Linear Algebra and its Applications 482 (2015) 207–220

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Three dimensional strongly symmetric circulant tensors

Liqun Qi [a,*,1], Qun Wang [a], Yannan Chen [b,2]

[a] *Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong*
[b] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, China*

ARTICLE INFO

Article history:
Received 15 April 2015
Accepted 19 May 2015
Available online 18 June 2015
Submitted by R. Brualdi

MSC:
15A18
15A69

Keywords:
H-eigenvalue
Strongly symmetric tensor
Circulant tensor
Sum of squares
Positive semi-definiteness

ABSTRACT

In this paper, we give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite. We show that this condition can be a sufficient condition for such a tensor to be sum-of-squares in some cases. There are no PNS strongly symmetric circulant tensors to be found in numerical tests.

* Corresponding author.
E-mail addresses: maqilq@polyu.edu.hk (L. Qi), wangqun876@gmail.com (Q. Wang), ynchen@zzu.edu.cn (Y. Chen).
[1] The author's work was partially supported by the Hong Kong Research Grant Council (Grant Nos. PolyU 502111, 501212, 501913 and 15302114).
[2] The author's work was supported by the National Natural Science Foundation of China (Grant No. 11401539) and the Development Foundation for Excellent Youth Scholars of Zhengzhou University (Grant No. 1421315070).

http://dx.doi.org/10.1016/j.laa.2015.05.024
0024-3795/

1. Introduction

Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an mth order n dimensional real symmetric tensor, where m is even, $i_1, \cdots, i_m = 1, \cdots, n$. Entries $a_{i_1 \cdots i_m}$ are real and invariant under any index permutation. Then $\mathcal{A}$ corresponds to a homogeneous polynomial $f(\mathbf{x})$ for $\mathbf{x} \in \Re^n$, defined by

$$f(\mathbf{x}) = \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$

If $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Re^n$, $\mathcal{A}$ is called a positive semi-definite (PSD) tensor, and f is accordingly called a PSD polynomial. If $f(\mathbf{x})$ can be decomposed to a sum of squares of polynomials of degree $\frac{m}{2}$, $\mathcal{A}$ is called a sum-of-squares (SOS) tensor, and f is accordingly called an SOS polynomial. It is clear that an SOS polynomial is positive semi-definite.

The identification of a PSD tensor is NP-hard and is also important both theoretically and practically [1,9,18]. Recently it was discovered that several easily checkable classes of special even order symmetric tensors are PSD, including even order symmetric diagonally dominated tensors [18], even order symmetric B_0 tensors [20], even order Hilbert tensors [23], even order symmetric M tensors [25], even order symmetric double B_0 tensors [11], even order symmetric strong H tensors [12,10], even order strong Hankel tensors [19], even order positive Cauchy tensors [3], etc. However, some kinds of tensors, such as Hankel tensors, are not easy to identify their positive semi-definiteness. So, we turn to find a sufficient and necessary condition of PSD for a new class of structured tensors.

In 1888, Hilbert [9] proved that only in the following three cases, a PSD homogeneous polynomial of degree m in n variables is an SOS polynomial: 1) $m = 2$; 2) $n = 2$; 3) $m = 4$ and $n = 3$. Hilbert proved that in all the other possible combinations of n and even m, there are PSD non-SOS (PNS) homogeneous polynomials.[3] However, Hilbert did not give an explicit example for PNS homogeneous polynomials. The first explicit example for PNS homogeneous polynomials was given by Motzkin [17] in 1967. More examples of PNS homogeneous polynomials can be found in [7,22]. In [13] and [4], sixth order three dimensional Hankel tensors and fourth order four dimensional Hankel tensors were studied respectively. No PNS Hankel tensors were found there.

Are there other special classes of even order symmetric PNS-free tensors, whose positive semi-definiteness is not easily checkable? A good candidate for such PNS-free tensors is the class of even order strongly symmetric circulant tensors.

Strongly symmetric tensors were introduced in [21]. An mth order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called a strongly symmetric tensor if

$$a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}$$

[3] Chesi [6] used the abbreviation PNS for PSD non-SOS in 2007.

as long as $\{i_1, \cdots, i_m\} = \{j_1, \cdots, j_m\}$. Note that a symmetric matrix is a strongly symmetric tensor of order 2. Hence, strongly symmetric tensors are also extensions of symmetric matrices.

Circulant tensors have applications in stochastic process and spectral hypergraph theory [5]. An mth order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called a circulant tensor if

$$a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}$$

as long as $j_l \equiv i_l + 1, (\bmod\ n)$ for $l = 1, \cdots, m$.

In this paper, we consider even order three dimensional strongly symmetric circulant tensors. A general three dimensional strongly symmetric circulant tensor has only three independent entries: the diagonal entry d, the off-diagonal entries of value u with two different indices in $a_{i_1 \cdots i_m}$, and the off-diagonal entries of value c with three different indices in $a_{i_1 \cdots i_m}$. Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an mth order three dimensional strongly symmetric circulant tensor. Here m can be even or odd. We denote

$$a_S \equiv a_{i_1 \cdots i_m}$$

if $S = \{i_1, \cdots, i_m\}$. Then there are seven cases for S: $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$, and $\{1, 2, 3\}$. Since $\mathcal{A}$ is also circulant, we have

$$a_{\{1\}} = a_{\{2\}} = a_{\{3\}}, \qquad \text{and} \qquad a_{\{1,2\}} = a_{\{2,3\}} = a_{\{1,3\}}.$$

Let $d = a_{\{1\}} = a_{\{2\}} = a_{\{3\}}$, $u = a_{\{1,2\}} = a_{\{2,3\}} = a_{\{1,3\}}$ and $c = a_{\{1,2,3\}}$. Then we see that d is the diagonal entry of $\mathcal{A}$: $d = a_{1 \cdots 1} = a_{2 \cdots 2} = a_{3 \cdots 3}$, and an mth order three dimensional strongly symmetric circulant tensor has only three independent entries d, u and c. Thus, we may denote a general three dimensional strongly symmetric circulant tensor $\mathcal{A} = \mathcal{A}(m, d, u, c)$, where m is its order. When the context is clear, we only use $\mathcal{A}$ to denote it.

In our discussion, we need the concept of H-eigenvalues of symmetric tensors, which was introduced in [18] and is closely related to positive semi-definiteness of even order symmetric tensors. Let $\mathcal{T} = (t_{i_1 \cdots i_m})$ be an mth order n dimensional real symmetric tensor and $\mathbf{x} \in \Re^n$. Then $\mathcal{T}\mathbf{x}^{m-1}$ is a vector in $\Re^n$, with its ith component defined as

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \sum_{i_2, \cdots, i_m = 1}^{n} t_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

If there is $\mathbf{x} \in \Re^n, \mathbf{x} \neq \mathbf{0}$ and $\lambda \in \Re$ such that for $i = 1, \cdots, n$,

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \lambda x_i^{m-1},$$

then λ is called an H-eigenvalue of $\mathcal{T}$ and $\mathbf{x}$ is called its associated H-eigenvector. When m is even, H-eigenvalues always exist. $\mathcal{T}$ is PSD if and only if its smallest H-eigenvalue is nonnegative [18]. From now on, we denote the smallest H-eigenvalue of $\mathcal{A}(m,d,u,c)$ as $\lambda_{\min}(m,d,u,c)$.

Now, let $m = 2k$ be even. In Section 2, we show that there are two one-variable functions $M_c(u)$ and $N_c(u)$, such that $M_c(u) \geq N_c(u) \geq 0$, $\mathcal{A}$ is SOS if and only if $d \geq M_c(u)$, and $\mathcal{A}$ is PSD if and only if $d \geq N_c(u)$. If $M_c(u) = N_c(u)$, then three dimensional strongly symmetric PNS circulant tensors do not exist for such u and c. We show that if $u, c \leq 0$ or $u = c > 0$, then $M_c(u) = N_c(u)$. Explicit formulae for $M_c(u) = N_c(u)$ are given there in these cases. Thus, it is PNS-free for such u and c.

Note that $\mathcal{A}$ is PSD or SOS if and only if $\alpha\mathcal{A}$ is PSD or SOS, respectively. Thus, we only need to consider three cases that $c = 0$, $c = 1$ and $c = -1$.

In Section 3, we discuss the case that $c = 0$. In this case, for $u > 0$, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We show that $-N_0(1)$ is the smallest H-eigenvalue of $\mathcal{A}(m, 0, 1, 0)$. Numerical tests show that $M_0(1) = N_0(1)$ for $m = 6, 8, 10, 12$ and 14.

In Section 4, we study the case that $c = -1$. We show that there is a $u_0 > 0$ such that if $u \leq u_0$, $N_{-1}(u)$ is linear and the explicit formula of $N_{-1}(u)$ can be given, and if $u > u_0$, $N_{-1}(u)$ is the smallest H-eigenvalue of a tensor with u as a parameter. Numerical tests show that for $u > 0$, we still have $M_{-1}(u) = N_{-1}(u)$ for $m = 6, 8, 10$ and 12.

In Section 5, we study the case that $c = 1$. We show that there is a $v_0 < 0$ such that if $u \leq v_0$, $N_1(u)$ is linear and the explicit formula of $N_1(u)$ can be given, and if $u > v_0$, $N_1(u)$ is the smallest H-eigenvalue of a tensor with u as a parameter. Numerical tests show that for $u \neq 1$, we still have $M_1(u) = N_1(u)$ for $m = 6, 8, 10$ and 12.

Some final remarks are made in Section 6.

2. Functions $M_c(u)$ and $N_c(u)$

In this section and the next three sections, we assume that $n = 3$ and $m = 2k$ is even. Let $\mathcal{A}$ be an mth order three dimensional strongly symmetric circulant tensor. Then we may write $f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ as

$$\begin{aligned} f(\mathbf{x}) &= d(x_1^m + x_2^m + x_3^m) + u\sum_{p=1}^{m-1}\binom{m}{p}(x_1^{m-p}x_2^p + x_1^{m-p}x_3^p + x_2^{m-p}x_3^p) \\ &\quad + c\sum_{p=1}^{m-2}\sum_{q=1}^{m-p-1}\binom{m}{p}\binom{m-p}{q}x_1^{m-p-q}x_2^p x_3^q. \end{aligned} \tag{1}$$

We now establish two functions $M_c(u)$ and $N_c(u)$, in the following theorem. Recall that for an mth order n dimensional tensor $\mathcal{A} = (a_{i_1\cdots i_m})$, the sum of the absolute values of its ith off-diagonal entries, i.e.,

$$r_i = \sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| - |a_{ii\cdots i}|.$$

If $a_{i\cdots i} \geq r_i$ for $i = 1, \cdots, n$, then $\mathcal{A}$ is called diagonally dominated. It is shown in [18] that all the H-eigenvalues of a diagonally dominated tensor, if there is, are nonnegative. Furthermore, an even order symmetric diagonally dominated tensor is PSD [18] and SOS [2].

Theorem 1. *Let $\mathcal{A}$ be an mth order three dimensional strongly symmetric circulant tensor. For given off-diagonal entries u and c, we define*

$$M_c(u) \equiv \inf\{d \in \Re \ : \ \mathcal{A}(m,d,u,c) \text{ is } SOS\},$$
$$N_c(u) \equiv \inf\{d \in \Re \ : \ \mathcal{A}(m,d,u,c) \text{ is } PSD\}.$$

Then, functions $M_c(u)$ and $N_c(u)$ are well-defined and convex. Furthermore, we have

$$0 \leq N_c(u) \leq M_c(u) \leq |u|(2^m - 2) + |c|(3^{m-1} - 2^m + 1). \tag{2}$$

Proof. Since $\mathcal{A}$ is a circulant tensor, then it has the same off-diagonal entry absolute value sum for different rows, i.e., $r_1 = r_2 = r_3$. By (1), this row sum is equal to the right hand side of (2). Thus, if d is greater than or equal to this value, $\mathcal{A}$ is diagonally dominated and thus PSD and SOS. This shows that functions $M_c(u)$ and $N_c(u)$ are well-defined and the inequalities in (2) hold. As the set of PSD tensors and the set of SOS tensors are convex [14], $M_c(u)$ and $N_c(u)$ are convex. Since a necessary condition for an even order circulant tensor to be PSD is that its diagonal entry to be nonnegative [5], we have $N_c(u) \geq 0$ for all u and c.

By definition, we have $N_c(u) \leq M_c(u)$. Clearly, if $M_c(u) = N_c(u)$, then mth order three dimensional PNS strongly symmetric circulant tensors do not exist for such u and c. The theorem is proved. □

Theorem 1 means that $\mathcal{A}$ is SOS if and only if $d \geq M_c(u)$, and $\mathcal{A}$ is PSD if and only if $d \geq N_c(u)$. If $M_c(u) = N_c(u)$, then mth order three dimensional PNS strongly symmetric circulant tensors do not exist for such u and c. If $M_c(u) = N_c(u)$, u is called a **PNS-free point** for c.

For the convenience, we present formally three ingredients used in theoretical proofs of PNS-free points. If a point u enjoys these ingredients, it is PNS-free.

Definition 1. Suppose that $n = 3$ and m is even. Suppose that there is a number M such that $f^*(\mathbf{x}) \equiv \mathcal{A}(m,M,u,c)\mathbf{x}^m$ is SOS for given u and c, and a nonzero vector $\bar{\mathbf{x}} \in \Re^3$ such that $f^*(\bar{\mathbf{x}}) = 0$. Then we call M the **critical value** of $\mathcal{A}$ at u and c, the SOS decomposition $f^*(\mathbf{x})$ the **critical SOS decomposition** of $\mathcal{A}$ at u and c, and $\bar{\mathbf{x}}$ the **critical minimizer** of $\mathcal{A}$ at u and c.

Theorem 2. *Let $u \in \Re$. Then u is PNS-free point for c if $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f^*(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at u and c.*

Proof. Suppose that $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f^*(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at u. Then we have $M \geq M_c(u)$ by the definition of $M_c(u)$. If $d < M$, then

$$f(\bar{\mathbf{x}}) = (d - M)(\bar{x}_1^m + \bar{x}_2^m + \bar{x}_3^m) + f^*(\bar{\mathbf{x}}) < 0.$$

This implies that $N_c(u) \geq M$ by the definition of $N_c(u)$. But $N_c(u) \leq M_c(u)$. Thus, $M_c(u) = N_c(u) = M$, i.e., u is PNS-free point for c. □

Corollary 1. *If $u, c \leq 0$, then*

$$M_c(u) = N_c(u) = -u(2^m - 2) - c(3^{m-1} - 2^m + 1). \tag{3}$$

Thus, it is PNS-free for such u and c.

Proof. Suppose that $u, c \leq 0$. Let M be the value of the right hand side of (2), and $\bar{\mathbf{x}} = (1, 1, 1)^\top$. If $d = M$, then $f(\mathbf{x}) = f^*(\mathbf{x})$ has an SOS decomposition as $\mathcal{A}$ is an even order diagonally dominated symmetric tensor [2]. We also see that $f^*(\bar{\mathbf{x}}) = 0$. The result follows. □

Corollary 2. *If $u = c > 0$, then*

$$M_c(u) = N_c(u) = u = c.$$

Thus, it is PNS-free for such u and c.

Proof. Suppose that $u = c > 0$. Let $M = u = c$, and $\bar{\mathbf{x}} = (2, -1, -1)^\top$. If $d = M$, then $f(\mathbf{x}) = f^*(\mathbf{x}) = (x_1 + x_2 + x_2)^m$ has an SOS decomposition. We also see that $f^*(\bar{\mathbf{x}}) = 0$. The result follows. □

Corollary 3. *If $u > 0$, then*

$$M_0(u) = uM_0(1)$$

and

$$N_0(u) = uN_0(1).$$

Hence, for $c = 0$, it is PNS-free if and only if $M_0(1) = N_0(1)$.

Proof. Suppose that $u > 0$ and $d \geq uM_0(1)$. By (1), we have

$$\begin{aligned} f(\mathbf{x}) &= (d - uM_0(1))(x_1^m + x_2^m + x_3^m) \\ &\quad + u\left(M_0(1)(x_1^m + x_2^m + x_3^m) + \sum_{p=1}^{m-1} \binom{m}{p} (x_1^{m-p}x_2^p + x_1^{m-p}x_3^p + x_2^{m-p}x_3^p)\right). \end{aligned}$$

Table 1
The values of $M_0(1)$ and $N_0(1)$.

m	$M_0(1)$	$N_0(1)$
6	1.737348471173345	1.737348471777547
8	1.882980354978972	1.882980356780414
10	1.947977161918168	1.947977172341075
12	1.976878006619490	1.976878047128592
14	1.989722829997529	1.989723542124766

We see that $f(\mathbf{x})$ is SOS. Hence, $M_0(u) = uM_0(1)$. Similarly, we may prove that $N_0(u) = uN_0(1)$. By these and Corollary 1, we have the last conclusion. $\square$

As discussed in the Introduction, for the PNS-free problem, we only need to consider three cases: $c = 0, 1$ and -1.

3. $c = 0$

If $u \leq 0$, by Corollary 1, we have $M_0(u) = N_0(u) = -u(2^m - 2)$. If $u > 0$, by Corollary 3, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We only need to consider the case that $u = 1$.

Proposition 1. *We have that* $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$.

Proof. By [18], $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $\lambda_{\min}(m, d, 1, 0) \geq 0$. By the structure of circulant tensors, $\lambda_{\min}(m, d, 1, 0) = d + \lambda_{\min}(m, 0, 1, 0)$. Thus, $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $d \geq -\lambda_{\min}(m, 0, 1, 0)$. By the definition of $N_c(u)$, we have $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$. $\square$

For $m = 6, 8, 10, 12$ and 14, we compute $M_0(1)$ and $N_0(1)$ by using softwares Matlab (YALMIP, GloptiPloy and SeDuMi) and Maple [8,15,16,24], respectively. We find for such m, $M_0(1) = N_0(1)$. The results are displayed in Table 1.

4. $c = -1$

If $u \leq 0$, then Corollary 1 indicates that $M_{-1}(u) = N_{-1}(u) = -u(2^m - 2) + (3^{m-1} - 2^m + 1)$. We now discuss the case that $u > 0$.

In this section and the next section, we denote that $\mathcal{B} = \mathcal{A}(m, 3^{m-1} - 2^m + 1, 0, -1)$ and $\mathcal{T} = \mathcal{A}(m, 2^m - 2, -1, 0)$. Then, $\mathcal{B}$ and $\mathcal{T}$ are obviously diagonally dominated. Hence, they are PSD and SOS [2]. And all of their H-eigenvalues are nonnegative.

Theorem 3. *Let*

$$\varphi(u) \equiv \lambda_{\min}(\mathcal{B} - u\mathcal{T}),$$

where $\lambda_{\min}(\cdot)$ denotes the smallest H-eigenvalue. Then, $\varphi(u) \le 0$. If $\varphi(u) = 0$, then we have

$$N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2). \tag{4}$$

If $\varphi(u) < 0$, then we have

$$\begin{aligned} N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2) \\ - \lambda_{\min}(m, 3^{m-1} - 2^m + 1 - u(2^m - 2), u, -1). \end{aligned} \tag{5}$$

Furthermore, the set $C = \{u : \varphi(u) = 0\}$ is a nonempty closed convex ray $(-\infty, u_0]$ for some $u_0 \ge 0$.

Proof. Let $\bar{x} = (1, 1, 1)^\top$. Then $\mathcal{B}\bar{\mathbf{x}}^m = 0$ and $\mathcal{T}\bar{\mathbf{x}}^m = 0$. Thus, $(\mathcal{B} - u\mathcal{T})\bar{\mathbf{x}}^m = 0$ for any u. By [18], we see that

$$\varphi(u) \equiv \lambda_{\min}(\mathcal{B} - u\mathcal{T}) \le 0.$$

If $\varphi(u) = 0$, we let $d = 3^{m-1} - 2^m + 1 - u(2^m - 2)$. Then $\mathcal{A}(m, d, u, -1) = \mathcal{B} - u\mathcal{T}$. We have $\lambda_{\min}(m, d, u, -1) = 0$. This implies (4).

If $\varphi(u) < 0$, then, because

$$f(\mathbf{x}) = (d - (3^{m-1} - 2^m + 1) + u(2^m - 2))\mathcal{I}\mathbf{x}^m + \mathcal{B}\mathbf{x}^m - u\mathcal{T}\mathbf{x}^m \ge 0,$$

we have

$$\begin{aligned} N_{-1}(u) &= \inf\{d \;:\; \lambda_{\min}((d - (3^{m-1} - 2^m + 1) + u(2^m - 2))\mathcal{I} + \mathcal{B} - u\mathcal{T}) \ge 0\} \\ &= (3^{m-1} - 2^m + 1) - u(2^m - 2) - \lambda_{\min}(\mathcal{B} - u\mathcal{T}) \\ &= 3^{m-1} - 2^m + 1 - u(2^m - 2) - \lambda_{\min}(m, 3^{m-1} - 2^m + 1 - u(2^m - 2), u, -1). \end{aligned}$$

We have (5).

By Corollary 1, C is nonempty and $u \in C$ as long as $u \le 0$. By Theorem 1, $N_{-1}(u)$ is a convex function. It follows, together with (4) and (5), that C is convex. Since $\lambda_{\min}$ is a continuous function [18], C is closed. Since $u \in C$ for any $u \le 0$, C is a ray, with the form $(-\infty, u_0]$ for some $u_0 \ge 0$. □

Corollary 4. *Let $u_0 \equiv \max\{\hat{u} \;:\; \varphi(\hat{u}) = 0\}$. Then u_0 is well-defined and $u_0 \ge 0$. Furthermore, for $u \le u_0$, we have (4), and for $u > u_0$, we have (5).*

Proposition 2. *If $M_{-1}(u_0) = N_{-1}(u_0) = 3^{m-1} - 2^m + 1 - u_0(2^m - 2)$, then for $u \le u_0$, we have $M_{-1}(u) = N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2)$.*

Table 2
The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6$.

u	$M_{-1}(u)$	$N_{-1}(u)$
0.1	173.799999999899	173.8
2	55.9999999995172	56
$\frac{45}{16}$	5.62499991033116	5.625
5	9.42544641511067	9.4254465011842588
10	18.1121860822789	18.112186280892696
40	70.2326321651344	70.232638183914150
300	521.943237017699	521.94324013633004

Proof. By Theorem 1, $M_{-1}(u)$ is convex. By Corollary 1, $M_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2)$ for $u \leq 0$. Since $u_0 \geq 0$, the conclusion follows. $\square$

Proposition 3. *Suppose* $u_0 = \max\{\hat{u} \ : \ \varphi(\hat{u}) = 0\}$*. Then, we have*

$$0 \leq u_0 \leq \bar{u}_0(m) \equiv \frac{3^{m-1}+1}{2^m} - 1. \tag{6}$$

Proof. Since $\mathcal{B}$ is PSD and has an H-eigenvalue 0, we have $\varphi(0) = 0$ and $u_0 \geq 0$.

On the other hand, we consider the case $u > \bar{u}_0$. Let $\mathbf{x}_0 = (1, 1, -3)^\top$. We have

$$(\mathcal{B} - \bar{u}_0\mathcal{T})\mathbf{x}_0^m = 0 \qquad \text{and} \qquad \mathcal{T}\mathbf{x}_0^m = 2^m(3^m - 1).$$

Then,

$$(\mathcal{B} - u\mathcal{T})\mathbf{x}_0^m = (\mathcal{B} - \bar{u}_0\mathcal{T})\mathbf{x}_0^m - (u - \bar{u}_0)\mathcal{T}\mathbf{x}_0^m = -(u - \bar{u}_0)2^m(3^m - 1) < 0.$$

Hence, we have $\varphi(u) = \lambda_{\min}(\mathcal{B} - u\mathcal{T}) < 0$ when $u > \bar{u}_0$. Therefore, $u_0 \leq \bar{u}_0$. $\square$

For $m = 6, 8, 10, 12$ and 14, we find that $\mathcal{B} - \bar{u}_0\mathcal{T}$ is PSD. This shows that for such m, $\varphi(\bar{u}_0) = 0$, i.e.,

$$u_0 = \bar{u}_0(m) \equiv \frac{3^{m-1}+1}{2^m} - 1. \tag{7}$$

It remains a further research topic to show that $\mathcal{B} - \bar{u}_0\mathcal{T}$ is PSD for all even m with $m \geq 16$. If this is true, then (7) is true for all even m with $m \geq 6$.

In Tables 2–5, the values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6, 8, 10, 12$ and $u = 0.1, 2, \frac{45}{16}, 5, 10, 40, 300$, $u = 1, 3, \frac{483}{64}, 10, 40, 140, 300$, $u = 1, 10, \frac{4665}{256}, 20, 40, 140, 300$ and $u = 1, 20, \frac{43\,263}{1024}, 60, 100, 140, 300$ are reported, respectively. We find for such m and u, $M_{-1}(u) = N_{-1}(u)$.

Table 3
The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 8$.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	1677.99999992219	1678
3	1169.99999999356	1170
$\frac{483}{64}$	15.0937478786308	15.09375
10	19.7129359300341	19.7129361640501
40	76.2023466001335	76.2023468071730
140	264.500365037152	264.500382469583
300	565.777184078832	565.777239551091

Table 4
The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 10$.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	17 637.9999999549	17 638
10	8439.99999783081	8440
$\frac{4665}{256}$	36.4452603485520	36.4453125
20	39.9075358817909	39.9075375522954
40	78.8670625326286	78.8670809985488
140	273.664775923815	273.664798232238
300	585.341085323688	585.341145806726

Table 5
The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 12$.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	168 957.999979042	168 958
20	91 171.9999996683	91 172
$\frac{43\,263}{1024}$	84.4971787852022	84.498046875
60	119.589505579120	119.589562756497
100	198.664532858285	198.664684641639
140	277.739708996851	277.739806526784
300	594.040191670531	594.040294067366

5. $c = 1$

Corollary 2 indicates that $M_1(1) = N_1(1) = 1$. Hence, we only need to consider the case that $u \neq 1$. Let $\mathcal{B}$ and $\mathcal{T}$ be the same as in the last section. We have the following theorem.

Theorem 4. *Let*

$$\psi(u) \equiv \lambda_{\min}(-u\mathcal{T} - \mathcal{B}).$$

Then, $\psi(u) \leq 0$. If $\psi(u) = 0$, then we have

$$N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2). \tag{8}$$

If $\psi(u) < 0$, then we have

$$\begin{aligned} N_1(u) = &-(3^{m-1} - 2^m + 1) - u(2^m - 2) \\ &- \lambda_{\min}(m, -(3^{m-1} - 2^m + 1) - u(2^m - 2), u, 1). \end{aligned} \tag{9}$$

Furthermore, if the set $C = \{u : \psi(u) = 0\}$ is nonempty, then it is a closed convex ray $(-\infty, v_0]$ for some $v_0 < 0$.

Proof. The proof of this theorem is similar to the proof of Theorem 3. However, we cannot apply Corollary 1 here. If C is nonempty, we may show that C is closed and convex as in the proof of Theorem 3.

If there is a $\hat{u} \le 0$ such that $\lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B}) = 0$, for $u \le \hat{u} \le 0$, we have

$$\begin{aligned}\psi(u) &= \lambda_{\min}(-u\mathcal{T} - \mathcal{B}) \\ &= \lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B} + (-u + \hat{u})\mathcal{T}) \\ &\ge \lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B}) \\ &= 0.\end{aligned}$$

Hence, $\psi(u) = 0$ for all $u \le \hat{u} \le 0$. So if C is not empty, it is a ray with the form $(-\infty, v_0]$ for some v_0. Clearly, $\psi(0) \equiv \lambda_{\min}(-\mathcal{B}) < 0$ as $\mathcal{B}$ is PSD and not a zero tensor. Hence, $v_0 < 0$.

The other parts of the proof are similar to the proof of Theorem 3. $\square$

Corollary 5. *If there is one point $\hat{u}$ such that $\psi(\hat{u}) = 0$, let $v_0 \equiv \max\{\hat{u} : \psi(\hat{u}) = 0\}$. Then for $u \le v_0$, we have (8), and for $u \ge v_0$, we have (9).*

We also have the following proposition.

Proposition 4. *If $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$, then for $u \le v_0$, we have $M_1(u) = N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$.*

Proof. Suppose that $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$. By (3), if $u \le v_0$ and $d = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$, we have

$$f^*(\mathbf{x}) = -\bar{u}g_1(\mathbf{x}) + g_2(\mathbf{x}),$$

where

$$\begin{aligned}g_1(\mathbf{x}) &= \mathcal{A}(m, 2^m - 2, -1, 0), \\ g_2(\mathbf{x}) &= \mathcal{A}(m, -(3^{m-1} - 2^m + 1) - v_0(2^m - 2), v_0, 1)\end{aligned}$$

and

$$\bar{u} \equiv u - v_0 \le 0.$$

We see that $g_2(\mathbf{x})$ is equal to the critical SOS decomposition of $\mathcal{A}$ at $c = 1$ and $u = v_0$, and $g_1(\mathbf{x})$ is equal to the critical SOS decomposition of $\mathcal{A}$ at $c = 0$ and $u = -1$. Hence both

Table 6
The values of $M_1(u)$ and $N_1(u)$ for $m = 6$.

u	$M_1(u)$	$N_1(u)$
-40	2299.99999993444	2300
-10	439.999999987168	440
$-\frac{70}{11}$	214.545454213196	214.54545454545454
-5	173.991050854352	173.99105151869704
-1	55.8846973214056	55.884697712412670
10	16.6347897201042	16.634789948247836
40	68.7552353830704	68.755241242186800

$g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ are SOS polynomials. This implies that $f^*(\mathbf{x})$ is an SOS polynomial. Let $\bar{\mathbf{x}} = (1,1,1)^\top$, we see that $f^*(\bar{\mathbf{x}}) = 0$. Then the conclusion follows from Theorem 2. □

Proposition 5. *Suppose that C is not empty. Let $v_0 = \max\{\hat{u} \ : \ \psi(\hat{u}) = 0\}$. Then, we have*

$$v_0 \le \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1}+1}. \tag{10}$$

Proof. Let $\mathbf{x}_0 = (1,1,-\frac{1}{2})^\top$. We have

$$(-\mathcal{B} - \bar{v}_0\mathcal{T})\mathbf{x}_0^m = 0 \qquad \text{and} \qquad \mathcal{T}\mathbf{x}_0^m = 2^m + 1 - 2^{1-m}.$$

Then,

$$(-\mathcal{B} - u\mathcal{T})\mathbf{x}_0^m = (-\mathcal{B} - \bar{v}_0\mathcal{T})\mathbf{x}_0^m - (u - \bar{v}_0)\mathcal{T}\mathbf{x}_0^m = -(u - \bar{v}_0)(2^m + 1 - 2^{1-m}) < 0.$$

Hence, we have $\psi(u) = \lambda_{\min}(-\mathcal{B} - u\mathcal{T}) < 0$ when $u > \bar{v}_0$. Therefore, $v_0 \le \bar{v}_0$. □

By the similar discussion on u_0, we find that $-\bar{v}_0\mathcal{T} - \mathcal{B}$ is PSD for $m = 6, 8, 10, 12$ and 14. This shows that for such m, $\psi(\bar{v}_0) = 0$, i.e.,

$$v_0 = \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1}+1}. \tag{11}$$

This also shows that C is not empty for such m. It remains a further research topic to show that $-\bar{v}_0\mathcal{T} - \mathcal{B}$ is PSD for all even m with $m \ge 16$. If this is true, then (11) is true for all even m with $m \ge 6$.

In Tables 6–9, the values of $M_1(u)$ and $N_1(u)$ for $m = 6, 8, 10, 12$ and $u = -40, -10, -\frac{70}{11}, -5, -1, 10, 40$, $u = -40, -20, -\frac{686}{43}, -10, -1, 10, 40$, $u = -100, -40, -\frac{710}{19}$, $-20, -1, 10, 40$ and $u = -140, -100, -\frac{58366}{683}, -60, -1, 10, 40$ are reported, respectively. We find for such m and u, $M_1(u) = N_1(u)$.

Table 7
The values of $M_1(u)$ and $N_1(u)$ for $m = 8$.

u	$M_1(u)$	$N_1(u)$
-40	8228.00000000754	8228
-20	3147.99999997053	3148
$-\frac{686}{43}$	2120.18604151092	2120.18604651163
-10	1371.80748977461	1371.80749709544
-1	243.740078469126	243.740080311110
10	17.9466697015668	17.9466711544215
40	74.4360734714431	74.4360817805826

Table 8
The values of $M_1(u)$ and $N_1(u)$ for $m = 10$.

u	$M_1(u)$	$N_1(u)$
-100	83 539.9999994888	83 540
-40	22 219.9999995661	22 220
$-\frac{710}{19}$	19 530.5255392283	19 530.5263157895
-20	10 678.1156381743	10 678.1156702343
-1	1004.40451207948	1004.40454284172
10	18.5317674915799	18.5317776218259
40	76.9710868541002	76.9710927899669

Table 9
The values of $M_1(u)$ and $N_1(u)$ for $m = 12$.

u	$M_1(u)$	$N_1(u)$
-140	400 107.999992756	400 108
-100	236 347.999998551	236 348
$-\frac{58\,366}{683}$	176 802.173593347	176 802.170881802
-60	124 727.840916646	124 727.840144917
-1	4063.38103939314	4063.38106552746
10	18.7918976770375	18.7919005425937
40	78.0982265029468	78.0982419563963

6. Final remarks

In Proposition 1, Theorems 3 and 4, we give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite. For $u, c \le 0$ and $u = c > 0$, we show that this condition is also sufficient for this tensor to be sum-of-squares. Numerical tests indicate that this is also true in the other cases.

How can $\mathcal{B} - \bar{u}_0\mathcal{T}$ and $-\bar{v}_0\mathcal{T} - \mathcal{B}$ be shown to be PSD for all even $m \ge 6$? If these are true, then (7) and (11) are true for all even $m \ge 6$.

Finally, more efforts are needed to prove that this problem is PNS-free eventually.

Acknowledgement

We are thankful to the referee for his or her comments, which helped us to improve our paper.

References

[1] A.A. Ahmadi, P.A. Parrilo, A convex polynomial that is not sos-convex, Math. Program. 135 (2012) 275–292.
[2] H. Chen, G. Li, L. Qi, Sum-of-squares tensors and their sum-of-squares ranks, arXiv:1504.03414, April 2015.
[3] H. Chen, L. Qi, Positive definiteness and semi-definiteness of even order symmetric Cauchy tensors, J. Ind. Manag. Optim. 11 (2015) 1263–1274.
[4] Y. Chen, L. Qi, Q. Wang, Positive semi-definiteness and sum-of-squares property of fourth order four dimensional Hankel tensors, arXiv:1502.04566, March 2015.
[5] Z. Chen, L. Qi, Circulant tensors with applications to spectral hypergraph theory and stochastic process, arXiv:1312.2752, November 2014.
[6] G. Chesi, On the gap between positive polynomials and SOS of polynomials, IEEE Trans. Automat. Control 52 (2007) 1066–1072.
[7] M.D. Choi, T.Y. Lam, Extremal positive semidefinite forms, Math. Ann. 231 (1977) 1–18.
[8] D. Henrion, J.B. Lasserre, J. Löfberg, GloptiPoly 3: moments, optimization and semidefinite programming, Optim. Methods Softw. 24 (2009) 761–779.
[9] D. Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten, Math. Ann. 32 (1888) 342–350.
[10] M.R. Kannan, N. Shaked-Monderer, A. Berman, Some properties of strong H-tensors and general H-tensors, Linear Algebra Appl. 476 (2015) 42–55.
[11] C. Li, Y. Li, Double B-tensors and quasi-double B-tensors, Linear Algebra Appl. 21 (2014) 39–50.
[12] C. Li, F. Wang, J. Zhao, Y. Zhu, Y. Li, Criterions for the positive definiteness of real supersymmetric tensors, J. Comput. Appl. Math. 255 (2014) 1–14.
[13] G. Li, L. Qi, Q. Wang, Are there sixth order three dimensional PNS Hankel tensors?, arXiv: 1411.2368, November 2014.
[14] G. Li, L. Qi, Y. Xu, SOS-Hankel tensors: theory and application, arXiv:1410.6989, October 2014.
[15] J. Löfberg, YALMIP: a toolbox for modeling and optimization in MATLAB, in: Proceedings of the CACSD Conference, Taipei, Taiwan, 2004.
[16] J. Löfberg, Pre- and post-processing sum-of-squares programs in practice, IEEE Trans. Automat. Control 54 (2004) 1007–1011.
[17] T.S. Motzkin, The arithmetic-geometric inequality, in: O. Shisha (Ed.), Inequalities, Academic Press, New York, 1967, pp. 205–224.
[18] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005) 1302–1324.
[19] L. Qi, Hankel tensors: associated Hankel matrices and Vandermonde decomposition, Commun. Math. Sci. 13 (2015) 113–125.
[20] L. Qi, Y. Song, An even order symmetric B tensor is positive definite, Linear Algebra Appl. 457 (2014) 303–312.
[21] L. Qi, C. Xu, Y. Xu, Nonnegative tensor factorization, completely positive tensors and a hierarchically elimination algorithm, SIAM J. Matrix Anal. Appl. 35 (2014) 1227–1241.
[22] B. Reznick, Some concrete aspects of Hilbert's 17th problem, Contemp. Math. 253 (2000) 251–272.
[23] Y. Song, L. Qi, Infinite and finite dimensional Hilbert tensors, Linear Algebra Appl. 451 (2014) 1–14.
[24] J.F. Sturm, SeDuMi version 1.1R3, Available at http://sedumi.ie.lehigh.edu, 2006.
[25] L. Zhang, L. Qi, G. Zhou, M-tensors and some applications, SIAM J. Matrix Anal. Appl. 35 (2014) 437–452.