathcal{A}^{(1)} - ((a_{122}^{(1)})^{\frac{1}{3}} v^{(1,2,2)})^3$. This implies that

$$a_{ijk}^{(2)} = \begin{cases} 1 & (i, j, k) = (1, 1, 1), \\ 4 & (i, j, k) = (2, 2, 2), \\ 0 & \text{otherwise}. \end{cases}$$

When $k = 2$, $\mathcal{I}_{(m-k)+}(\mathcal{A}^{(k)}) \equiv \mathcal{I}_{1+}(\mathcal{A}^{(2)}) = \{(1, 1, 1), (2, 2, 2)\}$, $v^{(1,1,1)} = \binom{1}{0}$, and $v^{(2,2,2)} = \binom{0}{1}$. Thus,

$$\mathcal{A}^{(3)} = \mathcal{A}^{(2)} - ((a_{111}^{(2)})^{\frac{1}{3}} v^{(1,1,1)})^3 - ((a_{222}^{(2)})^{\frac{1}{3}} v^{(2,2,2)})^3 = 0.$$

Then Algorithm 1 stops.

Now we get three vectors. They are

$$v^{(1)} = (a_{122}^{(1)})^{\frac{1}{3}} v^{(1,2,2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v^{(2)} = (a_{111}^{(2)})^{\frac{1}{3}} v^{(1,1,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$v^{(3)} = (a_{222}^{(2)})^{\frac{1}{3}} v^{(2,2,2)} = \begin{pmatrix} 0 \\ 1.5874 \end{pmatrix}.$$

We have

$$\mathcal{A} = \left(v^{(1)}\right)^3 + \left(v^{(2)}\right)^3 + \left(v^{(3)}\right)^3.$$

This shows that $\mathcal{A}$ is a completely positive tensor.

Table 1 indicates the result of this example. From Algorithm 1, we know that all of the nonzero entries of $v^{(i)}$ are the same. Thus, we put the common nonnegative value of the components of $v^{(i)}$ in the first row of Table 1, and put the indices of these nonzero components in the second row of Table 1; e.g., for $v^{(1)}$, $v_1^{(1)} = v_2^{(1)} = 1$. Thus, in the second column, we put 1 in the first row, i.e., the v-row, and $1, 2$ in the second row, i.e., the p-row. On the other hand, $v_1^{(2)} = 1$ and $v_2^{(2)} = 0$. Thus, in the third column, we put 1 in the first row and the second row. Similarly, since $v_2^{(3)} = 1.5874$ and $v_1^{(3)} = 0$, in the fourth column, we put 1.5874 in the first row and 2 in the second row. We see that $v^{(1)}, v^{(2)}, v^{(3)}$ are all nonnegative vectors. Thus, Table 1 also indicates that $\mathcal{A}$ is a completely positive tensor.

Table 1
$n = 2, m = 3$.

v	1	1	1.5874
p	1 2	1	2

In the following, we give the decomposition results of Algorithm 1 for three examples with $m = 3, n = 10$, and three examples with $m = 4, n = 10$. Instead of using a_{ijk} or a_{ijkl}, we use $\mathcal{A}(i, j, k)$ and $\mathcal{A}(i, j, k, l)$ below. As the numbers of nonzero entries are large now, for each similarity class of the index sets, we give only the value of one representative entry. For example, in the first example below, $\mathcal{A}(3, 4, 4) = 1$ implies that $\mathcal{A}(3, 3, 4) = \mathcal{A}(3, 4, 3) = \mathcal{A}(4, 3, 3) = \mathcal{A}(4, 3, 4) = \mathcal{A}(4, 4, 3) = 1$.

Example 2. Here, $\mathcal{A}$ is a strongly symmetric, hierarchically dominated nonnegative tensor. The entries of $\mathcal{A}$, whose index sets are not similar to the index sets of the entries defined below, are zero.

The $m = 3$, $n = 10$ case:

(1) $\mathcal{A}(1,1,1) = 1$, $\mathcal{A}(2,2,2) = 5$, $\mathcal{A}(3,3,3) = 3$, $\mathcal{A}(4,4,4) = 2$, $\mathcal{A}(5,5,5) = 4$, $\mathcal{A}(6,6,6) = 2$, $\mathcal{A}(7,7,7) = 2$, $\mathcal{A}(8,8,8) = 2$, $\mathcal{A}(9,9,9) = 5$, $\mathcal{A}(10,10,10) = 4$, $\mathcal{A}(1,5,5) = 1$, $\mathcal{A}(2,3,3) = 1$, $\mathcal{A}(2,6,6) = 1$, $\mathcal{A}(2,8,8) = 1$, $\mathcal{A}(3,4,4) = 1$, $\mathcal{A}(3,5,5) = 1$, $\mathcal{A}(4,5,5) = 1$, $\mathcal{A}(5,9,9) = 1$, $\mathcal{A}(6,9,9) = 1$, $\mathcal{A}(7,9,9) = 1$, $\mathcal{A}(7,10,10) = 1$, $\mathcal{A}(8,10,10) = 1$, $\mathcal{A}(9,10,10) = 1$, $\mathcal{A}(2,6,9) = 1$, $\mathcal{A}(2,8,10) = 1$, $\mathcal{A}(3,4,5) = 1$, $\mathcal{A}(7,9,10) = 1$.

(2) $\mathcal{A}(1,1,1) = 2$, $\mathcal{A}(2,2,2) = 5$, $\mathcal{A}(3,3,3) = 6$, $\mathcal{A}(4,4,4) = 2$, $\mathcal{A}(5,5,5) = 3$, $\mathcal{A}(8,8,8) = 6$, $\mathcal{A}(9,9,9) = 6$, $\mathcal{A}(10,10,10) = 4$, $\mathcal{A}(1,5,5) = 1$, $\mathcal{A}(1,10,10) = 1$, $\mathcal{A}(2,3,3) = 1$, $\mathcal{A}(2,8,8) = 1$, $\mathcal{A}(2,9,9) = 2$, $\mathcal{A}(2,10,10) = 1$, $\mathcal{A}(3,4,4) = 1$, $\mathcal{A}(3,8,8) = 2$, $\mathcal{A}(3,9,9) = 2$, $\mathcal{A}(4,8,8) = 1$, $\mathcal{A}(5,8,8) = 1$, $\mathcal{A}(5,10,10) = 1$, $\mathcal{A}(8,9,9) = 1$, $\mathcal{A}(9,10,10) = 1$, $\mathcal{A}(1,5,10) = 1$, $\mathcal{A}(2,3,9) = 1$, $\mathcal{A}(2,9,10) = 1$, $\mathcal{A}(3,4,8) = 1$, $\mathcal{A}(3,8,9) = 1$.

(3) $\mathcal{A}(2,2,2) = 4$, $\mathcal{A}(3,3,3) = 6$, $\mathcal{A}(4,4,4) = 7$, $\mathcal{A}(5,5,5) = 4$, $\mathcal{A}(7,7,7) = 4$, $\mathcal{A}(8,8,8) = 6$, $\mathcal{A}(9,9,9) = 4$, $\mathcal{A}(10,10,10) = 3$, $\mathcal{A}(2,3,3) = 1$, $\mathcal{A}(2,4,4) = 1$, $\mathcal{A}(2,5,5) = 1$, $\mathcal{A}(2,8,8) = 1$, $\mathcal{A}(3,4,4) = 1$, $\mathcal{A}(3,5,5) = 1$, $\mathcal{A}(3,7,7) = 1$, $\mathcal{A}(3,8,8) = 2$, $\mathcal{A}(4,5,5) = 2$, $\mathcal{A}(4,7,7) = 1$, $\mathcal{A}(4,9,9) = 1$, $\mathcal{A}(4,10,10) = 1$, $\mathcal{A}(7,8,8) = 1$, $\mathcal{A}(7,9,9) = 1$, $\mathcal{A}(8,9,9) = 1$, $\mathcal{A}(8,10,10) = 1$, $\mathcal{A}(9,10,10) = 1$, $\mathcal{A}(2,3,8) = 1$, $\mathcal{A}(2,4,5) = 1$, $\mathcal{A}(3,4,5) = 1$, $\mathcal{A}(3,7,8) = 1$, $\mathcal{A}(4,7,9) = 1$, $\mathcal{A}(8,9,10) = 1$.

TABLE 2
$n = 10, m = 3$ (1).

v	1	1	1	1	1	1	1	1.2599	1	1	1
p	2 6 9	2 8 10	3 4 5	7 9 10	1 5	2 3	5 9	2	3	4	5
v	1	1	1	1.2599	1.2599						
p	6	7	8	9	10						

TABLE 3
$n = 10, m = 3$ (2).

v	1	1	1	1	1	1	1	1	1.2599	1.4422
p	1 5 10	2 3 9	2 9 10	3 4 8	3 8 9	2 8	5 8	1	2	3
v	1	1	1.2599	1.4422	1.2599					
p	4	5	8	9	10					

The $m = 4, n = 10$ case:

(1) $\mathcal{A}(1,1,1,1) = 1$, $\mathcal{A}(2,2,2,2) = 6$, $\mathcal{A}(4,4,4,4) = 6$, $\mathcal{A}(5,5,5,5) = 2$, $\mathcal{A}(6,6,6,6) = 3$, $\mathcal{A}(7,7,7,7) = 4$, $\mathcal{A}(8,8,8,8) = 8$, $\mathcal{A}(9,9,9,9) = 12$, $\mathcal{A}(10,10,10,10) = 4$, $\mathcal{A}(1,10,10,10) = 1$, $\mathcal{A}(2,4,4,4) = 2$, $\mathcal{A}(2,8,8,8) = 2$, $\mathcal{A}(2,9,9,9) = 2$, $\mathcal{A}(4,8,8,8) = 2$, $\mathcal{A}(4,9,9,9) = 2$, $\mathcal{A}(5,7,7,7) = 1$, $\mathcal{A}(5,9,9,9) = 1$, $\mathcal{A}(6,7,7,7) = 1$, $\mathcal{A}(6,9,9,9) = 1$, $\mathcal{A}(6,10,10,10) = 1$, $\mathcal{A}(7,9,9,9) = 2$, $\mathcal{A}(8,9,9,9) = 3$, $\mathcal{A}(8,10,10,10) = 1$, $\mathcal{A}(9,10,10,10) = 1$, $\mathcal{A}(2,4,8,8) = 1$, $\mathcal{A}(2,4,9,9) = 1$, $\mathcal{A}(2,8,9,9) = 1$, $\mathcal{A}(4,8,9,9) = 1$, $\mathcal{A}(5,7,9,9) = 1$, $\mathcal{A}(6,7,9,9) = 1$, $\mathcal{A}(8,9,10,10) = 1$, $\mathcal{A}(2,4,8,9) = 1$.

(2) $\mathcal{A}(1,1,1,1) = 9$, $\mathcal{A}(2,2,2,2) = 6$, $\mathcal{A}(3,3,3,3) = 8$, $\mathcal{A}(4,4,4,4) = 1$, $\mathcal{A}(5,5,5,5) = 1$, $\mathcal{A}(6,6,6,6) = 4$, $\mathcal{A}(7,7,7,7) = 6$, $\mathcal{A}(8,8,8,8) = 6$, $\mathcal{A}(9,9,9,9) = 9$, $\mathcal{A}(10,10,10,10) = 2$, $\mathcal{A}(1,2,2,2) = 1$, $\mathcal{A}(1,3,3,3) = 2$, $\mathcal{A}(1,5,5,5) = 1$, $\mathcal{A}(1,7,7,7) = 1$, $\mathcal{A}(1,8,8,8) = 2$, $\mathcal{A}(1,9,9,9) = 2$, $\mathcal{A}(2,3,3,3) = 1$, $\mathcal{A}(2,6,6,6) = 2$, $\mathcal{A}(2,7,7,7) = 2$, $\mathcal{A}(3,6,6,6) = 1$, $\mathcal{A}(3,8,8,8) = 2$, $\mathcal{A}(3,9,9,9) = 2$, $\mathcal{A}(4,9,9,9) = 1$, $\mathcal{A}(6,7,7,7) = 1$, $\mathcal{A}(7,9,9,9) = 1$, $\mathcal{A}(7,10,10,10) = 1$, $\mathcal{A}(8,9,9,9) = 2$, $\mathcal{A}(9,10,10,10) = 1$, $\mathcal{A}(1,2,7,7) = 1$, $\mathcal{A}(1,3,8,8) = 1$, $\mathcal{A}(1,3,9,9) = 1$, $\mathcal{A}(1,8,9,9) = 1$, $\mathcal{A}(2,3,6,6) = 1$, $\mathcal{A}(2,6,7,7) = 1$, $\mathcal{A}(3,8,9,9) = 1$, $\mathcal{A}(7,9,10,10) = 1$, $\mathcal{A}(1,3,8,9) = 1$.

(3) $\mathcal{A}(1,1,1,1) = 18$, $\mathcal{A}(2,2,2,2) = 6$, $\mathcal{A}(3,3,3,3) = 4$, $\mathcal{A}(4,4,4,4) = 2$, $\mathcal{A}(5,5,5,5) = 26$, $\mathcal{A}(6,6,6,6) = 18$, $\mathcal{A}(8,8,8,8) = 8$, $\mathcal{A}(9,9,9,9) = 24$, $\mathcal{A}(10,10,10,10) = 8$, $\mathcal{A}(1,5,5,5) = 6$, $\mathcal{A}(1,6,6,6) = 4$, $\mathcal{A}(1,8,8,8) = 2$, $\mathcal{A}(1,9,9,9) = 4$, $\mathcal{A}(1,10,10,10) = 2$, $\mathcal{A}(2,5,5,5) = 2$, $\mathcal{A}(2,6,6,6) = 2$, $\mathcal{A}(2,9,9,9) = 2$, $\mathcal{A}(3,4,4,4) = 1$, $\mathcal{A}(3,9,9,9) = 2$, $\mathcal{A}(3,10,10,10) = 1$, $\mathcal{A}(4,9,9,9) = 1$, $\mathcal{A}(5,6,6,6) = 6$, $\mathcal{A}(5,8,8,8) = 3$, $\mathcal{A}(5,9,9,9) = 7$, $\mathcal{A}(5,10,10,10) = 2$, $\mathcal{A}(6,8,8,8) = 2$, $\mathcal{A}(6,9,9,9) = 4$, $\mathcal{A}(8,9,9,9) = 1$, $\mathcal{A}(9,10,10,10) = 3$, $\mathcal{A}(1,5,6,6) = 2$, $\mathcal{A}(1,5,8,8) = 1$, $\mathcal{A}(1,5,9,9) = 2$, $\mathcal{A}(1,5,10,10) = 1$, $\mathcal{A}(1,6,8,8) = 1$, $\mathcal{A}(1,6,9,9) = 1$, $\mathcal{A}(1,9,10,10) = 1$, $\mathcal{A}(2,5,6,6) = 1$, $\mathcal{A}(2,5,9,9) = 1$, $\mathcal{A}(2,6,9,9) = 1$, $\mathcal{A}(3,4,9,9) = 1$, $\mathcal{A}(3,9,10,10) = 1$, $\mathcal{A}(5,6,8,8) = 1$, $\mathcal{A}(5,6,9,9) = 2$, $\mathcal{A}(5,8,9,9) = 1$, $\mathcal{A}(5,9,10,10) = 1$, $\mathcal{A}(1,5,6,8) = 1$, $\mathcal{A}(1,5,6,9) = 1$, $\mathcal{A}(1,5,9,10) = 1$, $\mathcal{A}(2,5,6,9) = 1$.

The vectors decomposed from Algorithm 1 for these six examples are shown in Tables 2–7, in which v-rows are the values of the nonzero components of the vectors, p-rows are the indices of these nonzero components. They are in the same format as the simple example shown in Table 1.

Table 4
$n = 10, m = 3$ (3).

v	1	1	1	1	1	1	1	1.2599	1.4422
p	2 3 8	2 4 5	3 4 5	3 7 8	4 7 9	8 9 10	4 10	2	3
v	1.4422	1.2599	1.2599	1.4422	1.2599	1			
p	4	5	7	8	9	10			

Table 5
$n = 10, m = 4$ (1).

v	1	1	1	1	1	1	1	1	1	1	1
p	2 4 8 9	5 7 9	6 7 9	8 9 10	1 10	2 4	2 8	2 9	4 8	4 9	6 10
v	1	1.1892	1.1892	1	1	1.1892	1.3161	1.4953	1		
p	8 9	2	4	5	6	7	8	9	10		

Table 6
$n = 10, m = 4$ (2).

v	1	1	1	1	1	1	1	1	1	1	1
p	1 3 8 9	1 2 7	2 3 6	2 6 7	7 9 10	1 3	1 5	1 8	1 9	3 8	3 9
v	1	1	1.3161	1.3161	1.3161	1.1892	1.3161	1.1892	1.3161	1	
p	4 9	8 9	1	2	3	6	7	8	9	10	

Table 7
$n = 10, m = 4$ (3).

v	1	1	1	1	1	1	1	1.3161	1.1892	1
p	1 5 6 8	1 5 6 9	1 5 9 10	2 5 6 9	3 4 9	3 9 10	5 8 9	1 5	1 6	1 8
v	1.1892	1	1	1	1	1.3161	1	1.3161	1	1
p	1 9	1 10	2 5	2 6	2 9	5 6	5 8	5 9	5 10	6 8
v	1.1892	1	1.5651	1.1892	1.1892	1	1.7321	1.5651	1.3161	1.7321
p	6 9	9 10	1	2	3	4	5	6	8	9
v	1.3161									
p	10									

5. Further remarks. In this paper, we studied various properties of completely positive tensors, showed that a strongly symmetric, hierarchically dominated nonnegative tensor is a completely positive tensor, and presented a hierarchical elimination algorithm for checking this. These indicate that a rich theory for completely positive tensors can be established parallel to the theory of completely positive matrices [1, 6, 7, 14]. This theory will be a solid foundation for applications of nonnegative tensor factorization [3, 13]. Further research on topics such as CPranks is needed.

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