Properties of Tensor Complementarity Problem and Some Classes of Structured Tensors

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November 29, 2014

Abstract

This paper deals with the class of Q-tensors, that is, a Q-tensor is a real tensor A such that the tensor complementarity problem (q, A):

finding \( x \in \mathbb{R}^n \) such that \( x \geq 0, q + Ax^{m-1} \geq 0, \) and \( x^T(q + Ax^{m-1}) = 0, \)

has a solution for each vector \( q \in \mathbb{R}^n \). Several subclasses of Q-tensors are given: P-tensors, R-tensors, strictly semi-positive tensors and semi-positive R_0-tensors. We prove that a nonnegative tensor is a Q-tensor if and only if all of its diagonal entries are positive, and a symmetric tensor is a Q-tensor if and only if it is strictly copositive. We also show that the zero vector is the unique feasible solution of the tensor complementarity problem \((q, A)\) for \( q \geq 0 \) if \( A \) is a nonnegative Q-tensor.

Key words: Q-tensor, R-tensor, R_0-tensor, strictly semi-positive, tensor complementarity problem.


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1 Introduction

Throughout this paper, we use small letters $x, u, v, \cdots$, for scalars, small bold letters $x, y, u, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $A, B, \cdots$, for tensors. All the tensors discussed in this paper are real. Let $I_n := \{1, 2, \cdots, n\}$, and 

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n)^\top; x_i \in \mathbb{R}, i \in I_n\}, \mathbb{R}_n^+: = \{x \in \mathbb{R}^n; x \geq 0\}, \mathbb{R}_n^- := \{x \in \mathbb{R}^n; x \leq 0\}, \mathbb{R}_n^{++} := \{x \in \mathbb{R}^n; x > 0\}, e = (1, 1, \cdots, 1)^\top, \text{ and } x^{[m]} = (x_1^m, x_2^m, \cdots, x_n^m)^\top$$

for $x = (x_1, x_2, \cdots, x_n)^\top$, where $\mathbb{R}$ is the set of real numbers, $x^\top$ is the transposition of a vector $x$, and $x \geq 0$ ($x > 0$) means $x_i \geq 0$ ($x_i > 0$) for all $i \in I_n$.

Let $A = (a_{ij})$ be an $n \times n$ real matrix. $A$ is said to be a $\textbf{Q-matrix}$ iff the linear complementarity problem, denoted by $(q, A)$, finding $z \in \mathbb{R}^n$ such that $z \geq 0, q + Az \geq 0$, and $z^\top (q + Az) = 0 \quad (1.1)$ has a solution for each vector $q \in \mathbb{R}^n$. We say that $A$ is a $\textbf{P-matrix}$ iff for any nonzero vector $x$ in $\mathbb{R}^n$, there exists $i \in I_n$ such that $x_i(Ax)_i > 0$. It is well-known that $A$ is a P-matrix if and only if the linear complementarity problem $(q, A)$ has a unique solution for all $q \in \mathbb{R}^n$. Xiu and Zhang [1] also gave the necessary and sufficient conditions of P-tensors.

A good review of P-matrices and Q-matrices may be found in the books by Berman and Plemmons [2], and Cottle and Pang [3].


On the other hand, motivated by the discussion on positive definiteness of multivariate homogeneous polynomial forms [12, 13, 14], in 2005, Qi [15] introduced the concept of positive (semi-)definite symmetric tensors. In the same time, Qi also introduced eigenvalues, H-eigenvalues, E-eigenvalues and Z-eigenvalues for symmetric tensors. It was shown that an even order symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues or Z-eigenvalues are positive (nonnegative) [15, Theorem 5]. Recently, miscellaneous structured tensors are widely studied, for example, Zhang, Qi and Zhou [16] and Ding, Qi and Wei [17] for M-tensors, Song and Qi [18] for $P$-($P_0$)tensors and $B$-($B_0$)tensors, Qi and Song
for positive (semi-)definition of B-(B₀) tensors, Song and Qi [20] for infinite and finite dimensional Hilbert tensors, Song and Qi [21] for structure properties and an equivalent definition of (strictly) copositive tensors, Chen and Qi [22] for Cauchy tensor, Song and Qi [23] for E-eigenvalues of weakly symmetric nonnegative tensors and so on. Beside automatical control, positive semi-definite tensors also found applications in magnetic resonance imaging [24, 25, 26, 27] and spectral hypergraph theory [28, 29, 30].

The following questions are natural. Can we extend the concept of Q-matrices to Q-tensors? If this can be done, are those nice properties of Q-matrices still true for Q-tensors?

In this paper, we will introduce the concept of Q-tensors (Q-hypermatrices) and will study some subclasses and nice properties of such tensors.

In Section 2, we will extend the concept of Q-matrices to Q-tensors. Several main subclass of Q-matrices also are extended to the corresponding subclass of Q-tensors: R-tensor, R₀-tensor, semi-positive tensor, strictly semi-positive tensor. We will give several examples to verify that the class of R-(R₀-)tensors properly contains strictly semi-positive tensors as a subclass, while the class of P-tensors is a subclass of strictly semi-positive tensors. Some basic definitions and facts also are given in this section.

In Section 3, we will study some properties of Q-tensors. Firstly, we will prove that each R-tensor is certainly a Q-tensor and each semi-positive R₀-tensor is a R-tensor. Thus, we obtain that each P-tensor is a Q-tensor. We will show that a nonnegative tensor is a Q-tensor if and only if its all diagonal elements are positive and a nonnegative symmetric tensor is a Q-tensor if and only if it is strictly copositive. It will be proved that 0 is the unique feasible solution of the tensor complementarity problem (q, A) for q ≥ 0 if A is a non-negative Q-tensor.

2 Preliminaries

In this section, we will define the notation and collect some basic definitions and facts, which will be used later on.

A real mth order n-dimensional tensor (hypermatrix) $\mathbf{A} = (a_{i_1\cdots i_m})$ is a multi-array of real entries $a_{i_1\cdots i_m}$, where $i_j \in I_n$ for $j \in I_m$. Denote the set of all real mth order n-dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension $n^m$. Let $\mathbf{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. If the entries $a_{i_1\cdots i_m}$ are invariant under any permutation of their indices, then $\mathbf{A}$ is called a symmetric tensor. Denote the set of all real mth order n-dimensional tensors by $S_{m,n}$. Then $S_{m,n}$ is a linear subspace of $T_{m,n}$. We denote the zero tensor in $T_{m,n}$ by $\mathbf{O}$. Let $\mathbf{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ with its ith component as

$$(\mathbf{A}\mathbf{x}^{m-1})_i := \sum_{i_2, \ldots, i_m = 1}^n a_{i_1i_2\cdots i_m}x_{i_2}\cdots x_{i_m}$$
for $i \in I_n$. Then $Ax^m$ is a homogeneous polynomial of degree $m$, defined by

$$Ax^m := x^\top (Ax^{m-1}) = \sum_{i_1, \ldots, i_m = 1}^n a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$  

$x \in \mathbb{R}^n$. A tensor $A \in T_{m,n}$ is called positive semi-definite if for any vector $x \in \mathbb{R}^n$, $Ax^m \geq 0$, and is called positive definite if for any nonzero vector $x \in \mathbb{R}^n$, $Ax^m > 0$. Clearly, if $m$ is odd, there is no nontrivial positive semi-definite tensors. We now give the definition of Q-tensors, which are natural extensions of Q-matrices.

**Definition 2.1.** Let $A = (a_{i_1 \cdots i_m}) \in T_{m,n}$. We say that $A$ is a Q-tensor iff the tensor complementarity problem, denoted by $(q; A)$, finding $x \in \mathbb{R}^n$ such that $x \geq 0, q + Ax^{m-1} \geq 0,$ and $x^\top (q + Ax^{m-1}) = 0,$ has a solution for each vector $q \in \mathbb{R}^n$.

**Definition 2.2.** Let $A = (a_{i_1 \cdots i_m}) \in T_{m,n}$. We say that $A$ is

(i) a $R$-tensor iff the following system is inconsistent

$$\begin{cases} 0 \neq x \geq 0, \ t \geq 0 \\ (Ax^{m-1})_i + t = 0 \text{ if } x_i > 0, \\ (Ax^{m-1})_j + t \geq 0 \text{ if } x_j = 0; \end{cases}$$

(ii) a $R_0$-tensor iff the system (2.2) is inconsistent for $t = 0$.

Clearly, this definition 2.2 is a natural extension of the definition of Karamardian’s class of regular matrices [31].

**Definition 2.3.** Let $A = (a_{i_1 \cdots i_m}) \in T_{m,n}$. $A$ is said to be

(i) semi-positive iff for each $x \geq 0$ and $x \neq 0$, there exists an index $k \in I_n$ such that $x_k > 0$ and $(Ax^{m-1})_k \geq 0$;

(ii) strictly semi-positive iff for each $x \geq 0$ and $x \neq 0$, there exists an index $k \in I_n$ such that $x_k > 0$ and $(Ax^{m-1})_k > 0$;

(iii) a $P$-tensor (Song and Qi [18]) iff for each $x$ in $\mathbb{R}^n$ and $x \neq 0$, there exists $i \in I_n$ such that $x_i (Ax^{m-1})_i > 0$;
(iv) a $\mathbf{P}_0$-tensor (Song and Qi [18]) iff for every $\mathbf{x}$ in $\mathbb{R}^n$ and $\mathbf{x} \neq 0$, there exists $i \in I_n$ such that $x_i \neq 0$ and

$$x_i \left( \mathbf{A} \mathbf{x}^{m-1} \right)_i \geq 0.$$ 

Clearly, each $\mathbf{P}_0$-tensor is certainly semi-positive. The concept of $\mathbf{P}$-($\mathbf{P}_0$-)tensor is introduced by Song and Qi [18]. Furthermore, Song and Qi [18] studied some nice properties of such a class of tensors. The definition of (strictly) semi-positive tensor is a natural extension of the concept of (strictly) semi-positive (or semi-monotone) matrices [11, 32].

It follows from Definition 2.2 and 2.3 that each $\mathbf{P}$-tensor must be strictly semi-positive and every strictly semi-positive tensor is certainly both $\mathbf{R}$-tensor and $\mathbf{R}_0$-tensor. Now we give several examples to demonstrate that the above inclusions are proper.

**Example 2.1.** Let $\hat{\mathbf{A}} = (a_{i_1\cdots i_m}) \in T_{m,n}$ and $a_{i_1\cdots i_m} = 1$ for all $i_1, i_2, \cdots, i_m \in I_n$. Then

$$\left( \hat{\mathbf{A}} \mathbf{x}^{m-1} \right)_i = (x_1 + x_2 + \cdots + x_n)^{m-1}$$

for all $i \in I_n$ and hence $\hat{\mathbf{A}}$ is strictly semi-positive. However, $\hat{\mathbf{A}}$ is not a $\mathbf{P}$-tensor (for example, $x_i \left( \hat{\mathbf{A}} \mathbf{x}^{m-1} \right)_i = 0$ for $\mathbf{x} = (1, -1, 0, \cdots, 0)^\top$ and all $i \in I_n$).

**Example 2.2.** Let $\hat{\mathbf{A}} = (a_{i_1i_2i_3}) \in T_{3,2}$ and $a_{111} = 1, a_{122} = -1, a_{211} = -2, a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then

$$\hat{\mathbf{A}} \mathbf{x}^2 = \begin{pmatrix} x_1^2 - x_2^2 \\ -2x_1^2 + x_2^2 \end{pmatrix}.$$ 

Clearly, $\hat{\mathbf{A}}$ is not strictly semi-positive (for example, $(\hat{\mathbf{A}} \mathbf{x}^2)_1 = 0$ and $(\hat{\mathbf{A}} \mathbf{x}^2)_2 = -1$ for $\mathbf{x} = (1, 1)^\top$).

$\hat{\mathbf{A}}$ is a $\mathbf{R}_0$-tensor. In fact,

(i) if $x_1 > 0$, $(\hat{\mathbf{A}} \mathbf{x}^2)_1 = x_1^2 - x_2^2 = 0$. Then $x_2 = x_1^2$, and so $x_2 > 0$, but $(\hat{\mathbf{A}} \mathbf{x}^2)_2 = -2x_1^2 + x_2^2 = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $(\hat{\mathbf{A}} \mathbf{x}^2)_2 = -2x_1^2 + x_2^2 = 0$. Then $x_1^2 = \frac{1}{2}x_2^2 > 0$, but $(\hat{\mathbf{A}} \mathbf{x}^2)_1 = x_1^2 - x_2^2 = -\frac{1}{2}x_2^2 < 0$.

$\hat{\mathbf{A}}$ is not a $\mathbf{R}$-tensor. In fact, if $x_1 > 0$, $(\hat{\mathbf{A}} \mathbf{x}^2)_1 + t = x_1^2 - x_2^2 + t = 0$. Then $x_2^2 = x_1^2 + t > 0$, and so $x_2 > 0$, $(\hat{\mathbf{A}} \mathbf{x}^2)_2 + t = -2x_1^2 + x_2^2 + t = -x_1^2 + 2t$. Taking $x_1 = a > 0$, $t = \frac{1}{2}a^2$ and $x_2 = \frac{\sqrt{6}}{2}a$. That is, $\mathbf{x} = a(1, \frac{\sqrt{6}}{2})^\top$ and $t = \frac{1}{2}a^2$ solve the system (2.2).

**Example 2.3.** Let $\tilde{\mathbf{A}} = (a_{i_1i_2i_3}) \in T_{3,2}$ and $a_{111} = -1, a_{122} = 1, a_{211} = -2, a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then

$$\tilde{\mathbf{A}} \mathbf{x}^2 = \begin{pmatrix} -x_1^2 + x_2^2 \\ -2x_1^2 + x_2^2 \end{pmatrix}.$$ 

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Clearly, $\bar{A}$ is not strictly semi-positive (for example, $\mathbf{x} = (1, 1)^\top$).

$\bar{A}$ is a $R$-tensor. In fact,

(i) if $x_1 > 0$, $(\bar{A}\mathbf{x}^2)_1 + t = -x_1^2 + x_2^2 + t = 0$. Then $x_2^2 = x_1^2 - t$, but $(\bar{A}\mathbf{x}^2)_2 + t = -2x_1^2 + x_2^2 + t = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $(\bar{A}\mathbf{x}^2)_2 + t = -2x_1^2 + x_2^2 + t = 0$. Then $x_1^2 = \frac{1}{2}(x_2^2 + t) > 0$, but $(\bar{A}\mathbf{x}^2)_1 + t = -x_1^2 + x_2^2 + t = \frac{1}{2}(x_2^2 + t) > 0$.

$\bar{A}$ is a $R_0$-tensor. In fact,

(i) if $x_1 > 0$, $(\bar{A}\mathbf{x}^2)_1 = -x_1^2 + x_2^2 = 0$. Then $x_2^2 = x_1^2$, and so $x_2 > 0$, but $(\bar{A}\mathbf{x}^2)_2 = -2x_1^2 + x_2^2 = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $(\bar{A}\mathbf{x}^2)_2 = -2x_1^2 + x_2^2 = 0$. Then $x_1^2 = \frac{1}{2}x_2^2 > 0$, but $(\bar{A}\mathbf{x}^2)_1 = -x_1^2 + x_2^2 = \frac{1}{2}x_2^2 > 0$.

**Lemma 2.1.** ([2, Corollary 3.5]) Let $S = \{\mathbf{x} \in \mathbb{R}^{n+1}_+; \sum_{i=1}^{n+1} x_i = 1\}$. Assumed that $F : S \to \mathbb{R}^{n+1}$ is continuous on $S$. Then there exists $\bar{\mathbf{x}} \in S$ such that

$$\mathbf{x}^\top F(\bar{\mathbf{x}}) \geq \bar{\mathbf{x}}^\top F(\bar{\mathbf{x}}) \quad \text{for all } \mathbf{x} \in S \quad (2.3)$$

$$(F(\bar{\mathbf{x}}))_{i_k} = \min_{i \in I_{n+1}} (F(\bar{\mathbf{x}}))_{i} = \omega \text{ if } x_k > 0, \quad (2.4)$$

$$(F(\bar{\mathbf{x}}))_{i_k} \geq \omega \text{ if } x_k = 0. \quad (2.5)$$

Recall that a tensor $C \in T_{m,r}$ is called a **principal sub-tensor** of a tensor $A = (a_{i_1...i_m}) \in T_{m,n}$ $(1 \leq r \leq n)$ iff there is a set $J$ that composed of $r$ elements in $I_n$ such that

$$C = (a_{i_1...i_m}) \text{, for all } i_1, i_2, \cdots, i_m \in J.$$ 

The concept were first introduced and used in [15] for symmetric tensor. We denote by $A^r_J$ the principal sub-tensor of a tensor $A \in T_{m,n}$ such that the entries of $A^r_J$ are indexed by $J \subset I_n$ with $|J| = r$ $(1 \leq r \leq n)$, and denote by $\mathbf{x}_J$ the $r$-dimensional sub-vector of a vector $\mathbf{x} \in \mathbb{R}^n$, with the components of $\mathbf{x}_J$ indexed by $J$. Note that for $r = 1$, the principal sub-tensors are just the diagonal entries.

**Definition 2.4.** (Song and Qi [21]) Let $A = (a_{i_1...i_m}) \in S_{m,n}$. $A$ is said to be

(i) **copositive** if $Ax^m \geq 0$ for all $x \in \mathbb{R}^n_+$;

(ii) **strictly copositive** if $Ax^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$.

The concept of (strictly) copositive were first introduced and used by Song and Qi in [21]. They showed their equivalent definition and some special structures. The following lemma is one of the structure conclusions of (strictly) copositive in [21].
Lemma 2.2. ([21, Corollary 4.6]) Let \( A = (a_{i_1 \ldots i_m}) \in S_{m,n} \). Then

(i) If \( A \) is copositive, then \( a_{ii} \geq 0 \) for all \( i \in I_n \).

(ii) If \( A \) is strictly copositive, then \( a_{ii} > 0 \) for all \( i \in I_n \).

3 Main results

Theorem 3.1. Let \( A = (a_{i_1 \ldots i_m}) \in T_{m,n} \) be a R-tensor. Then \( A \) is a Q-tensor. That is, the tensor complementarity problem \((q, A)\) (2.1) has a solution for all \( q \in \mathbb{R}^n \).

Proof. Let the mapping \( F : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+^{n+1} \) be defined by

\[
F(y) = \begin{pmatrix} Ax^{m-1} + sq + se \\ s \end{pmatrix},
\]

where \( y = (x, s)^T, x \in \mathbb{R}_+^n, s \in \mathbb{R}_+ \) and \( e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n, q \in \mathbb{R}^n \). Obviously, \( F : S \rightarrow \mathbb{R}_+^{n+1} \) is continuous on the set \( S = \{x \in \mathbb{R}_+^{n+1}; \sum_{i=1}^{n+1} x_i = 1\} \). It follows from Lemma 2.1 that there exists \( \tilde{y} = (\tilde{x}, \tilde{s})^T \in S \) such that

\[
y^T F(y) \geq \tilde{y}^T F(\tilde{y}) \text{ for all } y \in S
\]

(3.2)

\[
(F(\tilde{y}))_k = \min_{i \in I_{n+1}} (F(\tilde{y}))_i = \omega \text{ if } \tilde{y}_k > 0,
\]

(3.3)

\[
(F(\tilde{y}))_k \geq \omega \text{ if } \tilde{y}_k = 0.
\]

(3.4)

We claim \( \tilde{s} > 0 \). In fact, suppose \( \tilde{s} = 0 \), then the fact that \( \tilde{y}_{n+1} = \tilde{s} = 0 \) together with (3.4) implies that

\[
\omega \leq (F(\tilde{y}))(n+1) = \tilde{s} = 0,
\]

and so for \( k \in I_n \),

\[
(F(\tilde{y}))_k = (Ax^{m-1})_k = \omega \text{ if } \tilde{x}_k > 0,
\]

(3.5)

\[
(F(\tilde{y}))_k \geq \omega \text{ if } \tilde{x}_k = 0.
\]

(3.6)

That is, for \( t = -\omega \geq 0 \),

\[
(Ax^{m-1})_k + t = 0 \text{ if } \tilde{x}_k > 0,
\]

(3.7)

\[
(Ax^{m-1})_k + t \geq 0 \text{ if } \tilde{x}_k = 0.
\]

(3.8)

This obtain a contradiction with the definition of R-tensor \( A \), which completes the proof of the claim.
Now we show that the tensor complementarity problem \((\mathbf{q}, \mathbf{A})\) has a solution for all \(\mathbf{q} \in \mathbb{R}^n\). In fact, if \(\mathbf{q} \geq 0\), clearly \(\mathbf{z} = 0\) and \(\mathbf{w} = \mathbf{A}\mathbf{z}^{m-1} + \mathbf{q} = \mathbf{q}\) solve \((\mathbf{q}, \mathbf{A})\). Next we consider \(\mathbf{q} \in \mathbb{R}^n/\mathbb{R}_+^n\). It follows from (3.1) and (3.3) and (3.4) that we must have

\[
(F(\tilde{\mathbf{y}}))_{n+1} = \min_{i \in I_{n+1}} (F(\tilde{\mathbf{y}}))_i = \omega = \tilde{s} = \tilde{y}_{n+1} > 0
\]

and for \(i \in I_n\),

\[
(F(\tilde{\mathbf{y}}))_i = (\mathbf{A}\tilde{\mathbf{x}}^{m-1})_i + \tilde{s}q_i + \tilde{s} = \omega = \tilde{s} \quad \text{if} \quad \tilde{y}_i = \tilde{x}_i > 0,
\]

\[
(F(\tilde{\mathbf{y}}))_i = (\mathbf{A}\tilde{\mathbf{x}}^{m-1})_i + \tilde{s}q_i + \tilde{s} \geq \omega = \tilde{s} \quad \text{if} \quad \tilde{y}_i = \tilde{x}_i = 0.
\]

Thus for \(\mathbf{z} = \frac{\tilde{s}}{\tilde{s}^{m-1}}\) and \(i \in I_n\), we have

\[
(\mathbf{A}\mathbf{z}^{m-1})_i + q_i = 0 \quad \text{if} \quad z_i > 0,
\]

\[
(\mathbf{A}\mathbf{z}^{m-1})_i + q_i \geq 0 \quad \text{if} \quad z_i = 0,
\]

and hence,

\[
\mathbf{z} \geq 0, \quad \mathbf{w} = \mathbf{q} + \mathbf{A}\mathbf{z}^{m-1} \geq 0, \quad \text{and} \quad \mathbf{z}^\top \mathbf{w} = 0.
\]

So we obtain a feasible solution \((\mathbf{z}, \mathbf{w})\) of the tensor complementarity problem \((\mathbf{q}, \mathbf{A})\), and then \(\mathbf{A}\) is a Q-tensor. The theorem is proved.

**Corollary 3.2.** Each strictly semi-positive tensor is a Q-tensor, and so is P-tensor. That is, the tensor complementarity problem \((\mathbf{q}, \mathbf{A})\) has a solution for all \(\mathbf{q} \in \mathbb{R}^n\) if \(\mathbf{A}\) is either a P-tensor or a strictly semi-positive tensor.

**Theorem 3.3.** Let a \(\mathbb{R}_0\)-tensor \(\mathbf{A}(\in T_{m,n})\) be semi-positive. Then \(\mathbf{A}\) is a R-tensor, and hence \(\mathbf{A}\) is a Q-tensor.

**Proof.** Suppose \(\mathbf{A}\) is not a R-tensor. Let the system (2.2) has a solution \(\bar{\mathbf{x}} \geq 0\) and \(\bar{\mathbf{x}} \neq 0\). If \(t = 0\), this contradicts the assumption that \(\mathbf{A}\) is a \(\mathbb{R}_0\)-tensor. So we must have \(t > 0\). Then for \(i \in I_n\), we have

\[
(\mathbf{A}\mathbf{x}^{m-1})_i + t = 0 \quad \text{if} \quad x_i > 0,
\]

and hence,

\[
(\mathbf{A}\mathbf{x}^{m-1})_i = -t < 0 \quad \text{if} \quad x_i > 0,
\]

which contradicts the assumption that \(\mathbf{A}\) is semi-positive. So \(\mathbf{A}\) is a R-tensor, and hence \(\mathbf{A}\) is a Q-tensor by Theorem 3.1.

**Theorem 3.4.** Let \(\mathbf{A} = (a_{i_1\cdots i_m}) \in T_{m,n}\) with \(\mathbf{A} \succeq \mathbf{O}\) \((a_{i_1\cdots i_m} \geq 0 \text{ for all } i_1 \cdots i_m \in I_n)\). Then \(\mathbf{A}\) is a Q-tensor if and only if \(a_{ii_1\cdots i} > 0\) for all \(i \in I_n\).
Proof. Sufficiency. If \( a_{ii...i} > 0 \) for all \( i \in I_n \) and \( \mathcal{A} \geq \mathcal{O} \), then it follows from the definition 2.3 of the strictly semi-positive tensor that \( \mathcal{A} \) is strictly semi-positive, and hence \( \mathcal{A} \) is a Q-tensor by Corollary 3.2.

Necessity. Suppose that there exists \( k \in I_n \) such that \( a_{kk...k} = 0 \). Let \( \mathbf{q} = (q_1, \ldots, q_n)^\top \) with \( q_k < 0 \) and \( q_i > 0 \) for all \( i \in I_n \) and \( i \neq k \). Since \( \mathcal{A} \) is a Q-tensor, the tensor complementarity problem \((\mathbf{q}, \mathcal{A})\) has at least a solution. Let \( \mathbf{z} \) is be a feasible solution to \((\mathbf{q}, \mathcal{A})\). Then

\[
\mathbf{z} \geq 0, \quad \mathbf{w} = \mathcal{A} \mathbf{z}^{m-1} + \mathbf{q} \geq 0 \quad \text{and} \quad \mathbf{z}^\top \mathbf{w} = 0.
\]

Clearly, \( \mathbf{z} \neq \mathbf{0} \). Since \( \mathbf{z} \geq 0 \) and \( \mathcal{A} \geq \mathcal{O} \) together with \( q_i > 0 \) for each \( i \in I_n \) with \( i \neq k \), we must have

\[
w_i = (\mathcal{A} \mathbf{z}^{m-1})_i + q_i = \sum_{i_2, \ldots, i_m=1}^n a_{ii_2 \cdots i_m} z_{i_2} \cdots z_{i_m} + q_i > 0 \quad \text{for} \quad i \neq k \quad \text{and} \quad i \in I_n.
\]

It follows from (3.5) that

\[
z_i = 0 \quad \text{for} \quad i \neq k \quad \text{and} \quad i \in I_n.
\]

Thus, we have

\[
w_k = (\mathcal{A} \mathbf{z}^{m-1})_k + q_k = \sum_{i_2, \ldots, i_m=1}^n a_{k_{i_2} \cdots i_m} z_{i_2} \cdots z_{i_m} + q_k = a_{kk \cdots k} z_k^{m-1} + q_k < 0
\]

since \( a_{kk \cdots k} = 0 \). This contradicts the fact that \( \mathbf{w} \geq \mathbf{0} \), so \( a_{ii...i} > 0 \) for all \( i \in I_n \). \( \Box \)

Corollary 3.5. Let a non-negative tensor \( \mathcal{A} \) be a Q-tensor. Then all principal sub-tensors of \( \mathcal{A} \) are also Q-tensors.

Corollary 3.6. Let a non-negative tensor \( \mathcal{A} \) be a Q-tensor. Then \( \mathbf{0} \) is the unique feasible solution to the tensor complementarity problem \((\mathbf{q}, \mathcal{A})\) for \( \mathbf{q} \geq \mathbf{0} \).

Proof. It follows from Theorem 3.4 that \( a_{ii...i} > 0 \) for all \( i \in I_n \), and hence

\[
(\mathcal{A} \mathbf{x}^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n a_{ii_2 \cdots i_m} x_{i_1} \cdots x_{i_m} = a_{ii...i} x_i^{m-1} + \sum_{(i_2, \ldots, i_m) \neq (i, \ldots, i)} a_{ii_2 \cdots i_m} x_{i_1} \cdots x_{i_m}.
\]

If \( \mathbf{x} = (x_1, \ldots, x_n)^\top \) is any feasible solution of the tensor complementarity problem \((\mathbf{q}, \mathcal{A})\), then we have

\[
\mathbf{x} \geq 0, \quad \mathbf{w} = \mathcal{A} \mathbf{x}^{m-1} + \mathbf{q} \geq 0 \quad \text{and} \quad \mathbf{x}^\top \mathbf{w} = \mathcal{A} \mathbf{x}^m + \mathbf{x}^\top \mathbf{q} = 0.
\]

Suppose \( x_i > 0 \) for some \( i \in I_n \). Then

\[
w_i = (\mathcal{A} \mathbf{x}^{m-1})_i + q_i = a_{ii...i} x_i^{m-1} + \sum_{(i_2, \ldots, i_m) \neq (i, \ldots, i)} a_{ii_2 \cdots i_m} x_{i_1} \cdots x_{i_m} + q_i > 0,
\]

and hence, \( \mathbf{x}^\top \mathbf{w} = x_i w_i + \sum_{k \neq i} x_k w_k > 0 \). This contradicts the fact that \( \mathbf{x}^\top \mathbf{w} = 0 \). Consequently, \( x_i = 0 \) for all \( i \in I_n \). \( \Box \)
Proposition 3.7. Let $A \in S_{m,n}$ be non-negative. Then $A$ is strictly copositive if and only if $a_{ii \ldots} > 0$ for all $i \in I_n$.

Proof. The necessity follows from Lemma 2.2. Now we show the sufficiency. Suppose $A$ is not strictly copositive. Then there exists $x \in \mathbb{R}_+^n \setminus \{0\}$ such that

$$x^\top (Ax^{m-1}) = Ax^m \leq 0.$$ 

Since $x \in \mathbb{R}_+^n \setminus \{0\}$, without loss of generality, we may assume $x_1 > 0$. Then by $A \geq 0$, we must have

$$a_{11 \ldots} x_1^m \leq \sum_{i_1, \ldots, i_m = 1}^n a_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m} = Ax^m \leq 0.$$ 

Thus, $a_{11 \ldots} \leq 0$. The contradiction establishes the proposition.

Corollary 3.8. Let $A \in S_{m,n}$ be non-negative. Then $A$ is a Q-tensor if and only if $A$ is strictly copositive.

References


19. Qi, L., Song, Y.: An even order symmetric B tensor is positive definite. Linear Algebra Appl. 457, 303-312 (2014)


