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RESEARCH ARTICLE

Conditions for strong ellipticity and M-eigenvalues*

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Abstract The strong ellipticity condition plays an important role in nonlinear elasticity and in materials. In this paper, we define M-eigenvalues for an elasticity tensor. The strong ellipticity condition holds if and only if the smallest M-eigenvalue of the elasticity tensor is positive. If the strong ellipticity condition holds, then the elasticity tensor is rank-one positive definite. The elasticity tensor is rank-one positive definite if and only if the smallest Z-eigenvalue of the elasticity tensor is positive. A Z-eigenvalue of the elasticity tensor is an M-eigenvalue but not vice versa. If the elasticity tensor is second-order positive definite, then the strong ellipticity condition holds. The converse conclusion is not right. Computational methods for finding M-eigenvalues are presented.

Keywords Elasticity tensor, strong ellipticity, M-eigenvalue, Z-eigenvalue
MSC 74B99, 15A18, 15A69

1 Introduction

The two/three-dimensional field equations for a homogeneous compressible nonlinearly elastic material for static problems without body forces can be written as (see, e.g., Knowles and Sternberg Refs. [8,22])

$$a_{ijkl}(1 + \nabla \boldsymbol{u})u_{k,lj} = 0, \tag{1}$$

where $u_i(\boldsymbol{X})$ ($i = 1, 2$ or $1, 2, 3$) is the displacement field ($\boldsymbol{X}$ is the coordinate of a material point in the reference configuration), $\mathscr{A} = (a_{ijkl})$ is the elasticity

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tensor, and

$$a_{ijkl}(\boldsymbol{F}) = a_{klij}(\boldsymbol{F}) = \frac{\partial^2 W(\boldsymbol{F})}{\partial F_{ij} \partial F_{kl}}, \quad \boldsymbol{F} = 1 + \nabla \boldsymbol{u}, \tag{2}$$

where W is the strain energy function. We say that the above equations are elliptic at the point $\boldsymbol{X}$ if and only if

$$\det[a_{ijkl}(\boldsymbol{F}(\boldsymbol{X}))y_j y_l] \neq 0 \tag{3}$$

for all unit vectors $y \in \mathbb{R}^n$, $n = 2$ or 3. On the other hand, we say that they are strongly elliptic if and only if

$$f(x, y) \equiv \mathscr{A}xyxy \equiv \sum_{i,j,k,l=1}^{n} a_{ijkl} x_i y_j x_k y_l > 0$$

for all unit vectors $x, y \in \mathbb{R}^n$, $n = 2$ or 3. While for an isotropic material, some inequalities have been established to judge strong ellipticity (see, Refs. [7,8,22,23,26]), for a general nonlinearly elastic material, a workable criterion to make a judgement is still lacking. Also, it is important to find out along which of the directions x and y the strong ellipticity fails, as this is related to the appearance of solutions with discontinuous strain gradients; see Refs. [7,8]. In this paper, we shall take a different approach from those in the literature mentioned above to tackle the problem of the strong ellipticity and also to give an algorithm for computing the most possible directions along which the strong ellipticity can fail.

Clearly, the strong ellipticity condition holds if and only if the optimal value of the following global polynomial optimization problem is positive:

$$\begin{aligned} \min \quad & f(x, y) \equiv \mathscr{A}xyxy \equiv \sum_{i,j,k,l=1}^{n} a_{ijkl} x_i y_j x_k y_l \\ \text{s.t.} \quad & x^{\mathrm{T}}x = 1, \quad y^{\mathrm{T}}y = 1, \end{aligned} \tag{4}$$

where $x, y \in \mathbb{R}^n$, $n = 2$ or 3. Because of the form of (4), we may also assume that for any i, j, k, l, we have

$$a_{ijkl} = a_{kjil} = a_{ilkj}.$$

Hence, $\mathscr{A}$ has 9 independent elements for $n = 2$ and 36 independent elements for $n = 3$. Denote $\mathscr{A}\cdot yxy$ as a vector whose ith component is

$$\sum_{j,k,l=1}^{n} a_{ijkl} y_j x_k y_l,$$

and $\mathscr{A}xyx\cdot$ as a vector whose lth component is

$$\sum_{i,j,k=1}^{n} a_{ijkl} x_i y_j x_k.$$

The optimality condition of (4) is

$$\begin{cases} \mathscr{A}\cdot yxy = \lambda x, \\ \mathscr{A}xyx\cdot = \mu y, \\ x^{\mathrm{T}}x = 1, \\ y^{\mathrm{T}}y = 1. \end{cases} \tag{5}$$

Suppose λ, μ, x and y satisfy (5). It is easy to see that

$$\lambda = \mathscr{A}xyxy = \mu.$$

Thus, we may rewrite (5) as

$$\begin{cases} \mathscr{A}\cdot yxy = \lambda x, \\ \mathscr{A}xyx\cdot = \lambda y, \\ x^{\mathrm{T}}x = 1, \\ y^{\mathrm{T}}y = 1. \end{cases} \tag{6}$$

If $\lambda \in \mathbb{R},\ x, y \in \mathbb{R}^n$ satisfy (6), we call λ an M-eigenvalue of $\mathscr{A}$, and call x and y left and right M-eigenvectors of $\mathscr{A}$, associated with the M-eigenvalue λ. Here, the letter 'M' stands for mechanics.

It is easy to see that M-eigenvalues always exist and the strong ellipticity condition holds if and only if the smallest M-eigenvalue of $\mathscr{A}$ is positive. In Section 2, we study properties of M-eigenvalues.

The strong ellipticity is related to two kinds of positive definiteness of $\mathscr{A}$.

The elasticity tensor $\mathscr{A}$ is called rank-one positive definite if for all $x \in \mathbb{R}^n,\ x \neq 0$,

$$f(x, x) \equiv \mathscr{A}x^4 \equiv \mathscr{A}xxxx \equiv \sum_{i,j,k,l=1}^{n} a_{ijkl}x_i x_j x_k x_l > 0.$$

Clearly, if the strong ellipticity holds, then $\mathscr{A}$ is rank-one positive definite. In Section 3, we study the relationships between the strong ellipticity and the rank-one positive definiteness, and their relationships with eigenvalues of $\mathscr{A}$. The elasticity tensor $\mathscr{A}$ is rank-one positive definite if and only if its smallest Z-eigenvalue is positive. The Z-eigenvalues of a tensor were introduced in the fully symmetric case in Ref. [16], extended to a nonsymmetric case in Ref. [18], and further discussed in Refs. [14,17]. In Section 3, we will see that all the Z-eigenvalues of $\mathscr{A}$ are M-eigenvalues of $\mathscr{A}$, but not vice versa. This reveals their relationships.

We may also call $\mathscr{A}$ positive definite if for any $D = (d_{ij}) \in \mathbb{R}^{n\times n},\ D \neq 0$, we have

$$\mathscr{A}D^2 \equiv \sum_{i,j,k,l=1}^{n} a_{ijkl}d_{ij}d_{kl} > 0.$$

To distinguish it from the rank-one positive definiteness, we call it the second-order positive definiteness. It is clear that if $\mathscr{A}$ is second-order positive definite, then the strong ellipticity holds as we may regard $d_{ij} \equiv x_i y_j$ to connect these two concepts. In Section 4, we study the relationship between strong ellipticity and the second-order positive definiteness.

For $n = 2$, all the M-eigenvalues of $\mathscr{A}$ and their corresponding left and right M-eigenvectors can be calculated by a direct method. We present such a direct method in Section 5.

For $n = 3$, we may find the smallest M-eigenvalue and its corresponding left and right M-eigenvectors by the SOS (sum of squares) method. We describe the SOS method for this problem in Section 6.

Some final remarks are made in Section 7.

2 M-Eigenvalues of a fourth-order partially symmetric tensor

Suppose that $\mathscr{A} = (a_{ijkl})$ is a fourth-order real partially symmetric tensor, where

$$a_{ijkl} = a_{kjil} = a_{ilkj}, \quad i, k = 1, \ldots, m, \ j, l = 1, \ldots, n.$$

Here, the dimension of the first and the third indices is m, while the dimension of the second and the fourth indices is n. Such a fourth-order partially symmetric tensor is useful in quantum physics, see Refs. [12,27].

If $\lambda \in \mathbb{R}, \ x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ satisfy (6), we call λ an M-eigenvalue of $\mathscr{A}$, and call x and y left and right M-eigenvectors of $\mathscr{A}$, associated with the M-eigenvalue λ.

Let

$$f(x, y) \equiv \mathscr{A}xyxy \equiv \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l.$$

If $f(x, y) > 0$ for all $x \in \mathbb{R}^m, \ x \neq 0, \ y \in \mathbb{R}^n, \ y \neq 0$, then we say that $\mathscr{A}$ is positive definite.

Theorem 1 *M-eigenvalues always exist. If x and y are left and right M-eigenvectors of $\mathscr{A}$, associated with an M-eigenvalue λ, then*

$$\lambda = \mathscr{A}xyxy. \tag{7}$$

Proof The feasible region of

$$\begin{aligned} \min \quad & f(x, y) \equiv \mathscr{A}xyxy \equiv \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l \\ \text{s.t.} \quad & x^{\mathrm{T}} x = 1, \quad y^{\mathrm{T}} y = 1 \end{aligned} \tag{8}$$

is compact. The objective function of (8) is continuous. Hence, the optimization problem (8) has at least a maximizer and a minimizer. They are critical points of (8) and satisfy (6) with corresponding Lagrangian multipliers. Hence, M-eigenvalues always exist. By (6), we have (7). Clearly,

$\mathscr{A}$ is positive definite if and only if the optimal value of (8) is positive. Then the last conclusion of the theorem follows. □

We may use the last two equations of (6) to homogenize the first two equations of (6). Then we have

$$\begin{cases} \mathscr{A}\cdot yxy = \lambda(y^{\mathrm{T}}y)x, \\ \mathscr{A}xyx\cdot = \lambda(x^{\mathrm{T}}x)y. \end{cases} \tag{9}$$

According to algebraic geometry [5], the resultant of (9) is a one-dimensional polynomial ϕ of λ. We call $\phi(\lambda)$ the M-characteristic polynomial of $\mathscr{A}$.

Theorem 2 *An M-eigenvalue of* $\mathscr{A}$ *is always a real root of the M-characteristic polynomial* $\phi(\lambda)$.

Proof According to the resultant theory [5], (9) has a nonzero complex solution (x, y) if and only if λ is a root of the resultant. The conclusion follows. □

Define the Frobenius norm of $\mathscr{A}$ as

$$\|\mathscr{A}\|_F = \left(\sum_{i,k=1}^{m} \sum_{j,l=1}^{n} (a_{ijkl})^2 \right)^{1/2}.$$

For $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, $\lambda xyxy$ is a rank-one fourth-order partially symmetric tensor with elements $\lambda x_i y_j x_k y_l$. We say that $\lambda xyxy$ is the best rank-one approximation of $\mathscr{A}$, if $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^m$, $x^{\mathrm{T}}x = 1$ and $y \in \mathbb{R}^n$, $y^{\mathrm{T}}y = 1$ minimize $\|\mathscr{A} - \lambda xyxy\|_F$. The best rank-one approximation has wide applications in signal and image processing, wireless communication systems, and independent component analysis, etc. [1,6,9,20,28].

Theorem 3 *If* λ *is the M-eigenvalue of* $\mathscr{A}$ *with the largest absolute value,* x *and* y *are corresponding left and right M-eigenvectors, then* $\lambda xyxy$ *is the best rank-one approximation of* $\mathscr{A}$.

Proof Let $x^{\mathrm{T}}x = 1$ and $y^{\mathrm{T}}y = 1$. We have

$$\begin{aligned} \|\mathscr{A} - \lambda xyxy\|_F^2 &= \|A\|_F^2 - 2\lambda\mathscr{A}xyxy + \lambda^2(x^{\mathrm{T}}x)(y^{\mathrm{T}}y) \\ &= \|A\|_F^2 - 2\lambda\mathscr{A}xyxy + \lambda^2. \end{aligned}$$

Its minimum is attained when $\lambda = \mathscr{A}xyxy$. Hence

$$\begin{aligned} &\min\{\|\mathscr{A} - \lambda xyxy\|_F^2 : \ \lambda \in \mathbb{R}, x^{\mathrm{T}}x = 1, \ y^{\mathrm{T}}y = 1\} \\ &\quad = \min\{\|\mathscr{A}\|_F^2 - (\mathscr{A}xyxy)^2 : \ x^{\mathrm{T}}x = 1, \ y^{\mathrm{T}}y = 1\} \\ &\quad = \|\mathscr{A}\|_F^2 - \max\{(\mathscr{A}xyxy)^2 : \ x^{\mathrm{T}}x = 1, \ y^{\mathrm{T}}y = 1\}. \end{aligned}$$

The conclusion follows. □

Suppose that $P = (p_{ii'}) \in \mathbb{R}^{m\times m}$ and $Q = (q_{jj'}) \in \mathbb{R}^{n\times n}$ are orthogonal matrices. Let

$$b_{ijkl} = \sum_{i',k'=1}^{m} \sum_{j',l'=1}^{n} p_{ii'} q_{jj'} p_{kk'} q_{ll'} a_{i'j'k'l'}.$$

Then $\mathscr{B} = (b_{ijkl})$ is also a fourth-order partially symmetric tensor. We say that $\mathscr{A}$ and $\mathscr{B}$ are orthogonally similar.

Theorem 4 *If fourth-order partially symmetric tensors $\mathscr{A}$ and $\mathscr{B}$ are orthogonally similar, then they have the same M-eigenvalues. In particular, if they are orthogonally similar via orthogonal matrices P and Q as above, and λ is an M-eigenvalue of $\mathscr{A}$ with left and right M-eigenvectors x and y, then λ is also an M-eigenvalue of $\mathscr{B}$ with left and right M-eigenvectors Px and Qy.*

The proof of this theorem is similar to the proof of Theorem 7 of Ref. [16]. We omit its proof here.

3 Strong ellipticity and rank-one positive definiteness

In this section, we assume that $m = n$.

As the strong ellipticity is related with the M-eigenvalues of $\mathscr{A}$, the rank-one positive definiteness of $\mathscr{A}$ is related with the Z-eigenvalues of $\mathscr{A}$. We now review the definition of Z-eigenvalues of $\mathscr{A}$. Clearly, the elasticity tensor $\mathscr{A}$ is rank-one positive definite if the optimal value of the following global polynomial optimization problem is positive:

$$\begin{aligned} \min \quad & f(x,x) \equiv \mathscr{A}x^4 \equiv \mathscr{A}xxxx \equiv \sum_{i,j,k,l=1}^{n} a_{ijkl}x_i x_j x_k x_l \\ \text{s.t.} \quad & x^{\mathrm{T}}x = 1, \end{aligned} \tag{10}$$

where $x \in \mathbb{R}^n$. Denote $\mathscr{A}x^3$ as a vector whose ith component is

$$\sum_{j,k,l=1}^{n} a_{ijkl}x_j x_k x_l.$$

The optimality condition of (10) is

$$\begin{cases} \mathscr{A}x^3 = \lambda x, \\ x^{\mathrm{T}}x = 1. \end{cases} \tag{11}$$

If $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ satisfy (11), we call λ a Z-eigenvalue of $\mathscr{A}$, and call x a Z-eigenvector of $\mathscr{A}$, associated with the Z-eigenvalue λ. Thus, Z-eigenvalues always exist, and $\mathscr{A}$ is rank-one positive definite if and only if its smallest Z-eigenvalue is positive.

It is easy to see that a Z-eigenvalue is an M-eigenvalue. However, an M-eigenvalue is not necessarily a Z-eigenvalue. We may see this from the following example for $n = 2$.

For $n = 2$, we have

$$\begin{aligned} f(x, y) &= \mathscr{A}xyxy \\ &= a_{1111}x_1^2y_1^2 + 2a_{1112}x_1^2y_1y_2 + a_{1212}x_1^2y_2^2 \\ &\quad + 2a_{1121}x_1x_2y_1^2 + 4a_{1122}x_1x_2y_1y_2 + 2a_{1222}x_1x_2y_2^2 \\ &\quad + a_{2121}x_2^2y_1^2 + 2a_{2122}x_2^2y_1y_2 + a_{2222}x_2^2y_2^2. \end{aligned} \tag{12}$$

Consider the example where

$$a_{1111} = 1, \quad a_{1112} = 2, \quad a_{1121} = 2, \quad a_{1122} = 4, \quad a_{1212} = 3,$$

$$a_{2121} = 3, \quad a_{1222} = 5, \quad a_{2122} = 5, \quad a_{2222} = 6.$$

Then (12) becomes

$$\begin{aligned} f(x, y) &= \mathscr{A}xyxy \\ &= x_1^2y_1^2 + 4x_1^2y_1y_2 + 3x_1^2y_2^2 + 4x_1x_2y_1^2 + 16x_1x_2y_1y_2 \\ &\quad + 10x_1x_2y_2^2 + 3x_2^2y_1^2 + 10x_2^2y_1y_2 + 6x_2^2y_2^2. \end{aligned} \tag{13}$$

$\mathscr{A}$ has eight M-eigenvector pairs:

$$\begin{cases} x = (-0.9639, 0.2664)^{\mathrm{T}}, \\ y = (-0.9639, 0.2664)^{\mathrm{T}}, \end{cases} \qquad \begin{cases} x = (0.5774, 0.8165)^{\mathrm{T}}, \\ y = (0.5774, 0.8165)^{\mathrm{T}}, \end{cases}$$

$$\begin{cases} x = (-0.8101, 0.5862)^{\mathrm{T}}, \\ y = (-0.8101, 0.5862)^{\mathrm{T}}, \end{cases} \qquad \begin{cases} x = (-0.5774, 0.8165)^{\mathrm{T}}, \\ y = (-0.5774, 0.8165)^{\mathrm{T}}, \end{cases}$$

$$\begin{cases} x = (0.5894, 0.8079)^{\mathrm{T}}, \\ y = (-0.8168, 0.5769)^{\mathrm{T}}, \end{cases} \qquad \begin{cases} x = (-0.8168, 0.5769)^{\mathrm{T}}, \\ y = (0.5894, 0.8079)^{\mathrm{T}}, \end{cases}$$

$$\begin{cases} x = (-0.9326, 0.3609)^{\mathrm{T}}, \\ y = (-0.6402, 0.7682)^{\mathrm{T}}, \end{cases} \qquad \begin{cases} x = (-0.6402, 0.7682)^{\mathrm{T}}, \\ y = (-0.9326, 0.3609)^{\mathrm{T}}, \end{cases}$$

and the corresponding M-eigenvalues are, respectively,

$$0.0710, \quad 15.2091, \quad 0.3437, \quad 0.1242,$$

$$-1.2765, \quad -1.2765, \quad 0.2765, \quad 0.2765.$$

We see that the first four M-eigenvalues are Z-eigenvalues, but the last four M-eigenvalues are not Z-eigenvalues. Thus, the smallest M-eigenvalue is $\lambda = -1.2765$, taken at

$$x = (-0.8168, 0.5769)^{\mathrm{T}}, \quad y = (0.5894, 0.8079)^{\mathrm{T}}$$

and

$$x = (0.5894, 0.8079)^{\mathrm{T}}, \quad y = (-0.8168, 0.5769)^{\mathrm{T}}.$$

The strong ellipticity condition does not hold. But the smallest Z-eigenvalue is 0.1242. $\mathscr{A}$ is rank-one positive definite.

We may substitute these M-eigenvalues and M-eigenvectors to (6) to verify them. We may also solve (11) exactly to find the Z-eigenvalues. In fact, (11) is now

$$\begin{cases} 2x_1^3 + 12x_1^2x_2 + 22x_1x_2^2 + 10x_2^3 = \lambda x_1, \\ 4x_1^3 + 22x_1^2x_2 + 30x_1x_2^2 + 12x_2^3 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

From the second and the third equations, it is easy to see that $x_2 \neq 0$. Thus, eliminating λ from the first two equations and using the third equation, we have

$$2t^4 + 10t^3 + 9t^2 - 5t - 5 = (2t^2 - 1)(t^2 + 5t + 5) = 0, \tag{14}$$

where $t = x_1/x_2$. This polynomial equation has four roots:

$$\frac{\sqrt{2}}{2}, \quad -\frac{\sqrt{2}}{2}, \quad \frac{-5 + \sqrt{5}}{2}, \quad -\frac{-5 + \sqrt{5}}{2}.$$

For any t, the Z-eigenvector of $\mathscr{A}$ can be obtained via the relation

$$x = \frac{1}{(1 + t^2)^{1/2}} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Then, the corresponding Z-eigenvalues can be calculated by

$$\lambda = \mathscr{A}x^4 = \frac{t^4 + 8t^3 + 22t^2 + 20t + 6}{(1 + t^2)^2}.$$

We may easily check that the four Z-eigenvalues of $\mathscr{A}$ given earlier are right and there are no other Z-eigenvalues.

Summarizing the above results, we have the following theorem.

Theorem 5 *The strong ellipticity condition holds if and only if the smallest M-eigenvalue of the elasticity tensor $\mathscr{A}$ is positive. The elasticity tensor $\mathscr{A}$ is rank-one positive definite if and only if its smallest Z-eigenvalue is positive. All the Z-eigenvalues are M-eigenvalues, but not vice versa.*

In some cases, it is possible that all the M-eigenvalues are Z-eigenvalues. If, furthermore, for all i, j, k, l, we have $a_{ijkl} = a_{jikl}$, then $\mathscr{A}$ is fully symmetric. In this case, we may denote

$$\mathscr{A}xy \equiv \mathscr{A} \cdot \cdot\, xy = \left(\sum_{k,l=1}^{n} a_{ijkl} x_k y_l \right),$$

which is a symmetric matrix. We have the following theorem.

Theorem 6 *Suppose that $n = 2$ and $\mathscr{A}$ is fully symmetric. Then all the M-eigenvalues of $\mathscr{A}$ are Z-eigenvalues if there are no $x, y \in \mathbb{R}^2$, satisfying the following three conditions*:

(i) $x^{\mathrm{T}}x = 1,\ y^{\mathrm{T}}y = 1$;

(ii) *x and y are linearly independent;*

(iii) $(\mathscr{A}xy)^2 = \lambda^2 I$, *where I is the 2×2 unit matrix.*

Proof Suppose that λ is an M-eigenvalue with x and y as its left and right M-eigenvectors. Then $x^{\mathrm{T}}x = 1,\ y^{\mathrm{T}}y = 1$. Suppose that λ is not a Z-eigenvalue. Then we may assume that x and y are linearly independent. From (6), we have

$$\begin{cases} (\mathscr{A}xy)y = \lambda x, \\ (\mathscr{A}xy)x = \lambda y, \\ x^{\mathrm{T}}x = 1, \\ y^{\mathrm{T}}y = 1. \end{cases}$$

Then we have

$$\begin{cases} (\mathscr{A}xy)^2 x = \lambda^2 x, \\ x^{\mathrm{T}}x = 1, \end{cases}$$

and

$$\begin{cases} (\mathscr{A}xy)^2 y = \lambda^2 y, \\ y^{\mathrm{T}}y = 1. \end{cases}$$

If $(\mathscr{A}xy)^2 \neq \lambda^2 I$, then $x = y$ or $x = -y$, contradicting the assumption that x and y are linearly independent. Hence, $(\mathscr{A}xy)^2 = \lambda^2 I$. The conclusion follows now. $\square$

Theorem 6 implies that when $\mathscr{A}$ is fully symmetric and $n = 2$, it is very possible that all the M-eigenvalues are Z-eigenvalues. It is also possible to extend this theorem to the case that $n = 3$. However, it is hard to verify the conditions of this theorem. Hence it is not so useful in practice.

4 Strong ellipticity and second-order positive definiteness

The second-order positive definiteness of the elasticity tensor $\mathscr{A}$ has been considered by Lord Kelvin [24,25] 150 years ago , also see Refs. [4,13]. We say that a fourth-order partially symmetric tensor $\mathscr{A}$ is second-order positive definite if for any matrix $D = (d_{ij}) \in \mathbb{R}^{m\times n},\ D \neq 0$, we have

$$\mathscr{A}D^2 = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} d_{ij} d_{kl} > 0.$$

Actually, in this case, we may regard $D = (d_{ij})$ an mn-dimensional vector d and $\mathscr{A}$ an $mn \times mn$ symmetric matrix A. Then $\mathscr{A}$ is second-order positive definite if and only if the smallest eigenvalue $\mu_{\min}$ of this $mn \times mn$ symmetric matrix A is positive. As we discussed in the introduction, when $m = n = 3$, the strong ellipticity condition holds if $\mathscr{A}$ is second-order positive definite. However, if the strong ellipticity condition holds, $\mathscr{A}$ may not be second-order positive definite. The following example shows this.

Let $n = 2$ and

$$a_{1111} = 1, \quad a_{1112} = 2, \quad a_{1122} = 4, \quad a_{1212} = 12, \quad a_{2121} = 12,$$
$$a_{1222} = 1, \quad a_{1121} = 2, \quad a_{2122} = 1, \quad a_{2222} = 2.$$

Then (12) becomes

$$\begin{aligned} f(x,y) &= \mathscr{A}xyxy \\ &= x_1^2y_1^2 + 4x_1^2y_1y_2 + 12x_1^2y_2^2 + 4x_1x_2y_1^2 + 16x_1x_2y_1y_2 \\ &\quad + 2x_1x_2y_2^2 + 12x_2^2y_1^2 + 2x_2^2y_1y_2 + 2x_2^2y_2^2. \end{aligned} \tag{15}$$

$\mathscr{A}$ has six M-eigenvector pairs:

$$\begin{cases} x = (-0.9946, 0.1040)^{\mathrm{T}}, \\ y = (-0.9946, 0.1040)^{\mathrm{T}}, \end{cases} \quad \begin{cases} x = (-0.6723, 0.7403)^{\mathrm{T}}, \\ y = (-0.6723, 0.7403)^{\mathrm{T}}, \end{cases}$$

$$\begin{cases} x = (0.6309, 0.7759)^{\mathrm{T}}, \\ y = (-0.6167, 0.7872)^{\mathrm{T}}, \end{cases} \quad \begin{cases} x = (0.7153, 0.6989)^{\mathrm{T}}, \\ y = (0.7153, 0.6989)^{\mathrm{T}}, \end{cases}$$

$$\begin{cases} x = (-0.6167, 0.7872)^{\mathrm{T}}, \\ y = (0.6309, 0.7759)^{\mathrm{T}}, \end{cases} \quad \begin{cases} x = (-0.0562, 0.9984)^{\mathrm{T}}, \\ y = (-0.0562, 0.9984)^{\mathrm{T}}, \end{cases}$$

and the corresponding M-eigenvalues are, respectively,

$$0.5837, \quad 7.8222, \quad 2.7964, \quad 13.7558, \quad 2.7964, \quad 1.8882.$$

Since the smallest M-eigenvalue is $0.5837 > 0$, we see that the strong ellipticity condition holds.

The corresponding 4×4 matrix of $\mathscr{A}$ is

$$\begin{pmatrix} a_{1111} & a_{1112} & a_{1121} & a_{1122} \\ a_{1112} & a_{1212} & a_{1122} & a_{1222} \\ a_{1121} & a_{1222} & a_{2121} & a_{2122} \\ a_{1122} & a_{1122} & a_{2122} & a_{2222} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 12 & 4 & 1 \\ 2 & 4 & 12 & 1 \\ 4 & 1 & 1 & 2 \end{pmatrix},$$

whose four eigenvalues are

$$-2.6110, \quad 4.7779, \quad 8.0000, \quad 16.8331.$$

Since the smallest eigenvalue $-2.6110 < 0$, we can see that $\mathscr{A}$ is not second-order positive definite.

5 A direct method for M-eigenvalues and M-eigenvectors when $m = n = 2$

In this section, we present a direct method to find all the M-eigenvalues and M-eigenvector pairs. The key idea here is to reduce the five variables system (6) to a system involving only two variables. Then, for this system of two variables, we may use the Sylvester formula of the resultant to find the solutions.

We have the following theorem.

Theorem 7 *We have the following results on the M-eigenvalues and their corresponding M-eigenvector pairs.*

(a) *If $a_{1112} = a_{1121} = 0$, then $\lambda = a_{1111}$ is an M-eigenvalue of $\mathscr{A}$ and the corresponding M-eigenvector pair is $x = y = (1,0)^{\mathrm{T}}$.*

(b) *For any real roots $(u,v)^{\mathrm{T}}$ of the following equations:*

$$\begin{cases} a_{1121}u^2 + (a_{2121} - a_{1111})uv - a_{1121}v^2 = 0, \\ a_{1112}u^2 + 2a_{1122}uv + a_{2122}v^2 = 0, \end{cases} \tag{16}$$

$$\lambda = a_{1111}u^2 + 2a_{1121}uv + a_{2121}v^2$$

is an M-eigenvalue with the corresponding eigenvector pair

$$x = \frac{(u,v)^{\mathrm{T}}}{\sqrt{u^2+v^2}}, \quad y = (\pm 1, 0)^{\mathrm{T}}.$$

(c) *For any real roots $(u,v)^{\mathrm{T}}$ of the following equations:*

$$\begin{cases} a_{1121}u^2 + 2a_{1122}uv + a_{1222}v^2 = 0, \\ a_{1112}u^2 + (a_{1212} - a_{1111})uv - a_{1112}v^2 = 0, \end{cases} \tag{17}$$

$$\lambda = a_{1111}u^2 + 2a_{1112}uv + a_{1212}v^2$$

is an M-eigenvalue with the corresponding eigenvector pair

$$x = (\pm 1, 0)^{\mathrm{T}}, \quad y = \frac{(u,v)^{\mathrm{T}}}{\sqrt{u^2+v^2}}.$$

(d) *$\lambda = \mathscr{A}xyxy$ is an M-eigenvalue and*

$$x = \pm\frac{1}{\sqrt{u^2+1}}(u,1)^{\mathrm{T}}, \quad y = \pm\frac{1}{\sqrt{v^2+1}}(v,1)^{\mathrm{T}} \tag{18}$$

constitute an M-eigenvector pair, where u and v are real solutions of the following system of polynomial equations:

$$\begin{cases} a_{1121}u^2v^2 + 2a_{1122}u^2v + a_{1222}u^2 + (a_{2121} - a_{1111})uv^2 - a_{1121}v^2 \\ \qquad +2(a_{2122} - a_{1112})uv + (a_{2222} - a_{1212})u - 2a_{1122}v - a_{1222} = 0, \\ a_{1112}u^2v^2 + (a_{1212} - a_{1111})u^2v - a_{1112}u^2 + 2(a_{1222} - a_{1121})uv \\ \qquad +2a_{1122}uv^2 - 2a_{1122}u + a_{2122}v^2 + (a_{2222} - a_{2121})v - a_{2122} = 0. \end{cases} \tag{19}$$

All the M-eigenvalues and the associated M-eigenvector pairs are given by (a)–(d) *if* $a_{1112} = a_{1121} = 0$, *and by* (b)–(d) *otherwise.*

Proof (a) If $a_{1112} = a_{1121} = 0$, it is direct to check that (a) holds.

(b) If $y_2 = 0$, then $y_1 = \pm 1$ and (6) becomes

$$\begin{cases} a_{1111}x_1 + a_{1121}x_2 = \lambda x_1, \\ a_{1121}x_1 + a_{2121}x_2 = \lambda x_2, \\ a_{1111}x_1^2 + 2a_{1121}x_1x_2 + a_{2121}x_2^2 = \lambda, \\ a_{1112}x_1^2 + 2a_{1122}x_1x_2 + a_{2122}x_2^2 = 0, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

Eliminating λ from the first two equations, we get

$$\begin{cases} a_{1121}x_1^2 + (a_{2121} - a_{1111})x_1x_2 - a_{1121}x_2^2 = 0, \\ a_{1112}x_1^2 + 2a_{1122}x_1x_2 + a_{2122}x_2^2 = 0, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

Let

$$u = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad v = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}.$$

Then the results of (b) follows immediately.

(c) It can be proved in a similar way as (b).

(d) If $x_2 \neq 0$ and $y_2 \neq 0$, (6) becomes

$$\begin{cases} a_{1111}x_1y_1^2 + 2a_{1112}x_1y_1y_2 + a_{1212}x_1y_2^2 + a_{1121}x_2y_1^2 \\ \quad +2a_{1122}x_2y_1y_2 + a_{1222}x_2y_2^2 = \lambda x_1, \\ a_{1121}x_1y_1^2 + 2a_{1122}x_1y_1y_2 + a_{1222}x_1y_2^2 + a_{2121}x_2y_1^2 \\ \quad +2a_{2122}x_2y_1y_2 + a_{2222}x_2y_2^2 = \lambda x_2, \\ a_{1111}x_1^2y_1 + a_{1112}x_1^2y_2 + 2a_{1121}x_1x_2y_1 + 2a_{1122}x_1x_2y_2 \\ \quad +a_{2121}x_2^2y_1 + a_{2122}x_2^2y_2 = \lambda y_1, \\ a_{1112}x_1^2y_1 + a_{1212}x_1^2y_2 + 2a_{1122}x_1x_2y_1 + 2a_{1222}x_1x_2y_2 \\ \quad +a_{2122}x_2^2y_1 + a_{2222}x_2^2y_2 = \lambda y_2, \\ x_1^2 + x_2^2 = 1, \\ y_1^2 + y_2^2 = 1. \end{cases}$$

Let $u = x_1/x_2$ and $v = y_1/y_2$. Then from the first two equalities of the above system, we have

$$a_{1121}u^2v^2 + 2a_{1122}u^2v + a_{1222}u^2 + (a_{2121} - a_{1111})uv^2 - a_{1121}v^2 \\ +2(a_{2122} - a_{1112})uv + (a_{2222} - a_{1212})u - 2a_{1122}v - a_{1222} = 0,$$

and from the third and the fourth equalities, we have

$$\begin{aligned} &a_{1112}u^2v^2 + (a_{1212} - a_{1111})u^2v - a_{1112}u^2 + 2(a_{1222} - a_{1121})uv \\ &\quad +2a_{1122}uv^2 - 2a_{1122}u + a_{2122}v^2 + (a_{2222} - a_{2121})v - a_{2122} = 0. \end{aligned}$$

Combining the above two equalities and the assumption that $x_1^2 + x_2^2 = 1$ and $y_1^2 + y_2^2 = 1$, we get the assertion immediately. □

From Theorem 7, we can see that to find all the M-eigenvalues and the associated M-eigenvector pairs, we need to solve some systems of polynomial equations with two variables. To solve such systems, we can use the resultant method from algebraic geometry. For example, to solve (19), we may regard it as equations of u,

$$\begin{cases} \alpha_0 u^2 + \alpha_1 u + \alpha_2 = 0, \\ \beta_0 u^2 + \beta_1 u + \beta_2 = 0, \end{cases} \tag{20}$$

where

$$\begin{aligned} \alpha_0 &= a_{1121}v^2 + 2a_{1122}v + a_{1222}, \\ \alpha_1 &= (a_{2121} - a_{1111})v^2 + 2(a_{2122} - a_{1112})v + (a_{2222} - a_{1212}), \\ \alpha_2 &= -(a_{1121}v^2 + 2a_{1122}v + a_{1222}), \end{aligned} \tag{21}$$

$$\begin{aligned} \beta_0 &= a_{1112}v^2 + a_{1212}v - a_{1112}, \\ \beta_1 &= 2a_{1122}v^2 + 2(a_{1222} - a_{1121})v - 2a_{1122}, \\ \beta_2 &= a_{2122}v^2 + (a_{2222} - a_{2121})v - a_{2122}). \end{aligned} \tag{22}$$

System (20) has solutions if and only if its resultant vanishes [5]. By the Sylvester theorem [5], its resultant can be calculated as a determinant of the following 4×4 matrix:

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & 0 \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 & 0 \\ 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix},$$

which is a polynomial of v. We can use Matlab to find all of its real roots. After this, we substitute them to (19) to find all the real solutions of u. Correspondingly, all the M-eigenvalues and the associated M-eigenvector pairs can be found.

6 SOS method for smallest M-eigenvalue and its M-eigenvectors when $m = n = 3$

When $m = n = 3$, it is not easy to get all the M-eigenvalues and the associated M-eigenvector pairs. In some cases, we just want to know if the strong ellipticity holds or not, which is equivalent to find if the objective value of (4) is positive or not.

A sufficient condition to ensure the objective value of (4) is positive is that there is some positive number ε, such that for all $x,\ y$,

$$x_1^2 + x_2^2 = 1, \quad y_1^2 + y_2^2 = 1,$$

and

$$f(x, y) - \varepsilon = \mathscr{A}xyxy - \varepsilon$$

is the sum of squares of some polynomials, i.e., there are some polynomials $g_1(x, y),\ g_2(x, y)$, and $s_1(x, y),\ s_2(x, y), \ldots, s_m(x, y)$,

$$f(x,y) - \varepsilon = g_1(x,y)(x_1^2 + x_2^2 - 1) + g_2(x,y)(y_1^2 + y_2^2 - 1) + \sum_{i=1}^{t} s_i(x,y)^2. \quad (23)$$

Verifying the above condition is also a difficult task, since the degrees of the polynomials $g_1(x, y),\ g_2(x, y)$ and $s_i(x, y),\ i = 1, \ldots, m$, are unknown. If we further restrict the degree of the polynomials $g_1(x, y),\ g_2(x, y)$ and $s_i(x, y),\ i = 1, \ldots, t$, to be no larger than a positive number d, then (23) is equivalent to a semidefinite programming problem, which can be solved via interior point algorithms efficiently. For details of the SOS method solving optimization problems with polynomials, the readers are referred to Refs. [10,15].

7 Final remarks

In this paper, we studied the strong ellipticity condition via M-eigenvalues. The strong ellipticity condition holds if and only if the smallest M-eigenvalue of the elasticity tensor is positive. A Z-eigenvalue of the elasticity tensor is an M-eigenvalue but not vice versa. In fact, it is relatively easier to find all the Z-eigenvalues and the associated Z-eigenvectors than to find all the M-eigenvalues and the associated M-eigenvectors. In Section 5, we presented a direct method to find all the M-eigenvalues and the associated M-eigenvectors in the case $m = n = 2$. When $m = n = 3$, this task becomes difficult. However, we may use a direct method to find all the Z-eigenvalues and the associated Z-eigenvectors in the case $m = n = 3$. Note that if an M-eigenvalue is not a Z-eigenvalue, then it appears twice, as if we switch the left and the right M-eigenvectors, we get the same M-eigenvalue. Can we use this property to derive a direct method to find all the M-eigenvalues which are not Z-eigenvalues, when $m = n = 3$? We shall leave this for future investigation.

Problem (9) has applications in quantum physics and was proved to be NP-hard in Ref. [12]. It is called a bi-quadratical programming problem there. A practical method to solve it was presented in Ref. [27].

Recently, Chang, Pearson and Zhang [3] have unified the definitions of H-eigenvalues, Z-eigenvalues and D-eigenvalues in Refs. [11,16,19,21]. What is its relation with M-eigenvalues here? Chang, Pearson and Zhang [2] established the Perron-Frobenius theorem for nonnegative tensors. If tensor $\mathscr{A}$ is nonnegative, will problem (4) have some better solution methods?

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