# Properties of Tensor Complementarity Problem and Some Classes of Structured Tensors

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**Abstract**

This paper deals with the class of Q-tensors, that is, a Q-tensor is a real tensor $\mathcal{A}$ such that the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$:

$$\text{finding } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} \geq \mathbf{0}, \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \text{ and } \mathbf{x}^\top(\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}) = 0,$$

has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$. Several subclasses of Q-tensors are given: P-tensors, R-tensors, strictly semi-positive tensors and semi-positive $\mathrm{R}_0$-tensors. We prove that a nonnegative tensor is a Q-tensor if and only if all of its diagonal entries are positive, and a symmetric tensor is a Q-tensor if and only if it is strictly copositive. We also show that the zero vector is the unique feasible solution of the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ for $\mathbf{q} \geq \mathbf{0}$ if $\mathcal{A}$ is a nonnegative Q-tensor.

**Key words:** Q-tensor, R-tensor, $\mathrm{R}_0$-tensor, strictly semi-positive, tensor complementarity problem.

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# 1 Introduction

Throughout this paper, we use small letters $x, u, v, \alpha, \cdots$, for scalars, small bold letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. All the tensors discussed in this paper are real. Let $I_n := \{1, 2, \cdots, n\}$, and $\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n)^\top; x_i \in \mathbb{R}, i \in I_n\}$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n; x \geq \mathbf{0}\}$, $\mathbb{R}^n_- := \{\mathbf{x} \in \mathbb{R}^n; x \leq \mathbf{0}\}$, $\mathbb{R}^n_{++} := \{\mathbf{x} \in \mathbb{R}^n; x > \mathbf{0}\}$, $\mathbf{e} = (1, 1, \cdots, 1)^\top$, and $\mathbf{x}^{[m]} = (x_1^m, x_2^m, \cdots, x_n^m)^\top$ for $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top$, where $\mathbb{R}$ is the set of real numbers, $\mathbf{x}^\top$ is the transposition of a vector $\mathbf{x}$, and $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) means $x_i \geq 0$ ($x_i > 0$) for all $i \in I_n$.

Let $A = (a_{ij})$ be an $n \times n$ real matrix. $A$ is said to be a **Q-matrix** iff the linear complementarity problem, denoted by $(\mathbf{q}, A)$,

$$\text{finding } \mathbf{z} \in \mathbb{R}^n \text{ such that } \mathbf{z} \geq \mathbf{0}, \mathbf{q} + A\mathbf{z} \geq \mathbf{0}, \text{ and } \mathbf{z}^\top(\mathbf{q} + A\mathbf{z}) = 0 \tag{1.1}$$

has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$. We say that $A$ is a **P-matrix** iff for any nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$, there exists $i \in I_n$ such that $x_i(Ax)_i > 0$. It is well-known that $A$ is a P-matrix if and only if the linear complementarity problem $(\mathbf{q}, A)$ has a unique solution for all $\mathbf{q} \in \mathbb{R}^n$. Xiu and Zhang [1] also gave the necessary and sufficient conditions of P-tensors. A good review of P-matrices and Q-matrices may be found in the books by Berman and Plemmons [2], and Cottle and Pang [3].

Q-matrices and $P(P_0)$-matrices have a long history and wide applications in mathematical sciences. Pang [4] showed that each semi-monotone $R_0$-matrix is a Q-matrix. Pang [5] gave a class of Q-matrices which includes N-matrices and strictly semi-monotone matrices. Murty [6] showed that a nonnegative matrix is a Q-matrix if and only if its all diagonal elements are positive. Morris [7] presented two counterexamples of the Q-Matrix conjectures: a matrix is Q-matrix solely by considering the signs of its subdeterminants. Cuttle [8] studied some properties of complete Q-matrices, a subclass of Q-matrices. Kojima and Saigal [9] showed the number of solutions to a class of linear complementarity problems. Gowda [10] proved that a symmetric semi-monotone matrix is a Q-matrix if and only if it is an $R_0$-matrix. Eaves [11] obtained the equivalent definition of strictly semi-monotone matrices, a main subclass of Q-matrices.

On the other hand, motivated by the discussion on positive definiteness of multivariate homogeneous polynomial forms [12, 13, 14], in 2005, Qi [15] introduced the concept of positive (semi-)definite symmetric tensors. In the same time, Qi also introduced eigenvalues, H-eigenvalues, E-eigenvalues and Z-eigenvalues for symmetric tensors. It was shown that an even order symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues or Z-eigenvalues are positive (nonnegative) [15, Theorem 5]. Recently, miscellaneous structured tensors are widely studied, for example, Zhang, Qi and Zhou [16] and Ding, Qi and Wei [17] for M-tensors, Song and Qi [18] for $P$-($P_0$)tensors and $B$-($B_0$)tensors, Qi and Song

[19] for positive (semi-)definition of B-($B_0$)tensors, Song and Qi [20] for infinite and finite dimensional Hilbert tensors, Song and Qi [21] for structure properties and an equivalent definition of (strictly) copositive tensors, Chen and Qi [22] for Cauchy tensor, Song and Qi [23] for E-eigenvalues of weakly symmetric nonnegative tensors and so on. Beside automatical control, positive semi-definite tensors also found applications in magnetic resonance imaging [24, 25, 26, 27] and spectral hypergraph theory [28, 29, 30].

The following questions are natural. Can we extend the concept of Q-matrices to Q-tensors? If this can be done, are those nice properties of Q-matrices still true for Q-tensors?

In this paper, we will introduce the concept of Q-tensors (Q-hypermatrices) and will study some subclasses and nice properties of such tensors.

In Section 2, we will extend the concept of Q-matrices to Q-tensors. Serval main subclass of Q-matrices also are extended to the corresponding subclass of Q-tensors: R-tensor, $R_0$-tensor, semi-positive tensor, strictly semi-positive tensor. We will give serval examples to verify that the class of R-($R_0$-)tensors properly contains strictly semi-positive tensors as a subclass, while the class of P-tensors is a subclass of strictly semi-positive tensors. Some basic definitions and facts also are given in this section.

In Section 3, we will study some properties of Q-tensors. Firstly, we will prove that each R-tensor is certainly a Q-tensor and each semi-positive $R_0$-tensor is a R-tensor. Thus, we obtain that each P-tensor is a Q-tensor. We will show that a nonnegative tensor is a Q-tensor if and only if its all diagonal elements are positive and a nonnegative symmetric tensor is a Q-tensor if and only if it is strictly copositive. It will be proved that $\mathbf{0}$ is the unique feasible solution of the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ for $\mathbf{q} \geq \mathbf{0}$ if $\mathcal{A}$ is a non-negative Q-tensor.

# 2 Preliminaries

In this section, we will define the notation and collect some basic definitions and facts, which will be used later on.

A real $m$th order $n$-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m}$, where $i_j \in I_n$ for $j \in I_m$. Denote the set of all real $m$th order $n$-dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension $n^m$. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. If the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a **symmetric tensor**. Denote the set of all real $m$th order $n$-dimensional tensors by $S_{m,n}$. Then $S_{m,n}$ is a linear subspace of $T_{m,n}$. We denote the zero tensor in $T_{m,n}$ by $\mathcal{O}$. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ with its $i$th component as

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i := \sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

for $i \in I_n$. Then $\mathcal{A}\mathbf{x}^m$ is a homogeneous polynomial of degree $m$, defined by

$$\mathcal{A}\mathbf{x}^m := \mathbf{x}^\top(\mathcal{A}\mathbf{x}^{m-1}) = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1}\cdots x_{i_m}.$$

$\mathbf{x} \in \mathbb{R}^n$. A tensor $\mathcal{A} \in T_{m,n}$ is called **positive semi-definite** if for any vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{A}\mathbf{x}^m \geq 0$, and is called **positive definite** if for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{A}\mathbf{x}^m > 0$. Clearly, if $m$ is odd, there is no nontrivial positive semi-definite tensors. We now give the definition of Q-tensors, which are natural extensions of Q-matrices.

**Definition 2.1.** *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. We say that $\mathcal{A}$ is a* **Q-tensor** *iff the tensor complementarity problem, denoted by $(\mathbf{q}, \mathcal{A})$,*

$$\text{finding } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} \geq \mathbf{0}, \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \text{ and } \mathbf{x}^\top(\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}) = 0, \tag{2.1}$$

*has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$.*

**Definition 2.2.** Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. We say that $\mathcal{A}$ is

(i) a **R-tensor** iff the following system is inconsistent

$$\begin{cases} 0 \neq \mathbf{x} \geq 0,\ t \geq 0 \\ (\mathcal{A}\mathbf{x}^{m-1})_i + t = 0 \text{ if } x_i > 0, \\ (\mathcal{A}\mathbf{x}^{m-1})_j + t \geq 0 \text{ if } x_j = 0; \end{cases} \tag{2.2}$$

(ii) a $\mathbf{R}_0$**-tensor** iff the system (2.2) is inconsistent for $t = 0$.

Clearly, this definition 2.2 is a natural extension of the definition of Karamardian's class of regular matrices [31].

**Definition 2.3.** Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. $\mathcal{A}$ is said to be

(i) **semi-positive** iff for each $\mathbf{x} \geq 0$ and $\mathbf{x} \neq \mathbf{0}$, there exists an index $k \in I_n$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k \geq 0;$$

(ii) **strictly semi-positive** iff for each $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists an index $k \in I_n$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k > 0;$$

(iii) a **P-tensor**(Song and Qi [18]) iff for each $\mathbf{x}$ in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$, there exists $i \in I_n$ such that

$$x_i \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i > 0;$$

(iv) a $\mathbf{P}_0$**-tensor**(Song and Qi [18]) iff for every $\mathbf{x}$ in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$,, there exists $i \in I_n$ such that $x_i \neq 0$ and

$$x_i \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i \geq 0.$$

Clearly, each $\mathrm{P}_0$-tensor is certainly semi-positive. The concept of P-($\mathrm{P}_0$-)tensor is introduced by Song and Qi [18]. Furthermore, Song and Qi [18] studied some nice properties of such a class of tensors. The definition of (strictly) semi-positive tensor is a natural extension of the concept of (strictly) semi-positive (or semi-monotone) matrices [11, 32].

It follows from Definition 2.2 and 2.3 that each P-tensor must be strictly semi-positive and every strictly semi-positive tensor is certainly both R-tensor and $\mathrm{R}_0$-tensor. Now we give serval examples to demonstrate that the above inclusions are proper.

**Example 2.1.** Let $\hat{\mathcal{A}} = (a_{i_1\cdots i_m}) \in T_{m,n}$ and $a_{i_1\cdots i_m} = 1$ for all $i_1, i_2, \cdots, i_m \in I_n$. Then

$$\left(\hat{\mathcal{A}}\mathbf{x}^{m-1}\right)_i = (x_1 + x_2 + \cdots + x_n)^{m-1}$$

for all $i \in I_n$ and hence $\hat{\mathcal{A}}$ is strictly semi-positive. However, $\hat{\mathcal{A}}$ is not a P-tensor (for example, $x_i \left(\hat{\mathcal{A}}\mathbf{x}^{m-1}\right)_i = 0$ for $\mathbf{x} = (1, -1, 0, \cdots, 0)^\top$ and all $i \in I_n$).

**Example 2.2.** Let $\tilde{\mathcal{A}} = (a_{i_1i_2i_3}) \in T_{3,2}$ and $a_{111} = 1$, $a_{122} = -1$,$a_{211} = -2$, $a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then

$$\tilde{\mathcal{A}}\mathbf{x}^2 = \begin{pmatrix} x_1^2 - x_2^2 \\ -2x_1^2 + x_2^2 \end{pmatrix}.$$

Clearly, $\tilde{\mathcal{A}}$ is not strictly semi-positive (for example, $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_1 = 0$ and $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_2 = -1$ for $\mathbf{x} = (1, 1)^\top$).

$\tilde{\mathcal{A}}$ is a $\mathrm{R}_0$-tensor. In fact,

(i) if $x_1 > 0$, $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_1 = x_1^2 - x_2^2 = 0$. Then $x_2^2 = x_1^2$, and so $x_2 > 0$, but $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = 0$. Then $x_1^2 = \frac{1}{2}x_2^2 > 0$, but $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_1 = x_1^2 - x_2^2 = -\frac{1}{2}x_2^2 < 0$.

$\tilde{\mathcal{A}}$ is not a R-tensor. In fact, if $x_1 > 0$, $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_1 + t = x_1^2 - x_2^2 + t = 0$. Then $x_2^2 = x_1^2 + t > 0$, and so $x_2 > 0$, $\left(\tilde{\mathcal{A}}\mathbf{x}^2\right)_2 + t = -2x_1^2 + x_2^2 + t = -x_1^2 + 2t$. Taking $x_1 = a > 0$, $t = \frac{1}{2}a^2$ and $x_2 = \frac{\sqrt{6}}{2}a$. That is, $\mathbf{x} = a(1, \frac{\sqrt{6}}{2})^\top$ and $t = \frac{1}{2}a^2$ solve the system (2.2).

**Example 2.3.** Let $\bar{\mathcal{A}} = (a_{i_1i_2i_3}) \in T_{3,2}$ and $a_{111} = -1$, $a_{122} = 1$,$a_{211} = -2$, $a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then

$$\bar{\mathcal{A}}\mathbf{x}^2 = \begin{pmatrix} -x_1^2 + x_2^2 \\ -2x_1^2 + x_2^2 \end{pmatrix}.$$

Clearly, $\bar{\mathcal{A}}$ is not strictly semi-positive (for example, $\mathbf{x} = (1,1)^\top$).

$\bar{\mathcal{A}}$ is a R-tensor. In fact,

(i) if $x_1 > 0$, $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_1 + t = -x_1^2 + x_2^2 + t = 0$. Then $x_2^2 = x_1^2 - t$, but $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_2 + t = -2x_1^2 + x_2^2 + t = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_2 + t = -2x_1^2 + x_2^2 + t = 0$. Then $x_1^2 = \frac{1}{2}(x_2^2 + t) > 0$, but $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_1 + t = -x_1^2 + x_2^2 + t = \frac{1}{2}(x_2^2 + t) > 0$.

$\bar{\mathcal{A}}$ is a $\mathrm{R}_0$-tensor. In fact,

(i) if $x_1 > 0$, $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_1 = -x_1^2 + x_2^2 = 0$. Then $x_2^2 = x_1^2$, and so $x_2 > 0$, but $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = 0$. Then $x_1^2 = \frac{1}{2}x_2^2 > 0$, but $\left(\bar{\mathcal{A}}\mathbf{x}^2\right)_1 = -x_1^2 + x_2^2 = \frac{1}{2}x_2^2 > 0$.

**Lemma 2.1.** ([2, Corollary 3.5])Let $S = \{\mathbf{x} \in \mathbb{R}^{n+1}_+; \sum\limits_{i=1}^{n+1} x_i = 1\}$. Assumed that $F : S \to \mathbb{R}^{n+1}$ is continuous on $S$. Then there exists $\bar{\mathbf{x}} \in S$ such that

$$\mathbf{x}^\top F(\bar{\mathbf{x}}) \geq \bar{\mathbf{x}}^\top F(\bar{\mathbf{x}}) \textit{ for all } \mathbf{x} \in S \tag{2.3}$$

$$(F(\bar{\mathbf{x}}))_k = \min_{i \in I_{n+1}} (F(\bar{\mathbf{x}}))_i = \omega \textit{ if } x_k > 0, \tag{2.4}$$

$$(F(\bar{\mathbf{x}}))_k \geq \omega \textit{ if } x_k = 0. \tag{2.5}$$

Recall that a tensor $\mathcal{C} \in T_{m,r}$ is called **a principal sub-tensor** of a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ $(1 \leq r \leq n)$ iff there is a set $J$ that composed of $r$ elements in $I_n$ such that

$$\mathcal{C} = (a_{i_1 \cdots i_m}), \text{ for all } i_1, i_2, \cdots, i_m \in J.$$

The concept were first introduced and used in [15] for symmetric tensor. We denote by $\mathcal{A}^J_r$ the principal sub-tensor of a tensor $\mathcal{A} \in T_{m,n}$ such that the entries of $\mathcal{A}^J_r$ are indexed by $J \subset I_n$ with $|J| = r$ $(1 \leq r \leq n)$, and denote by $\mathbf{x}_J$ the $r$-dimensional sub-vector of a vector $\mathbf{x} \in \mathbb{R}^n$, with the components of $\mathbf{x}_J$ indexed by $J$. Note that for $r = 1$, the principal sub-tensors are just the diagonal entries.

**Definition 2.4.** (Song and Qi [21]) Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$. $\mathcal{A}$ is said to be

(i) **copositive** if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$;

(ii) **strictly copositive** if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$.

The concept of (strictly) copositive were first introduced and used by Song and Qi in [21]. They showed their equivalent definition and some special structures. The following lemma is one of the structure conclusions of (strictly) copositive in [21].

**Lemma 2.2.** ([21, Corollary 4.6]) Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$. Then

(i) If $\mathcal{A}$ is copositive, then $a_{ii\cdots i} \geq 0$ for all $i \in I_n$.

(ii) If $\mathcal{A}$ is strictly copositive, then $a_{ii\cdots i} > 0$ for all $i \in I_n$.

# 3 Main results

**Theorem 3.1.** Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ be a R-tensor. Then $\mathcal{A}$ is a Q-tensor. That is, the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ (2.1) has a solution for all $\mathbf{q} \in \mathbb{R}^n$.

*Proof.* Let the mapping $F : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$ be defined by

$$F(\mathbf{y}) = \begin{pmatrix} \mathcal{A}\mathbf{x}^{m-1} + s\mathbf{q} + s\mathbf{e} \\ s \end{pmatrix}, \tag{3.1}$$

where $\mathbf{y} = (\mathbf{x}, s)^\top$, $\mathbf{x} \in \mathbb{R}^n_+$, $s \in \mathbb{R}_+$ and $\mathbf{e} = (1, 1, \cdots, 1)^\top \in \mathbb{R}^n$, $\mathbf{q} \in \mathbb{R}^n$. Obviously, $F : S \to \mathbb{R}^{n+1}$ is continuous on the set $S = \{\mathbf{x} \in \mathbb{R}^{n+1}_+; \sum\limits_{i=1}^{n+1} x_i = 1\}$. It follows from Lemma 2.1 that there exists $\tilde{\mathbf{y}} = (\tilde{\mathbf{x}}, \tilde{s})^\top \in S$ such that

$$\mathbf{y}^\top F(\tilde{\mathbf{y}}) \geq \tilde{\mathbf{y}}^\top F(\tilde{\mathbf{y}}) \quad \text{for all} \quad \mathbf{y} \in S \tag{3.2}$$

$$(F(\tilde{\mathbf{y}}))_k = \min_{i \in I_{n+1}} (F(\tilde{\mathbf{y}}))_i = \omega \quad \text{if} \quad \tilde{y}_k > 0, \tag{3.3}$$

$$(F(\tilde{\mathbf{y}}))_k \geq \omega \quad \text{if} \quad \tilde{y}_k = 0. \tag{3.4}$$

We claim $\tilde{s} > 0$. In fact, suppose $\tilde{s} = 0$, then the fact that $\tilde{y}_{n+1} = \tilde{s} = 0$ together with (3.4) implies that

$$\omega \leq (F(\tilde{\mathbf{y}}))_{n+1} = \tilde{s} = 0,$$

and so for $k \in I_n$,

$$(F(\tilde{\mathbf{y}}))_k = \left(\mathcal{A}\tilde{\mathbf{x}}^{m-1}\right)_k = \omega \quad \text{if} \quad \tilde{x}_k > 0,$$
$$(F(\tilde{\mathbf{y}}))_k = \left(\mathcal{A}\tilde{\mathbf{x}}^{m-1}\right)_k \geq \omega \quad \text{if} \quad \tilde{x}_k = 0.$$

That is, for $t = -\omega \geq 0$,

$$\left(\mathcal{A}\tilde{\mathbf{x}}^{m-1}\right)_k + t = 0 \quad \text{if} \quad \tilde{x}_k > 0,$$
$$\left(\mathcal{A}\tilde{\mathbf{x}}^{m-1}\right)_k + t \geq 0 \quad \text{if} \quad \tilde{x}_k = 0.$$

This obtain a contradiction with the definition of R-tensor $\mathcal{A}$, which completes the proof of the claim.

Now we show that the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ has a solution for all $\mathbf{q} \in \mathbb{R}^n$. In fact, if $\mathbf{q} \geq \mathbf{0}$, clearly $\mathbf{z} = \mathbf{0}$ and $\mathbf{w} = \mathcal{A}\mathbf{z}^{m-1} + \mathbf{q} = \mathbf{q}$ solve $(\mathbf{q}, \mathcal{A})$. Next we consider $\mathbf{q} \in \mathbb{R}^n/\mathbb{R}^n_+$. It follows from (3.1) and (3.3) and (3.4) that we must have

$$(F(\tilde{\mathbf{y}}))_{n+1} = \min_{i \in I_{n+1}} (F(\tilde{\mathbf{y}}))_i = \omega = \tilde{s} = \tilde{y}_{n+1} > 0$$

and for $i \in I_n$,

$$\begin{aligned} (F(\tilde{\mathbf{y}}))_i &= \left(\mathcal{A}\tilde{\mathbf{x}}^{m-1}\right)_i + \tilde{s}q_i + \tilde{s} = \omega = \tilde{s} \ \text{ if } \ \tilde{y}_i = \tilde{x}_i > 0, \\ (F(\tilde{\mathbf{y}}))_i &= \left(\mathcal{A}\tilde{\mathbf{x}}^{m-1}\right)_i + \tilde{s}q_i + \tilde{s} \geq \omega = \tilde{s} \ \text{ if } \ \tilde{y}_i = \tilde{x}_i = 0. \end{aligned}$$

Thus for $\mathbf{z} = \frac{\tilde{\mathbf{x}}}{\tilde{s}^{\frac{1}{m-1}}}$ and $i \in I_n$, we have

$$\begin{aligned} \left(\mathcal{A}\mathbf{z}^{m-1}\right)_i + q_i &= 0 \ \text{ if } \ z_i > 0, \\ \left(\mathcal{A}\mathbf{z}^{m-1}\right)_i + q_i &\geq 0 \ \text{ if } \ z_i = 0, \end{aligned}$$

and hence,

$$\mathbf{z} \geq \mathbf{0}, \mathbf{w} = \mathbf{q} + \mathcal{A}\mathbf{z}^{m-1} \geq \mathbf{0}, \ \text{and } \mathbf{z}^\top \mathbf{w} = 0.$$

So we obtain a feasible solution $(\mathbf{z}, \mathbf{w})$ of the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$, and then $\mathcal{A}$ is a Q-tensor. The theorem is proved. □

**Corollary 3.2.** Each strictly semi-positive tensor is a Q-tensor, and so is P-tensor. That is, the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ has a solution for all $\mathbf{q} \in \mathbb{R}^n$ if $\mathcal{A}$ is either a P-tensor or a strictly semi-positive tensor.

**Theorem 3.3.** Let a $\text{R}_0$-tensor $\mathcal{A} (\in T_{m,n})$ be semi-positive. Then $\mathcal{A}$ is a R-tensor, and hence $\mathcal{A}$ is a Q-tensor.

*Proof.* Suppose $\mathcal{A}$ is not a R-tensor. Let the system (2.2) has a solution $\bar{\mathbf{x}} \geq 0$ and $\bar{\mathbf{x}} \neq 0$. If $t = 0$, this contradicts the assumption that $\mathcal{A}$ is a $\text{R}_0$-tensor. So we must have $t > 0$. Then for $i \in I_n$, we have

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i + t = 0 \ \text{ if } \ x_i > 0,$$

and hence,

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = -t < 0 \ \text{ if } \ x_i > 0,$$

which contradicts the assumption that $\mathcal{A}$ is semi-positive. So $\mathcal{A}$ is a R-tensor, and hence $\mathcal{A}$ is a Q-tensor by Theorem 3.1. □

**Theorem 3.4.** Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ with $\mathcal{A} \geq \mathcal{O}$ ($a_{i_1 \cdots i_m} \geq 0$ for all $i_1 \cdots i_m \in I_n$). Then $\mathcal{A}$ is a Q-tensor if and only if $a_{ii \cdots i} > 0$ for all $i \in I_n$.

*Proof.* Sufficiency. If $a_{ii\cdots i} > 0$ for all $i \in I_n$ and $\mathcal{A} \geq \mathcal{O}$, then it folows from the definition 2.3 of the strictly semi-positive tensor that $\mathcal{A}$ is strictly semi-positive, and hence $\mathcal{A}$ is a Q-tensor by Corollary 3.2.

Necessity. Suppose that there exists $k \in I_n$ such that $a_{kk\cdots k} = 0$. Let $\mathbf{q} = (q_1, \cdots, q_n)^\top$ with $q_k < 0$ and $q_i > 0$ for all $i \in I_n$ and $i \neq k$. Since $\mathcal{A}$ is a Q-tensor, the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ has at least a solution. Let $\mathbf{z}$ is be a feasible solution to $(\mathbf{q}, \mathcal{A})$. Then

$$\mathbf{z} \geq \mathbf{0},\ \mathbf{w} = \mathcal{A}\mathbf{z}^{m-1} + \mathbf{q} \geq \mathbf{0} \text{ and } \mathbf{z}^\top \mathbf{w} = 0. \tag{3.5}$$

Clearly, $\mathbf{z} \neq \mathbf{0}$. Since $\mathbf{z} \geq \mathbf{0}$ and $\mathcal{A} \geq 0$ together with $q_i > 0$ for each $i \in I_n$ with $i \neq k$, we must have

$$w_i = \left(\mathcal{A}\mathbf{z}^{m-1}\right)_i + q_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} z_{i_2} \cdots z_{i_m} + q_i > 0 \text{ for } i \neq k \text{ and } i \in I_n.$$

It follows from (3.5) that

$$z_i = 0 \text{ for } i \neq k \text{ and } i \in I_n.$$

Thus, we have

$$w_k = \left(\mathcal{A}\mathbf{z}^{m-1}\right)_k + q_k = \sum_{i_2,\cdots,i_m=1}^{n} a_{ki_2\cdots i_m} z_{i_2} \cdots z_{i_m} + q_k = a_{kk\cdots k} z_k^{m-1} + q_k = q_k < 0$$

since $a_{kk\cdots k} = 0$. This contradicts the fact that $\mathbf{w} \geq \mathbf{0}$, so $a_{ii\cdots i} > 0$ for all $i \in I_n$. □

**Corollary 3.5.** Let a non-negative tensor $\mathcal{A}$ be a Q-tensor. Then all principal sub-tensors of $\mathcal{A}$ are also Q-tensors.

**Corollary 3.6.** Let a non-negative tensor $\mathcal{A}$ be a Q-tensor. Then $\mathbf{0}$ is the unique feasible solution to the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ for $\mathbf{q} \geq \mathbf{0}$.

*Proof.* It follows from Theorem 3.4 that $a_{ii\cdots i} > 0$ for all $i \in I_n$, and hence

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_1} \cdots x_{i_m} = a_{ii\cdots i} x_i^{m-1} + \sum_{(i_2,\cdots,i_m)\neq(i,\cdots,i)} a_{ii_2\cdots i_m} x_{i_1} \cdots x_{i_m}.$$

If $\mathbf{x} = (x_1, \cdots, x_n)^\top$ is any feasible solution of the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$, then we have

$$\mathbf{x} \geq \mathbf{0},\ \mathbf{w} = \mathcal{A}\mathbf{x}^{m-1} + \mathbf{q} \geq \mathbf{0} \text{ and } \mathbf{x}^\top \mathbf{w} = \mathcal{A}\mathbf{x}^m + \mathbf{x}^\top \mathbf{q} = 0. \tag{3.6}$$

Suppose $x_i > 0$ for some $i \in I_n$. Then

$$w_i = \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i + q_i = a_{ii\cdots i} x_i^{m-1} + \sum_{(i_2,\cdots,i_m)\neq(i,\cdots,i)} a_{ii_2\cdots i_m} x_{i_1} \cdots x_{i_m} + q_i > 0,$$

and hence, $\mathbf{x}^\top \mathbf{w} = x_i w_i + \sum_{k \neq i} x_k w_k > 0$. This contradicts the fact that $\mathbf{x}^\top \mathbf{w} = 0$. Consequently, $x_i = 0$ for all $i \in I_n$. □

**Proposition 3.7.** Let $\mathcal{A} \in S_{m,n}$ be non-negative. Then $\mathcal{A}$ is strictly copositive if and only if $a_{ii\cdots i} > 0$ for all $i \in I_n$.

*Proof.* The necessity follows from Lemma 2.2. Now we show the sufficiency. Suppose $\mathcal{A}$ is not strictly copositive. Then there exists $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that

$$\mathbf{x}^\top \left(\mathcal{A}\mathbf{x}^{m-1}\right) = \mathcal{A}\mathbf{x}^m \leq 0.$$

Since $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, without loss of generality, we may assume $x_1 > 0$. Then by $\mathcal{A} \geq \mathcal{O}$, we must have

$$a_{11\cdots 1}x_1^m \leq \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m}x_{i_1}\cdots x_{i_m} = \mathcal{A}\mathbf{x}^m \leq 0.$$

Thus, $a_{11\cdots 1} \leq 0$. The contradiction establishes the proposition. □

**Corollary 3.8.** Let $\mathcal{A} \in S_{m,n}$ be non-negative. Then $\mathcal{A}$ is a Q-tensor if and only if $\mathcal{A}$ is strictly copositive.

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