An Eigenvalue Method for Testing Positive Definiteness of a Multivariate Form

Qin Ni, Liqun Qi, and Fei Wang

***Abstract*—In this paper, we present an eigenvalue method for testing positive definiteness of a multivariate form. This problem plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control. At first we apply the D'Andrea–Dickenstein version of the classical Macaulay formulas of the resultant to compute the symmetric hyperdeterminant of an even order supersymmetric tensor. By using the supersymmetry property, we give detailed computation procedures for the Bezoutians and specified ordering of monomials in this approach. We then use these formulas to calculate the characteristic polynomial of a fourth order three dimensional supersymmetric tensor and give an eigenvalue method for testing positive definiteness of a quartic form of three variables. Some numerical results of this method are reported.**

***Index Terms*—Eigenvalue method, positive definiteness, supersymmetric tensor, symmetric hyperdeterminant.**

I. Introduction

An mth degree homogeneous polynomial form of n variables $f(x)$, where $x \in \Re^n$ can be denoted as

$$f(x) = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m}.$$

It is called positive definite if

$$f(x) > 0, \qquad \forall x \in R^n, x \neq 0.$$

Clearly, in this case, m must be even.

The positive definiteness of an even-degree homogeneous polynomial form plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control. Stability analysis can be reduced, using Lyapunov's method, to extend a positive definite function, such that its time derivative along the trajectories of the system is negative. Concretely, for the system $\dot{x} = g(x)$, if a multivariate polynomial $f(x)$ can be found such that $f(x)$ is positive definite and

$$\left(\frac{\partial f}{\partial x}\right)^T g(x) < 0, \quad \forall x \in R^n, x \neq 0$$

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then the system $\dot{x} = g(x)$ is asympototically stable. Hence, ascertaining whether a multivariate polynomial $f(x)$ is positive definite for all real x is often crucial to the use of Lyapunov stability tests. In [2], Anderson and Jury showed that tests for n-dimensional filters involve tests for positive definiteness of a set of real polynomials in $n-1$ variables, also see [18]. There are more examples, such as the multivariate network realizability theory [9], a test for Lyapunov stability in multivariable filters [6], a test of existence of periodic oscillations using Bendixon's theorem [16], and the output feedback stabilization problems [1].

Researchers in automatic control studied the conditions of such positive definiteness intensively [5]–[8], [13], [15], [17], [21], [28]. An explicit condition in terms of the coefficients for quartic forms in two variables has been given in [21] (note the comments in [28]). A sufficient condition for multivariable positivity or nonnegativity has also been given in [7]. An implementation of the Gram matrix method for the positive definiteness of forms of even order is presented in [15] (note also the comments in [13]). For $n = 2$, the positive definiteness of a homogeneous polynomial form can be checked by methods based on Sturm's sequences [5], [17]. In [5], the reader may find a discursive documentation of Sturm's theorem and its generalization, resultants, theory and applications of tests for positive definiteness and other results of relevance in this paper.

For $n \geq 3$ and $m \geq 4$, this problem is a hard problem in mathematics. There are a few methods to answer the question, based in decision algebra [6], [8]. In practice, these methods are computationally expensive. This problem is also related with Hilbert's result on representation as sum of squares of forms (discussed in Bose's book [5]). A nonnegative form may not have a sum of squares representation. In that case, a method for testing positive definiteness of the form based upon the sum of squares representation approach cannot find an exact solution of this problem. In this paper, we seek a different approach based upon eigenvalues of tensors. Our approach works even the nonnegative form does not have a sum of squares representation.

The mth degree homogeneous polynomial form of n variables $f(x)$ is equivalent to the tensor product of an mth-order n-dimensional supersymmetric tensor A and x^m defined by

$$f(x) \equiv Ax^m = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m}.$$

The tensor A is called supersymmetric as its entries $a_{i_1,\ldots,i_m}$ are invariant under any permutation of their indexes $i_1, \ldots, i_m$, where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$ [19]. The supersymmetric tensor A is called positive definite if $f(x)$ is positive definite.

Recently, motivated by the study of positive definiteness of homogeneous polynomial, Qi [24] introduced the concepts of eigenvalues of a real supersymmetric tensor A. For a vector $x \in$

C^n, Qi [24] used x_i to denote its components and $x^{[m]}$ to denote a vector in C^n such that

$$(x^{[m]})_i = x_i^m$$

for all i. By the tensor product, Ax^{m-1} for a vector $x \in C^n$ denotes a vector in C^n, whose ith component is

$$\sum_{i_2,\ldots,i_m}^{n} a_{i,i_2,\ldots,i_m} x_{i_2} \cdots x_{i_m}.$$

Qi [24] called a number $\lambda \in C$ and a nonzero vector $x \in C^n$ an **eigenvalue** of A and an **eigenvector** of A associated with the eigenvalue λ respectively, if they are solutions of the following homogeneous polynomial equations:

$$Ax^{m-1} = \lambda x^{[m-1]}. \quad (1)$$

If x is real, then λ is also real. In this case, λ and x are called an **H-eigenvalue** of A and an **H-eigenvector** of A associated with the H-eigenvalue λ, respectively. Otherwise, λ is called an **N-eigenvalue** of A. In the case $m = 2$, (1) reduces to the definition of eigenvalues and corresponding eigenvectors of a square matrix.

It was proved in [24] that H-eigenvalues exist for a real supersymmetric tensor A of even order m, and A is positive definite if and only if all of its H-eigenvalues are positive. Thus, the smallest H-eigenvalue of an even-order supersymmetric tensor A is an indicator of positive definiteness of A. The values of the eigenvalues of A are directly connected with the computation of the **symmetric hyperdeterminant**. In [24], the symmetric hyperdeterminant of A is defined as the resultant of the system $\nabla f(x) = 0$. One may use formulas of the resultant to compute it [11], [14], [27], but so far there are no explicit formulas of the resultant for $n \geq 2$ in the general case. Classically, there are Macaulay formulas [22], which express the multivariate resultant as a quotient of two determinants. Recently, D'Andrea and Dickenstein [12] gave a new version of the classical Macaulay formulas, by involving matrices of considerably smaller size, whose nonzero entries include coefficients of the given polynomials and coefficients of their Bezoutians. However, how to calculate such coefficients of Bezoutians and how to order the monomials still need to be specified in the computation. When the resultant size is very small, these can be determined easily. In general, these are still implementation tasks ahead.

The following theorem given by Qi [24] reveals an important relation between the eigenvalues and the symmetric hyperdeterminant.

Theorem 1.1: Suppose that m is even. A number $\lambda \in C$ is an eigenvalue of A if and only if it is a root of the following one-dimensional polynomial in λ:

$$\Phi(\lambda) = \det(A - \lambda I)$$

where I is unit supersymmetric tensor whose entries are

$$\delta_{i_1,\ldots,i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

The one-dimensional polynomial Φ was called the **characteristic polynomial** of A.

In this paper, we apply the D'Andrea–Dickenstein version of the classical Macaulay formulas of the resultant to compute the symmetric hyperdeterminant of an even order supersymmetric tensor. By using the supersymmetry property, we give detailed computation procedures for the Bezoutians and specified ordering of the monomials in this approach. Furthermore, we implement our formulas to calculate the characteristic polynomial of a fourth-order three-dimensional supersymmetric tensor. We propose an eigenvalue method for testing positive definiteness of a quartic form of three variables.

This paper is organized as follows. We give preliminary statements about symmetric hyperdeterminants and resultants in Section II. We establish computable formulas of the symmetric hyperdeterminant of A in Section III when m is even. We discuss the detailed computation of the characteristic polynomial of a fourth order three dimensional supersymmetric tensor in Section IV. We discuss methods for testing positive definiteness of A in Section V. In Section VI, we give some preliminary numerical test results. Some final comments are made in Section VII.

II. Preliminary Statements

The following lemma, theorem, and proposition were given in [24].

Lemma 2.1: The symmetric hyperdeterminant of A, $\det(A)$, is the resultant of

$$Ax^{m-1} = \left(\sum_{i_1,\ldots,i_{m-1}=1}^{n} a_{i,i_1,\ldots,i_{m-1}} x_{i_1} x_{i_2} \cdots x_{i_{m-1}} \right)_{i=1}^{n} = 0$$

and is a homogeneous polynomial of the entries of A, with the degree $d = n(m-1)^{n-1}$.

Theorem 2. 1: The eigenvalues of the supersymmetric tensor A have the following properties.

a) (Gershgorin-type theorem) The eigenvalues of A lie in the following n disks:

$$|\lambda - a_{i\cdots i}| \leq \sum \{|a_{ii_2\cdots i_m}| : i_2, \cdots, i_m = 1, \cdots, n, \delta_{ii_2\cdots i_m} = 0\}$$

for $i = 1, \ldots, n$, where the symbol $\delta_{ii_2\cdots i_m}$ refers to (2).

b) The number of eigenvalues of A is $d = n(m-1)^{n-1}$ and the product of all eigenvalues of A is equal to det(A).

c) The summation of all the eigenvalues of A is $(m-1)^{n-1}\mathrm{tr}(A)$, where $\mathrm{tr}(A)$ denotes the trace of A which is the summation of all diagonal elements of A.

Proposition 2.1: Suppose that $B = a(A + bI)$, where B and A are supersymmetric tensors, a and b are two real numbers. Then μ is an eigenvalue (H-eigenvalue) of B if and only if

$$\mu = a(\lambda + b)$$

and λ is an eigenvalue (H-eigenvalue) of A. In this case, they have the same eigenvectors.

So far, there are no explicit formulas of the symmetric hyperdeterminant for a general tensor. We use the resultant theory in [12] to establish the formula of the symmetric hyperdeterminant of an even order supersymmetric tensor. It is also easy to extend this formula to odd order tensors.

Let $f_1, \ldots, f_n$ be n homogeneous polynomials in n variables with degree $d_1, \ldots, d_n$, respectively. In order to describe the results of the resultant theory in [12], we state the definition of the Bezoutian associated with $f_1, \ldots, f_n$ in [3]. For each pair

(i,j) with $1 \le i,j \le n$, denote $\Delta_{ij}(x,y)$ for the incremental quotient

$$\Delta_{ij}(x,y) = \frac{1}{x_j - y_j}(f_i(y_1,\ldots,y_{j-1},x_j,\cdots,x_n) - f_i(y_1,\ldots,y_j,x_{j+1},\ldots,x_n)).$$

It is remarked that $f_i(y_1,\ldots,y_{j-1},x_j,\ldots,x_n) - f_i(y_1,\ldots,y_j,x_{j+1},\ldots,x_n)$ can be divided by $x_j - y_j$. Hence, we may express $\Delta_{ij}(x,y)$ as this quotient no matter whether $x_j - y_j = 0$ or not. Then we define the $n \times n$ determinant

$$\begin{aligned}\Delta(x,y) &= \det(\Delta_{ij}(x,y))_{1\le i,j\le n}\\ &= \sum_{|\gamma|\le t_n} \Delta_\gamma(x) y^\gamma\\ &= \sum_{|\gamma|+|\kappa|=t_n} \beta_{\kappa\gamma} x^\kappa y^\gamma \end{aligned} \tag{3}$$

where $t_n = \sum_{i=1}^n (d_i - 1), y^\gamma = y_1^{\gamma_1}\cdots y_n^{\gamma_n}, \gamma = (\gamma_1,\ldots,\gamma_n)^T, \gamma_1,\ldots,\gamma_n$ are nonnegative integers, $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n, x^\kappa, \kappa$ and $|\kappa|$ are defined similarly. The determinant $\Delta(x,y)$ is a representative of the Bezoutian associated with $f_1,\ldots,f_n$. It is a homogeneous polynomial in $2n$ variables x and y of degree t_n.

Define some sets of monomials as follows:

$$\begin{aligned} S_u &= \{x^\gamma : |\gamma| = u\}\\ S^{t,i} &= \{x^\gamma : |\gamma| = t - d_i, \gamma_1 < d_1,\ldots,\gamma_{i-1} < d_{i-1}\}\\ E^{t,i} &= \{x^\gamma \in S^{t,i} : \text{there exists } j \ne i : \gamma_j \ge d_j\}\end{aligned}$$

where $i = 1,2,\ldots,n$, t is a nonnegative integer and u is an integer. Define by S_u^* a dual basis of S_u. Note that $E^{t,n} = \emptyset$ and $S^{t,1} = S_{t-d_1}$ for any nonnegative t. If u is negative, then S_u is the empty set. Let $j_u : S_u \to S_u^*$ be the isomorphism associated with the monomial bases in S_u and denote by $t_\gamma = j_u(x^\gamma)$ the elements in the dual basis. We use the convention that all spaces in this paper have a monomial basis, or a dual monomial basis, and all these bases have a fixed order (usually the grade lexicographic order; see Definition 1). Thus, there is no ambiguity when we define matrices in the monomial bases.

Define two linear maps

$$\psi_{1,t}: \quad S_{t_n-t}^* \to S_t, t_\gamma \mapsto \Delta_\gamma(x),$$

$$\psi_{2,t}: \quad S^{t,1}\oplus\cdots\oplus S^{t,n} \to S_t, (g_1,\ldots,g_n) \mapsto \sum_{i=1}^n g_i f_i$$

and let Δ_t and D_t denote the matrices of $\psi_{1,t}$ and $\psi_{2,t}$ in the monomial bases, respectively. Denote

$$\begin{aligned}\Psi_t: \quad & S_{t_n-t}^* \oplus (S^{t,1}\oplus\cdots\oplus S^{t,n})\\ &\to S_t \oplus (S^{t_n-t,1}\oplus\cdots\oplus S^{t_n-t,n})^*\\ &(T,g) \mapsto (\psi_{1,t}(T) + \psi_{2,t}(g), \psi_{2,t_n-t}^*(T))\end{aligned}$$

where $\psi_{2,t_n-t}^*(T)$ is the dual of $\psi_{2,t_n-t}(T)$, i.e.,

$$\psi_{2,t_n-t}^*(T): \; S_{t_n-t}^* \to (S^{t_n-t,1}\oplus\cdots\oplus S^{t_n-t,n})^*.$$

Let M_t be the matrix of Ψ_t in the monomial bases. Denote by E_t the submatrix of M_t whose columns are indexed by the monomials in $E^{t,1}\cup\cdots E^{t,n-1}$, and whose rows are indexed by the monomial x^γ in S_t for which there exist two different indexes i,j such that $\gamma_i \ge d_i$, $\gamma_j \ge d_j$.

The following lemma is from [12].

Lemma 2.2: For any $t \ge 0$, $\det(M_t)$, $\det(E_t)$, $\det(E_{t_n-t})$ are nonzero polynomials but that they might vanish for a given choice of coefficients for $f_1,\ldots,f_n$. Let $Res_{d_1,\ldots,d_n}(f_1,\ldots,f_n)$ be the resultant of $f_1,\ldots,f_n$. Then

$$\mathrm{Res}_{d_1,\ldots,d_n}(f_1,\ldots,f_n) = \pm\frac{\det(M_t)}{\det(E_t)\det(E_{t_n-t})}$$

where $M_t = \begin{bmatrix}\Delta_t & D_t\\ D_{t_n-t}^T & 0\end{bmatrix}$ and $\Delta_t = (\beta_{\kappa\gamma})_{|\kappa|=t,|\gamma|=t_n-t}$.

M_t is a square matrix of size $\rho(t)$,

$$\rho(t) = \binom{t+n-1}{n-1} + \binom{t_n-t+n-1}{n-1} - H_d(t_n-t)$$

where $H_d(t)$ can be computed by the following formula:

$$\frac{\prod_{i=1}^n (1-y^{d_i})}{(1-y)^n} = \sum_{t=0}^\infty H_d(t) y^t.$$

When $t = [t_n/2]$, the size of $\rho(t)$ is minimal.

By using this lemma, we can establish a formula of the symmetric hyperdeterminant of an even-order supersymmetric tensor.

III. The Symmetric Hyperdeterminant of an Even Order Supersymmetric Tensor

Assume that $m > 2$ is an even number and n is a positive integer. Denote $\Omega_{k,n}$ by

$$\Omega_{k,n} = \{(i_1 i_2\cdots i_k) : i_1 \le i_2 \le\cdots\le i_k,\; i_1,i_2,\ldots,i_k \in \{1,2,\ldots,n\}\}$$

for $k = 1,2,\ldots,m$. Let $(i_1 i_2\cdots i_k) \in \Omega_{k,n}$, denote by $\alpha_{i_1 i_2\cdots i_k}$ the number of all combinations with repetitions of $(i_1 i_2\cdots i_k)$. Then after combining like monomials, we have

$$\begin{aligned} f(x) &= Ax^m\\ &= \sum_{i_1=1}^n \sum_{i_2=i_1}^n \cdots \sum_{i_m=i_{m-1}}^n \alpha_{i_1 i_2\cdots i_m} a_{i_1 i_2\cdots i_m} x_{i_1} x_{i_2}\cdots x_{i_m}\\ &= \sum_{(i_1 i_2\cdots i_m)\in\Omega_{m,n}} \alpha_{i_1 i_2\cdots i_m} a_{i_1 i_2\cdots i_m} x_{i_1} x_{i_2}\cdots x_{i_m}.\end{aligned}$$

Let

$$f_i(x) = (Ax^{m-1})_i, \quad i = 1,2,\ldots,n. \tag{4}$$

Then

$$\begin{aligned} f_i(x) &= \sum_{i_1=1}^n \cdots \sum_{i_{m-1}=i_{m-2}}^n \alpha_{i_1\cdots i_{m-1}} a^i_{i_1\cdots i_{m-1}} x_{i_1}\cdots x_{i_{m-1}}\\ &= \sum_{(i_1\cdots i_{m-1})\in\Omega_{m-1,n}} \alpha_{i_1\cdots i_{m-1}} a^i_{i_1\cdots i_{m-1}} x_{i_1}\cdots x_{i_{m-1}}\end{aligned} \tag{5}$$

where $i = 1,2,\ldots,n$, and $a^i_{i_1 i_2\cdots i_{m-1}} = a_{i i_1 i_2\cdots i_{m-1}}$. Hence, $f_1,f_2,\ldots,f_n$ are homogeneous polynomials in n variables with degree $m-1$, i.e.,

$$d_1 = \cdots = d_n = m-1, t_n = n(m-2)$$

and they include

$$u_n = \binom{m+n-2}{n-1} \tag{6}$$

monomials. From Lemma 2.2, we obtain the following proposition.

Proposition 3.1: Let A be an mth-order supersymmetric tensor, m be an even number and $m > 2$, and $f_1, f_2, \ldots, f_n$ be defined by (4). Then the symmetric hyperdeterminant of A is computed by the following formula:

$$\det(A) = \mathrm{Res}(f_1, \ldots, f_n) = \pm \frac{\det(M_t)}{\det(E_t)\det(E_{t_n - t})}.$$

When $t = t_n/2$, the size of M_t is minimal and

$$\det(A) = \pm \frac{\det(M_{t'_n})}{(\det(E_{t'_n}))^2}$$

where $t'_n = t_n/2$.

Proof: Because m is an even number, t_n is also an even number. Hence, $[t_n/2] = t_n/2$. The proposition follows from Lemmas 2.1 and 2.2. ◇

At first, we discuss the computation of $\Delta_{t'_n}$ in $M_{t'_n}$ (see Lemma 2.2). $\Delta_{t'_n}$ is a matrix whose entries are Bezoutians. In order to compute the Bezoutian associated with $f_1, f_2, \ldots, f_n$, we define

$$\phi_{ij}(\kappa) = f_i(y_1, \ldots, y_{j-1}, \kappa, x_{j+1}, \ldots, x_n)$$

and have the following lemma.

Lemma 3.1: Let

$$\Delta_{ij} = \frac{\phi_{ij}(x_j) - \phi_{ij}(y_j)}{x_j - y_j}. \tag{7}$$

Then

$$\Delta_{ij} = \sum_{p=1}^{m-1} \Big\{ (x_j^{p-1} + y_j x_j^{p-2} + \cdots + y_j^{p-2} x_j + y_j^{p-1}) \sum_{(i_1 \cdots i_{m-1}) \in \Omega_{m-1,n}^{j,p}} \alpha_{i_1 \cdots i_{m-1}} a^i_{i_1 \cdots i_{m-1}} y_{i_1} \cdots y_{i_k} x_{i_{k+p+1}} \cdots x_{i_{m-1}} \Big\} \tag{8}$$

for $j = 1, 2, \ldots, n$, where $\Omega_{m-1,n}^{j,p}$ is a subset of $\Omega_{m-1,n}$ and its entries include pj' s, $p = 1 \ldots, m-1$.

Proof: From (5) and (7), it follows that

$$\Delta_{ij} = \frac{1}{x_j - y_j} \sum_{(i_1 i_2 \cdots i_{m-1}) \in \Omega_{m-1,n}} \alpha_{i_1 \cdots i_{m-1}} a^i_{i_1 \cdots i_{m-1}} \{ y_{i_1} \cdots y_{i_{j-1}} x_{i_j} \cdots x_{i_{m-1}} - y_{i_1} \cdots y_{i_j} x_{i_{j+1}} \cdots x_{i_{m-1}} \}.$$

As remarked before, $f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n)$ can be divided by $x_j - y_j$. Hence, we may express $\Delta_{ij}(x, y)$ as this quotient no matter whether $x_j - y_j = 0$ or not. If $(i_1 \cdots i_{m-1}) \in \Omega_{m-1,n}$ does not include j, then the terms with this index in $\phi_{ij}(x_j)$ and $\phi_{ij}(y_j)$ are identical, and they are canceled in (7). If $(i_1 \cdots i_{m-1}) \in \Omega_{m-1,n}^{j,p}$, then from

$$\frac{x_j^p - y_j^p}{x_j - y_j} = x_j^{p-1} + y_j x_j^{p-2} + \cdots + y_j^{p-2} x_j + y_j^{p-1}$$

it follows (8). The proof is complete. ◇

From Lemma 3.1, it is seen that the n components in the jth column of $(\Delta_{ij}(x, y))_{n \times n}$ have the same monomials and have only different coefficients in monomials, $\alpha_{\theta_j} a^i_{\theta_j}$. According to the properties of determinants, we conclude that each element in $det(\Delta_{ij}(x, y))_{n \times n}$ is a product of a constant, an $n \times n$ determinant and a monomial $x^\kappa y^\gamma$ with $|\kappa| + |\gamma| = t_n$. Define an $n \times n$ determinant

$$a(\theta_1, \theta_2, \ldots, \theta_n) = \begin{vmatrix} a^1_{\theta_1} & a^1_{\theta_2} & \cdots & a^1_{\theta_n} \\ a^2_{\theta_1} & a^2_{\theta_2} & \cdots & a^2_{\theta_n} \\ \cdots & \cdots & \cdots & \cdots \\ a^n_{\theta_1} & a^n_{\theta_2} & \cdots & a^n_{\theta_n} \end{vmatrix}$$

where $a^i_{\theta_j} = a_{i\theta_j} \in \Omega_{m,n}$. We have the following lemma.

Lemma 3.2: $\alpha_{\theta_1} \alpha_{\theta_2} \cdots \alpha_{\theta_n} a(\theta_1, \theta_2, \ldots, \theta_n)$ is the coefficient of a monomial in the determinant, $\det(\Delta_{ij}(x, y))_{n \times n}$.

Proof: Let $\alpha_{\theta_j} a^1_{\theta_j} p_j$ be an entry in $\Delta_{1j}(x, y)$ where p_j is a monomial, $j = 1, 2, \ldots, n$. It follows that from (8) that $\alpha_{\theta_j} a^i_{\theta_j} p_j$ is also an entry in $\Delta_{ij}(x, y)$, $i = 2, 3, \ldots, n$, $j = 1, 2, \ldots, n$. Hence, the coefficient of $p_1 p_2 \cdots p_n$ in $det(\Delta_{ij}(x, y))_{n \times n}$ is

$$\alpha_{\theta_1} \alpha_{\theta_2} \cdots \alpha_{\theta_n} a(\theta_1, \theta_2, \ldots, \theta_n).$$

The proof is complete. ◇

According to Lemma 2.2, we have that

$$\Delta_{t'_n} = (\beta_{\kappa_i \gamma_j})_{|\kappa_i| = |\gamma_j| = t'_n}$$

where $\beta_{\kappa_i \gamma_j}$ is the coefficient of $x^{\kappa_i} y^{\gamma_j}$ in $\det(\Delta_{ij}(x, y)_{n \times n})$. In order to determine the permutation of elements of $\Delta_{t'_n}$, we need to choose the proper order of monomials, and recall the definition of the lex (lexicographic) order and the grlex (graded lexicographic) order of monomials [10], where Z is the set of integers and $Z_{\geq 0}$ is the set of nonnegative integers.

Definition 1: Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $\kappa, \gamma \in Z^n_{\geq 0}$. We say $\kappa >_{\mathrm{lex}} \gamma$, if in the vector difference $\kappa - \gamma \in Z^n$ the leftmost nonzero entry is positive. We will write $x^\kappa >_{\mathrm{lex}} y^\gamma$ if $\kappa >_{\mathrm{lex}} \gamma$. We say $\kappa >_{\mathrm{grlex}} \gamma$, if $|\kappa| > |\gamma|$, or $|\kappa| = |\gamma|$ and $\kappa >_{\mathrm{lex}} \gamma$. We will write $x^\kappa >_{\mathrm{grlex}} y^\gamma$ if $\kappa >_{\mathrm{grlex}} \gamma$.

We let

$$v_n = \binom{t'_n + n - 1}{n-1}, \quad \Omega^j_{m-1,n} = \bigcup_{p=1}^{m-1} \Omega^{j,p}_{m-1,n}$$

and divide the monomials of $\Delta_{1j}(x, y)$ into $m-1$ sets $c_{j1}, \ldots, c_{j,m-1}$ by

$$c_{j1} = \left\{ x_{i_1} \cdots x_{i_{m-2}} : (j i_1 \cdots i_{m-2}) \in \Omega^j_{m-1,n} \right\}$$
$$c_{j2} = \left\{ y_{i_1} x_{i_2} \cdots x_{i_{m-2}} : (i_1 j i_2 \cdots i_{m-2}) \in \Omega^j_{m-1,n} \right\}$$
$$\cdots$$
$$c_{j,m-1} = \left\{ y_{i_1} \cdots y_{i_{m-2}} : (i_1 \cdots i_{m-2} j) \in \Omega^j_{m-1,n} \right\}$$

for $j = 1, 2, \ldots, n$ [see (8)]. Here c_{ji} includes all monomials of $m-1-i$ x-variables and $i-1 y$-variables in Δ_{1j}, $i = 1, \ldots, m-1$.

Define

$$\Lambda = \{c_{1l_1} \times \cdots \times c_{nl_n} : l_1, \ldots, l_n \in \{1, \ldots, m-1\} \text{ such that } |\kappa| = |\gamma| = t'_n \text{ for all } x^\kappa y^\gamma = p_1 \cdots p_n, \text{ where } p_j \in c_{jl_j}, j = 1, \ldots, n\}$$

and denote

$$\Delta_{t'_n} = (d_{ij})_{v_n \times v_n}$$

where $d_{ij} = \beta_{\kappa_i \gamma_j}$ [see (3)], $i, j = 1, 2, \ldots, v_n, \kappa_1, \ldots, \kappa_{v_n}$, and $\gamma_1, \ldots, \gamma_{v_n}$ are ordered by the grlex order. We present an algorithm for computing $\Delta_{t'_n}$ as follows.

Algorithm 3.1:

Step 1) Initialization. Set $d_{ij} = 0, i, j = 1, 2, \ldots, v_n$. $\Lambda_0 = \Lambda$ and $k = 0$.

Step 2) Check termination. If $\Lambda_k = \emptyset$, then stop. Otherwise choose $c_{1l_1} \times c_{2l_2} \times \cdots \times c_{nl_n}$ from Λ_k.

Step 3) For each combination $p_1 p_2 \cdots p_n$ in $c_{1l_1} \times c_{2l_2} \times \cdots \times c_{nl_n}$, assume that the coefficient of p_s in c_{sl_s} is $\alpha_{\theta_{l_s}} a^1_{\theta_{l_s}}$ with $\theta_{l_s} \in \Omega_{m-1,n}$, $s = 1, 2, \ldots, n$.

i) Determine κ_i and γ_j such that

$$x^{\kappa_i} y^{\gamma_j} = p_1 p_2 \cdots p_n.$$

ii) Let $d_{ij} = d_{ij} + \alpha_{\theta_{l_1}} \cdots \alpha_{\theta_{l_n}} a(\theta_{l_1}, \ldots, \theta_{l_n})$..

Step 4) Let $\Lambda_{k+1} = \Lambda_k \setminus \{c_{1l_1} \times c_{2l_2} \times \cdots \times c_{nl_n}\}$. Set $k = k+1$ and go to Step 2).

Now, we discuss the computation of $D_{t'_n}$ and $E_{t'_n}$. $D_{t'_n}$ is the matrix of the map ψ_{2,t'_n} in the monomial basis

$$\psi_{2,t'_n} : S^{t_n,1} \oplus \cdots \oplus S^{t_n,n} \to S_{t'_n}$$

and $E_{t'_n}$ is a submatrix of $D_{t'_n}$.

If $t'_n - m + 1 < 0$, then $S^{t_n,1}, \ldots, S^{t_n,n}$ are empty sets and $M_{t'_n}$ does not include $D_{t'_n}$. If $t'_n - m + 1 = 0$, then there is only a constant 1 in each base of $S^{t_n,1}, \ldots, S^{t_n,n}$, $v_n = u_n$ and $D_{t'_n}$ is $u_n \times n$ matrix

$$D_{t'_n} = \begin{bmatrix} \alpha_{\theta_1} a^1_{\theta_1} & \alpha_{\theta_1} a^2_{\theta_1} & \cdots & \alpha_{\theta_1} a^n_{\theta_1} \\ \alpha_{\theta_2} a^1_{\theta_2} & \alpha_{\theta_2} a^2_{\theta_2} & \cdots & \alpha_{\theta_2} a^n_{\theta_2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{\theta_{u_n}} a^1_{\theta_{u_n}} & \alpha_{\theta_{u_n}} a^2_{\theta_{u_n}} & \cdots & \alpha_{\theta_{u_n}} a^n_{\theta_{u_n}} \end{bmatrix}. \quad (9)$$

Lemma 3.3: Let $w_i = |S^{t'_n,i}|, i = 1, 2, \ldots, n$. Then

$$D_{t'_n} = \left[D^1_{t'_n}, D^2_{t'_n}, \ldots, D^n_{t'_n}\right]$$

where $D^i_{t'_n}$ is the $v_n \times w_i$ matrix of the mapping

$$\psi_i : S^{t'_n,i} \to S_{t'_n} \quad g_i \longmapsto g_i f_i.$$

Moreover, $D^i_{t'_n}$ is a sparse matrix, there are u_n identical nonzero entries

$$\alpha_{\theta_1} a^i_{\theta_1}, \alpha_{\theta_2} a^i_{\theta_2}, \ldots, \alpha_{\theta_{u_n}} a^i_{\theta_{u_n}}$$

in each of its columns.

Proof: Assume that all monomials in $S^{t'_n,i}$ by the grlex order are $p_1, p_2, \ldots, p_{w_i}$, and that all monomials in $S_{t'_n}$ by the grlex order are $p'_1, p'_2, \ldots, p'_{v_n}$. Let $D^i_{t'_n}$ be the matrix of ψ_i. Then

$$(p_1 f_i, p_2 f_i, \ldots, p_{w_i} f_i) = (p'_1, p'_2, \ldots, p'_{v_n}) D^i_{t'_n} \quad (10)$$

which implies that $D^i_{t'_n}$ is a $v_n \times w_i$ matrix.

It is seen from (5) and (6) that there are u_n terms in each $p_j f_i, j = 1, \ldots, w_i$ and the u_n coefficients of $p_j f_i, j = 1, \ldots, w_i$ are the same. This means that in each column of $D^i_{t'_n}$ there are u_n identical nonzero entries. ◇

According to Lemma 2.2, we know that $E_{t'_n}$ is a submatrix $D_{t'_n}$, whose columns are indexed by the monomials in $E^{t'_n,1} \cup E^{t'_n,2} \cup \cdots \cup E^{t'_n,n-1}$, and whose rows are indexed by the monomials in $T_{t'_n}$ where

$$T_{t'_n} = \{x^\gamma : |\gamma| = t'_n, \exists i, j, i \neq j, \gamma_i \geq m-1, \gamma_j \geq m-1\}.$$

For convenience, we denote

$$E_{t'_n} = [E^1_{t'_n}, E^2_{t'_n}, \ldots, E^{n-1}_{t'_n}]$$

where the columns of $E^i_{t'_n}$ are indexed by the monomials in $E^{t'_n,i}, i = 1, 2, \ldots, n-1$. If

$$t'_n - m + 1 \leq m - 2 \quad (11)$$

then $E_{t'_n}$ is indexed by an empty set, and according to the convention we define $\det(E_{t'_n}) = 1$.

In order to determine the exact position of the nonzero entries of $D_{t'_n}$ and exact indexes of row and column of $E_{t'_n}$, we need to discuss the monomial ordering in $S^{t'_n,i}$ and other sets. For the grlex order, we define the functions rank and unrank on some set, P, which consist of monomials. The concept of these functions refers to [20].

Let P be a set of finitely many monomials, $|P| = N$, and define

$$\textbf{rank}: P \to \{1, 2, \ldots, N\}$$
$$\textbf{unrank}: \{1, 2, \ldots, N\} \to P$$

where **rank** is a ranking function defined on P in the grlex order, and **unrank** is the inverse function of the function **rank**.

In the following lemma, we give a formula for calculating the rank of a term of $S_{t'_n}$.

Lemma 3.4: Let a term in $S_{t'_n}$ be expressed by

$$x^{\gamma_1}_{k_1} x^{\gamma_2}_{k_2} \cdots x^{\gamma_j}_{k_j}$$

with $\gamma_1 + \gamma_2 + \cdots + \gamma_j = t'_n, 1 \leq k_1 \leq k_2 \leq \cdots \leq k_j \leq n, \gamma_1 \geq 1, \ldots, \gamma_j \geq 1$. Then the rank of this term is

$$\text{rank}\left(x^{\gamma_1}_{k_1} x^{\gamma_2}_{k_2} \cdots x^{\gamma_j}_{k_j}\right) = 1 + \sum_{i=0}^{j-1} \left[\binom{n - k_i + t'_n - (\gamma_1 + \cdots + \gamma_i)}{n - k_i} - \binom{n - k_{i+1} + t'_n - (\gamma_1 + \cdots + \gamma_i)}{n - k_{i+1}}\right]$$

where $k_0 \equiv 1$.

Proof: We partition the change from the first term $x_1^{t'_n}$ to this term as follows:

$$x_1^{t'_n} \to x_{k_1}^{t'_n} \to x_{k_1}^{\gamma_1} x_{k_2}^{t'_n-\gamma_1} \to x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} x_{k_3}^{t'_n-\gamma_1-\gamma_2} \to \cdots \\ \to x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_{j-1}}^{t'_n-(\gamma_1+\cdots+\gamma_{j-2})} \to x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_j}^{\gamma_j}.$$

In the first stage (i.e., $i = 0$),

$$\begin{aligned} &\operatorname{rank}\left(x_{k_1}^{t'_n}\right) - \operatorname{rank}\left(x_1^{t'_n}\right) \\ =&\operatorname{rank}\left(x_n^{t'_n}\right) - \operatorname{rank}\left(x_1^{t'_n}\right) - \left[\operatorname{rank}\left(x_n^{t'_n}\right) - \operatorname{rank}\left(x_{k_1}^{t'_n}\right)\right] \\ =&\binom{n-1+t'_n}{n-1} - \binom{n-k_1+t'_n}{n-k_1}. \end{aligned} \tag{12}$$

For the $(i+1)$th stage,

$$\operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{\gamma_i} x_{k_{i+1}}^{t'_n-(\gamma_1+\cdots+\gamma_i)}\right) \\ -\operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{t'_n-(\gamma_1+\cdots+\gamma_{i-1})}\right)$$

is decomposed into two differences

$$\operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{\gamma_i} x_n^{t'_n-(\gamma_1+\cdots+\gamma_i)}\right) \\ -\operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{\gamma_i} x_{k_i}^{t'_n-(\gamma_1+\cdots+\gamma_{i-1}+\gamma_i)}\right)$$

and

$$\operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{\gamma_i} x_n^{t'_n-(\gamma_1+\cdots+\gamma_i)}\right) \\ -\operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{\gamma_i} x_{k_{i+1}}^{t'_n-(\gamma_1+\cdots+\gamma_i)}\right).$$

It is not difficult to obtain

$$\begin{aligned} &\operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{\gamma_i} x_{k_{i+1}}^{t'_n-(\gamma_1+\cdots+\gamma_i)}\right) \\ &- \operatorname{rank}\left(x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_i}^{t'_n-(\gamma_1+\cdots+\gamma_{i-1})}\right) \\ =&\binom{n-k_i+t'_n-(\gamma_1+\cdots+\gamma_i)}{n-k_i} \\ &-\binom{n-k_{i+1}+t'_n-(\gamma_1+\cdots+\gamma_i)}{n-k_{i+1}}. \end{aligned} \tag{13}$$

This lemma follows from (12), (13), and $\operatorname{rank}(x_1^{t'_n}) = 1$. ◇

Now we give an algorithm for determining the unrank function of the monomials in $S_{m-1}, S^{t'_n,i}, i = 1, 2, \ldots, n$, the indexes of columns in $E_{t'_n}^i, i = 1, 2, \ldots, n-1$, the indexes of rows of $E_{t'_n}$, and an array which stores the position of nonzero entries in $D_{t'_n}$.

Algorithm 3.2:

Step 1) Determine the unrank function of all monomial in S_{m-1} which are stored in unrank $(\cdot)$. Let $k = 0$. For $i_1 = 1, 2, \ldots, n; i_2 = i_1, i_1+1, \ldots, n; \cdots, i_{m-1} = i_{m-2}, i_{m-2}, \ldots, n$; do $k = k+1$, $\operatorname{unrank}(k) = x_{i_1} x_{i_2} \cdots x_{i_{m-1}}$.

Step 2) Determine the unrank function of all monomial in $S^{t'_n,i}$ which are stored in unrank $(i,\cdot)$, $i = 1, 2, \ldots, n$, the indexes of columns of $E^{t'_n,i}$ stored in $C(i,\cdot), i = 1, 2, \ldots, n-1$, and the indexes of the rows of $E_{t'_n}$ stored in $R(\cdot)$.

2.1) Set $k_1 = 0, k_2 = 0, \ldots, k_n = 0; j_1 = 0, j_2 = 0, \ldots, j_{n-1} = 0$; and $n_r = 0$.

2.2) For $i_1 = 1, 2, \cdot, n$; $i_2 = i_1, i_1+1, \ldots, n; \cdots; i_{t'_n} = i_{t'_n-1}, i_{t'_n-1}+1, \ldots, n$; do

$$\text{poly} = x_{i_1} x_{i_2} \cdots x_{i_{t'_n}} = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n};$$

1) $k_1 = k_1 + 1, \operatorname{unrank}(1, k_1) = \text{poly}$.
   i) if there exists $x_p^{\gamma_p}$ in poly such that $p \geq 2$, $\gamma_p \geq m-1$, then $j_1 = j_1 + 1, C(1, j_1) = k_1$;
   ii) if there exists γ_{p_1} and $\gamma_{p_2}, p_1 \neq p_2$ such that $\gamma_{p_1} \geq m-1$, $\gamma_{p_2} \geq m-1$, then $n_r = n_r + 1, R(n_r) = k_1$.

2) For $l = 1, 2, \ldots, n-2$, do
   if $\gamma_1 < m-1, \ldots, \gamma_l < m-1$, then
   i) $k_{l+1} = k_{l+1} + 1$, $\operatorname{unrank}(l+1, k_{l+1}) =$ poly;
   ii) If there exists $x_p^{\gamma_p}$ in poly such that $p \geq l+2$, $\gamma_p \geq m-1$, then $j_{l+1} = j_{l+1} + 1, C(l+1, j_{l+1}) = k_{l+1}$.

3) If $\gamma_1 < m-1, \ldots, \gamma_{n-1} < m-1$, then $k_n = k_n + 1, \operatorname{unrank}(n, k_n) = \text{poly}$.

Step 3) Determine the array $B \in \Re^{n \times w_m \times u_n}$ where $w_m = \max\{w_i : 1 \leq i \leq n\}$. For $i = 1, 2, \ldots, n; j = 1, 2, \ldots, w_i; k = 1, 2, \ldots, u_n$, do
   i) poly $= \operatorname{unrank}(i,j)\operatorname{unrank}(k)$;
   ii) $B(i,j,k) = \operatorname{rank}$ (poly), where the computation of rank is determined by the formula in Lemma 3.4.

In the following proposition we give an approach for determining $D_{t'_n}$ and $E_{t'_n}$.

Proposition 3.2: The matrix $D_{t'_n}$ is determined by Lemma 3.3 and the array B generated by Algorithm 3.2. In each $D_{t'_n}^i, i = 1, 2, \ldots, n$, there are u_n identical nonzero entries, $\alpha_{\theta_1} a_{\theta_1}^i, \alpha_{\theta_2} a_{\theta_2}^i, \ldots, \alpha_{\theta_{u_n}} a_{\theta_{u_n}}^i$ in each of its columns. While the positions of u_n nonzero entries in the jth column of $D_{t'_n}^i$ are $B(i,j,1), B(i,j,2), \ldots, B(i,j,u_n), j = 1, 2, \ldots, w_i$.

The matrix $E_{t'_n}$ is determined by $D_{t'_n}$, the array R and C. $E_{t'_n}^i$ is a submatrix of $D_{t'_n}^i, i = 1, 2, \ldots, n-1$, and

$$E_{t'_n}^i = D_{t'_n}^i \begin{pmatrix} R(1) & R(2) & \cdots & R(|T_{t'_n}|) \\ C(i,1) & C(i,2) & \cdots & C(i, |E^{t'_n,i}|) \end{pmatrix}.$$

Proof: From Step 1) of Algorithm 3.2, unrank (1), unrank(2), …, unrank(u_n) are all monomials in S_{m-1} by the grlex order, and they are all monomials in any $f_i, i = 1, 2, \ldots, n$. unrank (i,j) stores the jth monomial in $S^{t'_n,i}$. According to Lemma 3.3 and Step 3) of Algorithm 3.2, it follows that $B(i,j,k), k = 1, 2, \ldots, u_n$ are the rank of all monomials of $g_j f_i$ in $S_{t'_n}$ by grlex order. From (10) it is seen that $B(i,j,k)$, $k = 1, 2, \ldots, u_n$ are the position of u_n nonzero entries in the jth column of $D_{t'_n}^i$. From Algorithm 3.2, the indexes of columns of $E^{t'_n,i}$ are stored in $C(i,\cdot), i = 1, 2, \ldots, n-1$, and the indexes of the rows of $E_{t'_n}$ are stored in $R(\cdot)$. Thus, we obtain the form of $E_{t'_n}^i$ in this lemma. ◇

Hence, the symmetric hyperdeterminant of an even order supersymmetric tensor is completely determined by Proposition 3.2, Algorithms 3.1 and 3.2.

IV. The Characteristic Polynomial of a Fourth Order Three-Dimensional Supersymmetric Tensor

In this section, we consider the detailed computation of the characteristic polynomial of a fourth-order three-dimensional supersymmetric tensor. Let $m=4$ and $n=3$. By the definition of eigenvalues, an eigenvalue λ together with its eigenvector x satisfies the following homogeneous polynomial equation:

$$\begin{pmatrix} f_1(x,\lambda) \\ f_2(x,\lambda) \\ f_3(x,\lambda) \end{pmatrix} = Ax^3 - \lambda x^{[3]} = (A-\lambda I)x^3 = 0. \quad (14)$$

Let $\bar{A} = A - \lambda I$. Then

$$\bar{a}_{i_1 i_2 i_3 i_4} = \begin{cases} a_{i_1 i_2 i_3 i_4} - \lambda, & \text{if } i_1 = i_2 = i_3 = i_4 \\ a_{i_1 i_2 i_3 i_4}, & \text{otherwise.} \end{cases} \quad (15)$$

In order to give the expressions of $f_1(x,\lambda)$, $f_2(x,\lambda)$ and $f_3(x,\lambda)$, we denote

$$z = \left(x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^3, x_2^2x_3, x_2x_3^2, x_3^3\right)$$

where the monomials with degree 3 are ordered by the grlex order. Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be the subscripts of these monomials by the grlex order. Then

$$(\theta_1, \theta_2, \ldots, \theta_{10}) = (111, 112, 112, 122, 123, 133, 222, 223, 233, 333).$$

According to the definition, we know that $\alpha_{\theta_i} = \alpha_{i_1 i_2 i_3}$ denotes the number of all permutations of $(i_1 i_2 i_3)$ from which it follows that

$$(\alpha_{\theta_1}, \alpha_{\theta_2}, \ldots, \alpha_{\theta_{10}}) = (1,3,3,3,6,3,1,3,3,1).$$

Denote

$$B = \begin{pmatrix} \bar{a}^1_{\theta_1} & \bar{a}^1_{\theta_2} & \cdots & \bar{a}^1_{\theta_{10}} \\ \bar{a}^2_{\theta_1} & \bar{a}^2_{\theta_2} & \cdots & \bar{a}^2_{\theta_{10}} \\ \bar{a}^3_{\theta_1} & \bar{a}^3_{\theta_2} & \cdots & \bar{a}^3_{\theta_{10}} \end{pmatrix} \quad (16)$$

where $\bar{a}^i_{\theta_j} = \bar{a}_{i\theta_j}$ and

$$D = \operatorname{diag}(\alpha_{\theta_1}, \alpha_{\theta_2}, \ldots, \alpha_{\theta_{10}}). \quad (17)$$

Then from (14), we have

$$\begin{pmatrix} f_1(x,\lambda) \\ f_2(x,\lambda) \\ f_3(x,\lambda) \end{pmatrix} = (A-\lambda I)x^3 = \bar{A}x^3 = BDz^\top.$$

From Lemma 3.1, we obtain

$$\begin{aligned}
&\Delta_{i1}(x,y) \\
&= \bar{a}^i_{\theta_1}\left(x_1^2 + x_1y_1 + y_1^2\right) + 3\left(\bar{a}^i_{\theta_2}(x_1x_2 + y_1x_2)\right. \\
&\quad \left. + \bar{a}^i_{\theta_3}(x_1x_3 + y_1x_3) + \left(\bar{a}^i_{\theta_4}x_2^2 + 2\bar{a}^i_{\theta_5}x_2x_3 + \bar{a}^i_{\theta_6}x_3^2\right)\right) \\
&\Delta_{i2}(x,y) \\
&= \bar{a}^i_{\theta_7}\left(x_2^2 + x_2y_2 + y_2^2\right) + 3\left(\bar{a}^i_{\theta_4}(y_1x_2 + y_1y_2)\right. \\
&\quad \left. + \bar{a}^i_{\theta_8}(x_2x_3 + y_2x_3) + \left(\bar{a}^i_{\theta_2}y_1^2 + 2\bar{a}^i_{\theta_5}y_1x_3 + \bar{a}^i_{\theta_9}x_3^2\right)\right) \\
&\Delta_{i3}(x,y) \\
&= \bar{a}^i_{\theta_{10}}\left(x_3^2 + x_3y_3 + y_3^2\right) + 3(\bar{a}^i_{\theta_6}(y_1x_3 + y_1y_3) \\
&\quad + \bar{a}^i_{\theta_9}(y_2x_3 + y_2y_3) + \left(\bar{a}^i_{\theta_3}y_1^2 + 2\bar{a}^i_{\theta_5}y_1y_2 + \bar{a}^i_{\theta_8}y_2^2\right)
\end{aligned}$$

for $i = 1,2,3$. Define the 3×3 determinant

$$\Delta(x,y) = det(\Delta_{ij}(x,y))_{1\le i,j\le 3} = \sum_{|\kappa|+|\gamma|=6} \beta_{\kappa\gamma} x^\kappa y^\gamma.$$

Let $\Delta_3 = (\beta_{\kappa_i\gamma_j})_{10\times 10}$ where $\beta_{\kappa_i\gamma_j}, i,j = 1,2,\ldots,10$ are the coefficients of $x^{\kappa_i}y^{\gamma_j}$ with $|\kappa_i| = 3$ and $|\gamma_j| = 3$. Then the grlex orders of κ and γ are

$$(3,0,0),(2,1,0),(2,0,1),(1,2,0),(1,1,1), (1,0,2),(0,3,0),(0,2,1),(0,1,2),(0,0,3).$$

Denote

$$\Delta_3 = (d_{ij}) \in \Re^{10\times 10}, d_{ij} \equiv \beta_{\kappa_i\gamma_j}, \quad i,j = 1,2,\ldots,10$$

and

$$[i_1i_2i_3] = \begin{vmatrix} \bar{a}^1_{\theta_{i_1}} & \bar{a}^1_{\theta_{i_2}} & \bar{a}^1_{\theta_{i_3}} \\ \bar{a}^2_{\theta_{i_1}} & \bar{a}^2_{\theta_{i_2}} & \bar{a}^2_{\theta_{i_3}} \\ \bar{a}^3_{\theta_{i_1}} & \bar{a}^3_{\theta_{i_2}} & \bar{a}^3_{\theta_{i_3}} \end{vmatrix}.$$

Using Algorithm 3.1, we obtain the following 100 elements where some determinants are combined, zero determinants are deleted, and the subscripts are permuted in the grlex order according to the properties of determinants.

$$\begin{aligned}
d_{11} &= 0, d_{21} = -9[134], d_{31} = 9[126] - 18[135], \\
d_{41} &= -27[234] - 3[137], d_{51} = 9[146] - 9[138] - 54[235], \\
d_{61} &= 3[12,10] + 18[156] - 9[139] - 27[236], \\
d_{71} &= -9[237], d_{81} = -3[167] - 27[238], \\
d_{91} &= -27[239] + 3[14,10] - 9[168], \\
d_{10,1} &= 6[15,10] - 9[23,10] - 9[169], d_{12} = 0, \\
d_{22} &= -3[137] + 18[145], d_{32} = 9[146] - 9[138] + 9[129], \\
d_{42} &= -9[237] - 6[157] + 54[245], \\
d_{52} &= -3[167] + 27[246] - 27[238] + 9[149] - 18[158] \\
&\quad + 54[345], \\
d_{62} &= 3[14,10] - 9[168] + 27[346] - 27[239], \\
d_{72} &= -18[257] + 9[347], \\
d_{82} &= 3[179] - 9[276] - 54[258] + 27[348], \\
d_{92} &= 9[24,10] - 27[268] + [17,10] + 9[189] + 27[349] \\
&\quad - 54[259], \\
d_{10,2} &= 3[18,10] + 9[34,10] - 27[269], d_{13} = 0, \\
d_{23} &= 9[146], d_{33} = 3[12,10] + 18[156], \\
d_{43} &= -3[167] + 27[246], \\
d_{53} &= 3[14,10] - 9[168] + 54[256] + 27[346], \\
d_{63} &= 6[15,10] - 9[169] - 9[23,10] + 54[356], \\
d_{73} &= -9[267], d_{83} = [17,10] - 27[268] - 9[367], \\
d_{93} &= -27[269] + 3[18,10] + 9[34,10] - 27[368], \\
d_{10,3} &= 3[19,10] + 18[35,10] - 27[369] - 9[26,10], \\
d_{14} &= 0, d_{24} = -6[157] + 9[148], d_{34} = -3[167] + 9[149], \\
d_{44} &= 27[248] + 3[178] - 18[257], \\
d_{54} &= 27[348] - 18[357] + 27[249] - 9[267] + 3[179], \\
d_{64} &= [17,10] - 9[367] + 27[349], d_{74} = 9[278] - 18[457], \\
d_{84} &= 9[279] + 9[378] - 9[467] - 54[458],
\end{aligned}$$

$$\begin{aligned}
d_{94} &= 3[27,10] + 9[379] - 54[459] - 27[468],\\
d_{10,4} &= 3[37,10] - 27[469], d_{15} = 0,\\
d_{25} &= -3[167] + 9[149], d_{35} = 3[14,10] - 9[168] + 18[159],\\
d_{45} &= 3[179] - 9[267] + 27[249],\\
d_{55} &= [17,10] + 9[24,10] - 27[268] - 9[367] + 9[189]\\
&\quad + 54[259] + 27[349],\\
d_{65} &= 3[18,10] + 9[34,10] - 27[368] + 54[359],\\
d_{75} &= 9[279] - 9[467],\\
d_{85} &= 3[27,10] + 27[289] + 9[379] - 27[468] - 18[567],\\
d_{95} &= 9[28,10] - 18[45,10] - 54[568] + 3[37,10]\\
&\quad + 27[389] - 27[469],\\
d_{10,5} &= 9[38,10] - 9[46,10] - 54[569], d_{16} = 0,\\
d_{26} &= 3[14,10], d_{36} = 6[15,10], d_{46} = [17,10] + 9[24,10],\\
d_{56} &= 3[18,10] + 18[25,10] + 9[34,10],\\
d_{66} &= 3[19,10] + 18[35,10], d_{76} = 3[27,10],\\
d_{86} &= 9[28,10] + 3[37,10],\\
d_{96} &= 9[29,10] + 9[38,10] - 9[46,10],\\
d_{10,6} &= 9[39,10] - 18[56,10], d_{17} = 0, d_{27} = 3[178],\\
d_{37} &= 3[179], d_{47} = 9[278], d_{57} = 9[279] + 9[378],\\
d_{67} &= 9[379], d_{77} = 9[478], d_{87} = 9[479] + 18[578],\\
d_{97} &= 18[579] + 9[678], d_{10,7} = 9[679], d_{18} = 0,\\
d_{28} &= 3[179], d_{38} = [17,10] + 9[189], d_{48} = 9[279],\\
d_{58} &= 27[289] + 9[379] + 3[27,10], d_{68} = 3[37,10]\\
&\quad + 27[389],\\
d_{78} &= 9[479], d_{88} = 3[47,10] + 27[489] + 18[579],\\
d_{98} &= 6[57,10] + 54[589] + 9[679], d_{10,8} = 3[67,10]\\
&\quad + 27[689],\\
d_{19} &= 0, d_{29} = [17,10], d_{39} = 3[18,10], d_{49} = 3[27,10],\\
d_{59} &= 9[28,10] + 3[37,10], d_{69} = 9[38,10],\\
d_{79} &= 3[47,10],\\
d_{89} &= 9[48,10] + 6[57,10], d_{99} = 18[58,10] + 3[67,10],\\
d_{10,9} &= 9[68,10], d_{1,10} = d_{2,10} = \cdots = d_{10,10} = 0.
\end{aligned} \tag{18}$$

In this case, $t'_n = 3$, $m = 4$, i.e., $t'_n - m + 1 = 0$. From (9), we have

$$D_3 = \begin{bmatrix} \alpha_{\theta_1}\bar{a}^1_{\theta_1} & \alpha_{\theta_1}\bar{a}^2_{\theta_1} & \alpha_{\theta_1}\bar{a}^3_{\theta_1}\\ \alpha_{\theta_2}\bar{a}^1_{\theta_2} & \alpha_{\theta_2}\bar{a}^2_{\theta_2} & \alpha_{\theta_2}\bar{a}^3_{\theta_2}\\ \cdots & \cdots & \cdots\\ \alpha_{\theta_{10}}\bar{a}^1_{\theta_{10}} & \alpha_{\theta_{10}}\bar{a}^2_{\theta_{10}} & \alpha_{\theta_{10}}\bar{a}^3_{\theta_{10}} \end{bmatrix} = DB^T$$

where B and D are defined in (16) and (17). In addition, the condition (11) is satisfied which implies $\det(E_3) = 1$.

Hence from Proposition 3.1 we obtain the characteristic polynomial of a fourth-order 3-D tensor

$$\Phi(\lambda) = \det(A - \lambda I) = \det(M_3(\lambda))$$
$$M_3(\lambda) = (m_{ij})_{13\times 13} = \begin{bmatrix} \Delta_3 & DB^T\\ BD & 0 \end{bmatrix} \tag{19}$$

where the 100 elements in Δ_3 are computed by (18), B and D refer to (16) and (17).

V. Testing Positive Definiteness of a Fourth-Order Three-Dimensional Supersymmetric Tensor

Let

$$\phi(\lambda) = \sum_{l=0}^{d} a_l \lambda^l$$

where $d = n(m-1)^{(n-1)}$, be the characteristic polynomial of an mth-order n-dimensional supersymmetric tensor A. From Theorem 2.1, we have

$$a_d = (-1)^d, \quad a_{d-1} = (-1)^{d-1}(m-1)^{n-1}\mathrm{tr}(A)$$
$$a_0 = \det(A) = \phi(0).$$

When $m = 4$ and $n = 3$, $d = 27$. By directly computing, we have

$$\begin{aligned}
a_{d-2} &= a_{25}\\
&= 36a_{1113}a_{1333} - 81a_{1111}a_{3333} + 36a_{1112}a_{1222}\\
&\quad + 36a_{2223}a_{2333} - 81a_{2222}a_{3333} - 81a_{1111}a_{2222}\\
&\quad - 36a^2_{3333} - 36a^2_{2222} - 36a^2_{1111} + 54a^2_{1122}\\
&\quad + 54a^2_{2233} + 54a^2_{1133}.
\end{aligned}$$

However, other coefficients of $\phi(\lambda)$ are hard to be computed directly.

We choose 24 equidistant points

$$\lambda_i = s(1 - \frac{2(i-1)}{23}) \tag{20}$$

$i = 1, 2, \ldots, 24$, where s is a positive number, and have

$$\begin{aligned}
&a_1 + \lambda_i^1 a_2 + \ldots + \lambda_i^{24} a_{24}\\
&\quad = \left(\phi(\lambda_i) - \left(-\lambda_i^{27} + \lambda_i^{26} a_{26}\right.\right.\\
&\qquad \left.\left. + \lambda_i^{25} a_{25} + a_0\right)\right)/\lambda_i,
\end{aligned} \tag{21}$$

for $i = 1, 2, \ldots, 24$. It is seen that this is a Vandermonde system of linear equations. We use the Björck–Pereyra [4] algorithm to solve this system, and get $a_1, \ldots, a_{24}$.

For improving accurateness of computation, we may scale the characteristic polynomial. Proposition 2.1 implies that we can get all eigenvalues of A by computing all roots of the characteristic polynomial of B, denoted by $\psi(\mu) = \det(B - \mu I)$.

From Theorem 2.1 it follows that all nonpositive real roots of $\phi(\lambda)$ lie in $[L_1, 0]$ where

$$L_1 = \min\{a_{ii\cdots i} - \sum\{|a_{ii_2\cdots i_m}| : i_2, \ldots, i_m = 1, \ldots, n, \delta_{ii_2\cdots i_m} = 0\}.$$

If $L_1 < -1$, then $B = \frac{1}{|L_1|}A$ and $\psi(\mu) = \det(B - \mu I)$. If $L_1 \geq -1$, let $\psi \equiv \phi$. Let $L = \max\{L_1, -1\}$. Then ϕ has a nonpositive roots if and only if ψ has a nonpositive root, and all nonpositive real roots of $\psi(\mu)$ lie in $[L, 0]$. In order to find if $\psi(\lambda)$ has a nonpositive real root, we recall Sturm's theorem [5].

Theorem 5.1: Let $\psi(\lambda)$ be a nonconstant polynomial with real coefficients and let c_1 and c_2, with $c_1 < c_2$, be two real numbers such that $\psi(c_1) \cdot \psi(c_2) \neq 0$. If the sequence $\psi_0, \psi_1, \ldots, \psi_r$ is defined by the conditions

$$\psi_0 = \psi, \quad \psi_1 = \frac{d\psi}{d\lambda}, \psi_{i+1} = -\psi_{i-1} \bmod \psi_i$$

where $i = 1, 2, \ldots, r$ and $\psi_{r+1} \equiv 0$. The sequence $\psi_0, \psi_1, \ldots, \psi_r$ is called a **sequence of Sturm**. Denote by $v(x)$ the number of changes of signs in the sequence $\psi_0(x), \psi_1(x), \ldots, \psi_r(x)$. Then the number of **distinct zeros** of ψ on the interval (c_1, c_2) is equal to $v(c_1) - v(c_2)$.

For example, if $\psi(\lambda) = \lambda^2$, then we have $v(-1) - v(1) = 1$, i.e., ψ has one distinct root in $(-1, 1)$, though this root is a double root.

According to the conclusion in [24], if $\phi(\lambda)$ has at least a nonpositive real root with odd multiplicity, then A is not positive definite. Let V be the set of nonpositive real roots of ψ, which have even multiplicities. Since $a_d = (-1)^d$, $\lim_{\alpha pha \to \infty} \psi(\alpha) = +\infty$, i.e., $\psi(\alpha) > 0$ if $\alpha < L$, and $\psi(L) \geq 0$. If $\psi(\alpha) < 0$ for any $\alpha \in (L, 0]$, then $\psi(\lambda)$ has at least a nonpositive real root with odd multiplicity. This implies that A is not positive definite. If $\psi(\alpha) = 0$ for any $\alpha \in [L, 0]$, then α is a root of ψ, we may find a $k \geq 1$ such that $\psi^{(k-1)}(\alpha) = 0$ and $\psi^{(k)}(\alpha) \neq 0$. This implies that the multiplicity of α is k. If k is odd or $\psi^{(k)}(\alpha) < 0$, then A is not positive definite. If k is even and $\psi^{(k)}(\alpha) > 0$, then $\psi(\lambda) = \eta(\lambda)(\lambda - \alpha)^k$. We may record α in V and use η instead of ψ to check other nonpositive roots of ψ.

We now set $V = \emptyset$. Then we check $\psi(0)$ and $\psi(L)$. If $\psi(0) < 0$, or 0 or L is an odd-multiple root of ψ, then A is not positive definite. Otherwise, if 0 or L is an even-multiple root of ψ, then we may record it to V and replace ψ by η as described above. If $\eta(0) < 0$, then ψ has an odd-multiple nonpositive root and A is not positive definite. In the remaining case, both $\eta(0)$ and $\eta(L)$ are positive.

Then we may use the Sturm sequence of $\eta(\lambda)$ or $\psi(\lambda)$ if $V = \emptyset$, to know whether $\eta(\lambda)$ or $\psi(\lambda)$ has nonpositive real roots in $[L, 0]$ or not. If $\psi(\lambda)$ has no nonpositive real roots, then A is positive definite. Otherwise we check the value of $\eta(\lambda)$ or $\psi(\lambda)$ at the midpoint of $[L, 0]$ and use the Sturm sequence if necessary. We may repeat this process until either we find that A is not positive definite because ψ has an odd-multiple nonpositive root, or we have used the Sturm sequence to separate all **distinct nonpositive roots** of ψ. For a nonpositive root of ψ which has been separated in an interval (c_1, c_2), ψ or its reduced polynomial is positive at both c_1 and c_2 by the above procedures. Then we may easily conclude that ψ has an even-multiple root in (c_1, c_2). This also implies that $(d\psi/d\lambda)$ has an odd-multiple root in this interval. Then we may apply the bisection method to $(d\psi/d\lambda)$ or the derivative function of the reduced polynomial of ψ, to find an approximate value of this root if necessary.

If ϕ has nonpositive roots and all the nonpositive roots of $\phi(\lambda)$ are of even multiplicity, then we call this case the hard case. In this case we have to find if there exist real eigenvectors of A, associated with these nonpositive roots in order to determine the positive definiteness of A.

Let $\bar{\lambda}$ be a nonpositive real root of $\phi(\mu)$. Then the eigenvector associated with $\bar{\lambda}$ can be determined by the following equations:

$$\begin{pmatrix} f_1(x, \bar{\lambda}) \\ f_2(x, \bar{\lambda}) \\ f_3(x, \bar{\lambda}) \end{pmatrix} = 0 \tag{22}$$

$$x_1^4 + x_2^4 + x_3^4 - 1 = 0 \tag{23}$$

for $n = 3, m = 4$. Because f_i [see (5)] are homogeneous functions in 3 variables with degree 3, $i = 1, 2, 3$, we can obtain two systems of polynomial equations

$$f_i(x_1, x_2, 0, \bar{\lambda}) = 0, \ i = 1, 2, 3 \tag{24}$$

$$x_1^4 + x_2^4 - 1 = 0 \tag{25}$$

and

$$f_i(t_1, t_2, 1, \bar{\lambda}) = 0, \ i = 1, 2, 3 \tag{26}$$

$$(t_1^4 + t_2^4 + 1)x_3^4 - 1 = 0 \tag{27}$$

which are equivalent to (22)–(23). It is remarked that (24)–(25) is directly solved by eliminating x_2. While (26)–(27) can be solved by the Levenberg-Marquardt algorithm for solving nonlinear least squares problems. If we find a real eigenvector associated with a nonpositive real eigenvalue, then A is not positive definite.

Now we present an algorithm to test positive definiteness of a multivariate form in detail. In this algorithm, we use U to denote a set of intervals, each of which has more than one nonpositive distinct roots of ϕ, and W to denote a set of intervals, each of which has one nonpositive even-multiple distinct root of ϕ.

Algorithm 5.1: (An eigenvalue method for testing positive definiteness of a multivariate form)

Step 0) If $a_{ii \cdots i} \leq 0$ for some $i \in \{1, 2, \ldots, n\}$, A is not positive definite. Compute the lower bound of real eigenvalues, L_1 by the formula (21). If $L_1 > 0$, then A is positive definite, stop. If $L_1 < -1$, then set $A = (1/|L_1|)A$. Let $L = \max\{-1, L_1\}, V = \emptyset, U = \emptyset$ and $W = \emptyset$.

Step 1) Compute the matrices $M_{t'_n(\lambda)}$ and $E_{t'_n(\lambda)}$ by Proposition 3.2, Algorithms 3.1 and 3.2, where

$$\det(A - \lambda I) = \pm \frac{\det(M_{t'_n}(\lambda))}{(\det(E_{t'_n}(\lambda)))^2}$$

(see Proposition 3.1). When $m = 4$ and $n = 3$, $t'_n = 3$, $(M_3(\lambda))$ is defined in (19).

Step 2) Compute all coefficients in the characteristic polynomial of A, $\phi(\lambda) = \det(A - \lambda I)$, by the Björck–Pereyra algorithm. When $m = 4$ and $n = 3$, $\phi(\lambda) = \det(M_3(\lambda))$ [see (19)].

Step 3) If $\phi(0) < 0$, then A is not positive definite. If $\phi(0) = 0$ or $\phi(L) = 0$, check its multiplicity. If 0 or L is an odd-multiple root of ϕ, then A is not positive definite. Stop in these two cases. If 0 or L is an even-multiple root of ϕ, record it to V and replace ϕ by a reduced polynomial η which was described before. If the reduced polynomial ϕ is negative at 0, A is not positive definite, stop.

Step 4) Compute the Sturm sequence of $\phi(\lambda)$ according to Theorem 5.1. Use this sequence to check the number of distinct roots of $\phi(\lambda)$ in $(L, 0)$. If this number is zero and $V = \emptyset$, then A is positive definite and stop. If this number is 1, put $(L, 0)$ to W. If this number is bigger than 1, put $(L, 0)$ to U.

Step 5) If $U \neq \emptyset$, take an interval (c_1, c_2) from U. Let $c_3 = \frac{c_1+c_2}{2}$. If $\phi(c_3) < 0$, then A is not positive definite, stop. If $\psi(c_3) = 0$, find k such that $\psi^{(k-1)}(c_3) = 0$ and $\psi^{(k)}(c_3) \neq 0$. If k is odd or $\psi^{(k)}(c_3) < 0$, then A is not positive definite, stop. If k is even and $\psi^{(k)}(c_3) > 0$, record c_3 to V and replace ϕ by a reduced polynomial η as described before and compute the Sturm sequence for the reduced polynomial ϕ. Use the Sturm sequence to determine the numbers of distinct roots of ϕ in (c_1, c_3) and (c_3, c_2) respectively. Put (c_1, c_3) and (c_3, c_2) to U or W or discard one of them, depending these two numbers are bigger than one, exactly one, or zero. Repeat this step until either we find that A is not positive definite or $U = \emptyset$.

Step 6) If $V \neq \emptyset$, take a number, say $\bar{\lambda}$, from V. Then λ' is a nonpositive even-multiple root of ϕ. We may find if A has a real eigenvector associated with λ' or not. When $m = 4$ and $n = 3$, for $\bar{\lambda}$, solve the system (22)–(23). If there is a real solution in (22)–(23), then A is not positive definite, stop. Repeat this step until either we find that A is not positive definite or $V = \emptyset$.

Step 7) If $W \neq \emptyset$, take an interval, say (c_1, c_2), from W. Apply the bisection method to $(d\phi/d\lambda)$ on (c_1, c_2) to find an approximate root λ' of ϕ. Then $\bar{\lambda}$ is the approximate value of an nonpositive even-multiple root of ϕ. We may find if A has a real eigenvector associated with $\bar{\lambda}$ or not. When $m = 4$ and $n = 3$, for $\bar{\lambda}$, solve the system (22)–(23). If there is a real solution in (22)–(23), then A is not positive definite, stop. Repeat this step until either we find that A is not positive definite or $W = \emptyset$. In the latter case, A is positive definite.

Remarks:

1) In Step 2), it is not easy to get all coefficients of $\phi(\lambda)$ with good precision. From numerical test we find that $s = 2$ [see (20)] is a good choice.
2) In Steps 4) and 5), a modified Sturm function $\tilde{\phi}_i$, where $\tilde{\phi}_i = \phi_i/s_i$, s_i is a positive number, is generated such that the absolute value of leading coefficient of $\tilde{\phi}_i$ is 1, $i = 1, \ldots, r$.
3) In Step 7), the nonpositive even-multiple roots of $\phi(\lambda)$ in the intervals in W are approximately computed by the bisection method such that the error between approximated root and exact root is less than 10^{-6}. In addition, when (26)–(27) is solved, the minimal solution of $\tilde{f}(t_1, t_2, 1, \bar{\lambda}) = 0.5(f_1^2+f_2^2+f_3^2)$ is found. If there is a real t_1, t_2 such that $\tilde{f} = 0$, then $f_1 = f_2 = f_3 = 0$. By (27), set $\bar{x}_3 = (1/(t_1^4+t_2^4+1))^{1/4}$, and let $\bar{x}_1 = t_1\bar{x}_3, \bar{x}_2 = t_2\bar{x}_3$. It is easy to see that $\bar{x}_1, \bar{x}_2, \bar{x}_3$ is a real solution of (26)–(27). Hence, if the minimal value $\tilde{f}$ is less that 10^{-10}, then we think that there is a real solution.

VI. Numerical Test

In this section, we present some preliminary numerical tests for fourth order three dimensional supersymmetric tensors with Algorithm 5.1. The computation was done on a personal computer (Pentium IV, 2.8 GHz) running Matlab 7.0.

Because it is difficult to find test problems in the literature, we generate four kinds of problems by random approaches for testing the performance of Algorithm 5.1. In the following problems let $f(x) = Ax^4$.

TP I (general case)

$$\begin{aligned} f(x) = {} & a_{1111}x_1^4 + 4a_{1112}x_1^3x_2 + 4a_{1113}x_1^3x_3 + 6a_{1122}x_1^2x_2^2 \\ & + 12a_{1123}x_1^2x_2x_3 + 6a_{1133}x_1^2x_3^2 + 4a_{1222}x_1x_2^3 \\ & + 12a_{1223}x_1x_2^2x_3 + 12a_{1233}x_1x_2x_3^2 + 4a_{1333}x_1x_3^3 \\ & + a_{2222}x_2^4 + 4a_{2223}x_2^3x_3 + 6a_{2233}x_2^2x_3^2 \\ & + 4a_{2333}x_2x_3^3 + a_{3333}x_3^4 \end{aligned} \tag{28}$$

where a_{ijkl} is a random number in $[-l_b, l_b]$ with $l_b > 0$.

TP II (special case)

$$\begin{aligned} f(x) = {} & (b_1x_1 + b_2x_2 + b_3x_3)^2(b_4x_1 \\ & + b_5x_2 + b_6x_3)^2 \\ & + \lambda_0(x_1^4 + x_2^4 + x_3^4) \end{aligned} \tag{29}$$

where b_i is a random number in $[-5, 5]$ for $i = 1, 2, \ldots, 6$, and λ_0 is a parameter.

It is easy to know that when $\lambda_0 = 0, 0$ is the minimum H-eigenvalue of A. By Proposition 2.1, λ_0 is the minimum H-eigenvalue of f in (29).

TP III (hard case)

$$\begin{aligned} f(x) = {} & (b_1x_1 + b_2x_2 + b_3x_3)^2 \left(x_1^2 + x_2^2 + x_3^2\right) \\ & +\lambda_0 \left(x_1^4 + x_2^4 + x_3^4\right) \end{aligned} \tag{30}$$

where b_i is a random number in $[-10, 10]$ for all $i = 1, \ldots, 3$ and λ is a parameter. Similarly, λ_0 is also the minimum H-eigenvalue of f in (30). However the hard case can arise when Algorithm is used to solve this kind problem with negative λ_0. Numerical results show that λ_0 is almost always an even-multiple H-eigenvalue of f in the problems generated by (30).

TP IV (N-eigenvalue case)

$$f(x) = (b_1x_1^2 + b_2x_2^2 + b_3x_3^2)^2 + \lambda_0(x_1^4 + x_2^4 + x_3^4) \tag{31}$$

where b_1, b_2 and b_3 are positive random numbers in (1 100). In the problems generated by (31), λ_0 is an N-eigenvalue of A.

Tables I–IV show the performance of Algorithm 5.1 on the four kinds of problems where

NP: the number of the problem.

ALB: the absolute value of the lower bound of real eigenvalues [see (21)].

HE: current minimal nonpositive H-eigenvalue of A in output where “p” means that all H-eigenvalues of A are positive.

NE: minimal nonpositive N-eigenvalue of A (only for Table IV).

PD: the positive definiteness, where “y” means yes,“n” means no.

Time: the CPU time in seconds.

TABLE I
RESULTS OF TP I

NP	ALB	HE	PD	Time
1	20	-7.3137	n	0.0469
2	18.5	p	y	0.0215
3	13	-1.9140	n	0.4760
4	95.75	-18.7285	n	0.0625
5	214.5	-53.8265	n	0.0781
6	195	p	y	0.0156
7	2250	-804.72	n	0.0251
8	1541.5	-816.286	n	0.0788
9	1869	p	y	0.0176

TABLE II
RESULTS OF TP II

NP	ALB	HE	PD	Time
1	2529.5	p	y	0.0381
2	620	p	y	0.0156
3	524.5	p	y	0.469
4	1162	0	n	0.0283
5	604	0	n	0.0307
6	832	0	n	0.0212
7	2039	-1.010	n	0.0469
8	1539	-1.004	n	0.0781
9	1062	-0.99991	n	0.0572

TABLE III
RESULTS OF TP III

NP	ALB	HE	PD	Time
1	413	0	n	0.0469
2	112	0	n	0.0469
3	175	-0.00015	n	0.0156
4	529	-1.00084	n	0.109
5	449	-1.00234	n	0.156
6	154	-1.0011	n	0.0469
7	281.5	-2.00097	n	0.1250
8	132.5	-2.0011	n	0.0938
9	227.5	-1.9979	n	0.1406

TABLE IV
RESULTS OF TP IV

NP	ALB	NE	PD	Time
1	1848	-0.01181	y	0.0781
2	2565	0	y	0.0469
3	2640	-0.014443	y	0.0937
4	4418	-10.0297	y	0.141
5	2917	-10.0248	y	0.135
6	3115	-10.004	y	0.0469
7	2210	-10.0229	y	0.126
8	2008	-99.970	y	0.1875
9	3940	-100.061	y	0.1563

In Table I, we give the results of 9 different general test problems generated by (28) where we choose $l_b = 10$ in 1–3, $l_b = 100$ in 4–6 and $l_b = 1000$ in 7–9. Algorithm 5.1 can solve these problems very well in short time. The hard case does not arise in the computation process of Algorithm 5.1 for solving these nine problems.

In Table II, nine special test problems are generated by (29), and we choose $\lambda_0 = 1$ in 1–3, $\lambda_0 = 0$ in 4–6 and $\lambda_0 = -1$ in 7–9. These problems are solved by Algorithm 5.1 and the correct results are obtained. The hard case does not arise also in the computation process of Algorithm 5.1 for solving these nine special problems.

In Table III, nine test problems which are not positive definite are generated by (30), and we choose $\lambda_0 = 0$ in . 1–3, $\lambda_0 = -1$ in the problems 4–6 and $\lambda_0 = -2$ in the problems 7–9. The hard case arises when Algorithm 5.1 is used to solve these nine problems. From the results in Table III we know that Algorithm 5.1 can handle the hard case in short time.

In Tables IV, 9 test problems which are positive definite are generated by (31), and we choose $\lambda_0 = 0$ in 1–3, $\lambda_0 = -10$ in the problems 4–7 and $\lambda_0 = -100$ in the problems 8–9. λ_0 is N-eigenvalue of these problems. From Table IV, we know that the correct results can be obtained by Algorithm 5.1.

Numerical results show that Algorithm 5.1 is a feasible and efficient eigenvalue method for testing positive definiteness of a quartic form of three variables.

VII. FINAL COMMENTS

In this paper we propose an eigenvalue method for testing positive definiteness of a multivariate form. At first we give a frame of method for computing the symmetric hyperdeterminant and the characteristic polynomial of a supersymmetric tensor for the general case. Then we propose Algorithm 5.1 which can be carried out when $n = 3$ and $m = 4$. A possible improvement of this method is to use the E-eigenvalues and the E-characteristic polynomial of A instead of the eigenvalues and the characteristic polynomial of A in the algorithm. The E-eigenvalues and the E-characteristic polynomial of A were also introduced in [24], and studied further in [25], [26], [23]. They may also be used to test positive definiteness of a multivariate form. An advantage of the E-eigenvalues and the E-characteristic polynomial is that the degree of the E-characteristic polynomial is much lower than the degree of the characteristic polynomial. When $n = 3$ and $m = 4$, the degree of the characteristic polynomial is 27 as indicated in Section V, while the degree of the E-characteristic polynomial is at most 13 [23]. However, unlike the degree of the characteristic polynomial, which is fixed when m and n are fixed, the degree of the E-characteristic polynomial is not fixed. For example, when $n = 3$ and $m = 4$, the degree of the E-characteristic polynomial may be 13 or may be less than 13. This creates a difficulty to identify that degree for a particular problem. This is why we study the eigenvalue method first in this paper.

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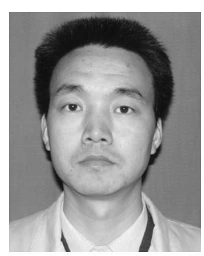

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