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# Positive Definite Tensors to Nonlinear Complementarity Problems

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**Abstract** The main purpose of this paper is to investigate some kinds of nonlinear complementarity problems (NCP). For the structured tensors, such as, symmetric positive definite tensors and copositive tensors, we derive the existence theorems on a solution of these kinds of nonlinear complementarity problems. We prove that a unique solution of the NCP exists under the condition of diagonalizable tensors.

**Keywords** Copositive tensor · Symmetric tensor · Positive definite tensor · Diagonalizable tensors · Nonlinear complementarity problems

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## 1 Introduction

Over the past decade, the research of finite-dimensional variational inequality and complementarity problems [1–6] has been rapidly developed in the theory of existence, uniqueness and sensitivity of solutions, theory of algorithms, and the application of these techniques to transportation planning, regional science, socio-economic analysis, energy modeling and game theory.

Qi [7] defined two kinds of eigenvalues and described some relative results similar to the matrix eigenvalues. Lim [8] proposed another definition of eigenvalues, eigenvectors, singular values, and singular vectors for tensors based on a constrained

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variational approach, much like the Rayleigh quotient for symmetric matrix eigenvalues, independently.

Linear complementarity problems can be settled by the criss-cross algorithm [9]. Conversely, for linear complementarity problems, the criss-cross algorithm terminates finitely only if the matrix is a sufficient matrix [9]. A sufficient matrix is a generalization both of a positive definite matrix [10, Section 4.2] and of a P-matrix [11], whose principal minors are each positive. Similarly, for a symmetric tensor, Qi [7] gave the definition of a positive definite tensor and derived a method to check whether a symmetric tensor is positive definite or not. The concept of copositive matrices [12] is an important concept in applied mathematics, with applications in control theory, optimization modeling, linear complementarity problems, graph theory and linear evolution variational inequalities [13]. Qi [14] extended this concept to tensors.

The rest of this paper is organized as follows. Section 2 introduces notations and definitions of basic preliminaries. Some existence and uniqueness theorems of solutions of nonlinear complementarity problems and two problems that we consider in this paper are given in Section 3. The main results in Section 4 is to study the existence and uniqueness of solution(s) of Problems 3.1 and 3.2. In Section 5, we present two conjectures and two open questions. We conclude our paper in Section 6. Finally, the Reference section contains the most comprehensive bibliography in this area to date.

## 2 Notation and Definitions

In this section, we define the notations and collect some basic definitions and facts, which will be used later on.

Throughout this paper, we assume that $m,\ n\ (\geq 2)$ are positive integers and $m$ is even. We use small letters $x,u,v,\dots,$ for scalars, small bold letters $\mathbf{x},\mathbf{u},\mathbf{v},\dots,$ for vectors, capital letters $A,B,C,\dots,$ for matrices, calligraphic letters $\mathscr{A},\mathscr{B},\mathscr{C},\dots,$ for tensors, and $\mathfrak{A},\mathfrak{B},\mathfrak{C},\dots,$ for the subsets in $\mathbb{R}^n$. Denote $[n]=\{1,2,\dots,n\}$. $\mathbf{0}$ means a column vector in $\mathbb{R}^n$, where its all entries are zeros. $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. Given a column vector $\mathbf{x}\in\mathbb{R}^n$, $\mathbf{x}^\top$ represents the transpose of $\mathbf{x}$, that is, $\mathbf{x}^\top$ is a row vector.

The set $T_{m,n}$ consists of all order $m$ dimension $n$ tensors and every element in $\mathscr{A}\in T_{m,n}$ is real, that is, $\mathscr{A}_{i_1i_2\dots i_m}\in\mathbb{R}$ where $i_k\in[n]$ with $k\in[m]$. $\mathscr{D}\in T_{m,n}$ is diagonal if all off-diagonal entries are zero. Particularly, when the diagonal entries of $\mathscr{D}$ are 1, then $\mathscr{D}$ is called the identity tensor and denote it by $\mathscr{I}$ [7]. Given a vector $\mathbf{x}\in\mathbb{R}^n$, $\|\mathbf{x}\|_\alpha^\alpha=x_1^\alpha+x_2^\alpha+\dots+x_n^\alpha$ with $\alpha$ is an positive integers.

In following, we present five definitions about tensors and nonlinear mappings. The mode-$k$ tensor-matrix product and mode-$k$ tensor-vector product of $\mathscr{A}\in T_{m,n}$ are defined as follows.

**Definition 2.1 ([15])** The mode-$k$ product of a tensor $\mathscr{A} \in T_{m,n}$ by a matrix $B \in \mathbb{R}^{n\times n}$, denoted by $\mathscr{A} \times_k B$ is a tensor $\mathscr{C} \in T_{m,n}$ of which are given by

$$\mathscr{C}_{i_1\dots i_{k-1} j i_{k+1}\dots i_m} = \sum_{i_k=1}^{n} \mathscr{A}_{i_1\dots i_{k-1} i_k i_{k+1}\dots i_m} b_{j i_k}, \quad k \in [m].$$

Particularly, the mode-$k$ multiplication of a tensor $\mathscr{A} \in T_{m,n}$ by a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted by $\mathscr{A} \bar{\times}_k \mathbf{x}$. Set $\mathscr{C} = \mathscr{A} \bar{\times}_k \mathbf{x}$, then, element-wise, we have

$$\mathscr{C}_{i_1\dots i_{k-1} i_{k+1}\dots i_m} = \sum_{i_k=1}^{n} \mathscr{A}_{i_1\dots i_{k-1} i_k i_{k+1}\dots i_m} x_{i_k}.$$

According to Definition 2.1, let $m$ vectors $\mathbf{x}_k \in \mathbb{R}^n$, $\mathscr{A} \bar{\times}_1 \mathbf{x}_1 \dots \bar{\times}_m \mathbf{x}_m$ is easy to define. If these $m$ vectors are the same vector, denoted by $\mathbf{x}$, then $\mathscr{A} \bar{\times}_1 \mathbf{x} \dots \bar{\times}_m \mathbf{x}$ can be simplified as $\mathscr{A}\mathbf{x}^m$.

Given a mapping $F : \mathfrak{X} \subset \mathbb{R}^k \to \mathbb{R}^n$, we suppose that $F(\mathbf{x}) \in \mathbb{R}^n$ is a column vector in this paper for all $\mathbf{x} \in \mathfrak{X}$. Now, given two column vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle$ represents the inner product of $\mathbf{x}$ and $\mathbf{y}$, i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^\top \mathbf{x}$. All properties of an inner product can be found in [16].

Our next definition is motivated by the class of copositive matrices [11], which in turn generalizes that of nonnegative matrices.

**Definition 2.2 ([3])** A mapping $F : \mathfrak{X} \to \mathbb{R}^n$ is said to be

(a) Copositive with respect to $\mathfrak{X}$, iff

$$\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x} \rangle \geq 0, \quad \forall \mathbf{x} \in \mathfrak{X}.$$

(b) Strictly copositive with respect to $\mathfrak{X}$, iff

$$\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x} \rangle > 0, \quad \forall \mathbf{x} \in \mathfrak{X},\ \mathbf{x} \neq \mathbf{0}.$$

(c) Strongly copositive with respect to $\mathfrak{X}$, iff there exists a scalar $\alpha > 0$ such that

$$\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x} \rangle \geq \alpha \|\mathbf{x}\|_2^2, \quad \forall \mathbf{x} \in \mathfrak{X}.$$

The definition of a symmetric tensor [8, 7] is stated as follows.

**Definition 2.3** Suppose that $\mathscr{A} \in T_{m,n}$. $\mathscr{A}$ is called symmetric iff $\mathscr{A}_{i_1 i_2 \dots i_m}$ is invariant by any permutation $\pi$, that is $\mathscr{A}_{i_1 i_2 \dots i_m} = \mathscr{A}_{\pi(i_1, i_2, \dots, i_m)}$ where all $i_k \in [n]$ with $k \in [m]$. We denote all symmetric tensors by $ST_{m,n}$.

When $m$ is even and $\mathscr{A}$ is symmetric, we say that

(a) $\mathscr{A}$ is positive definite [7], iff $\mathscr{A}\mathbf{x}^m > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$,
(b) $\mathscr{A}$ is positive semidefinite [7], iff $\mathscr{A}\mathbf{x}^m \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,
(c) $\mathscr{A}$ is a copositive tensor [14], iff for any $\mathbf{x} \in \mathbb{R}^n_+$, then $\mathscr{A}\mathbf{x}^m \geq 0$,
(d) $\mathscr{A}$ is a strictly copositive tensor [14], iff for any nonzero $\mathbf{x} \in \mathbb{R}^n_+$, then $\mathscr{A}\mathbf{x}^m > 0$.

The set of all positive definite tensors is denoted by $SPT_{m,n}$.

The mapping

$$G(\mathbf{x}) = F(\mathbf{x}) - F(\mathbf{0}) \tag{1}$$

plays an important role in the nonlinear complementarity problem; this again is motivated by the linear complementarity problem.

The strict copositivity of a mapping can be relaxed through the introduction of the class of $\mathbf{d}$-regular mappings.

**Definition 2.4 ([3])**
For any vector $\mathbf{x} \in \mathbb{R}^n_+$, we define the index sets

$$I_+(\mathbf{x}) = \{i : x_i > 0\} \quad \text{and} \quad I_0(\mathbf{x}) = \{i : x_i = 0\}.$$

Let $\mathbf{d} > \mathbf{0}$ be an arbitrary vector in $\mathbb{R}^n$. A mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $\mathbf{d}$-regular, if the following system of equations has no solution in $(\mathbf{x},t) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ with $\mathbf{z} \neq \mathbf{0}$,

$$\begin{aligned} G_i(\mathbf{x}) + td_i &= 0, \quad i \in I_+(\mathbf{x}), \\ G_i(\mathbf{x}) + td_i &\geq 0, \quad i \in I_0(\mathbf{x}). \end{aligned} \tag{2}$$

Equivalently, $G$ is $\mathbf{d}$-regular if, for any scalar $r > 0$, the augmented nonlinear complementarity problem NCP$(H)$ defined by $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$H \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} = \begin{pmatrix} G(\mathbf{x}) + t\mathbf{d} \\ r - \langle \mathbf{d}, \mathbf{x} \rangle \end{pmatrix},$$

has no solution $(\mathbf{x},t)$ with $\mathbf{x} \neq \mathbf{0}$.

Similar to the diagonalizable matrices [10], the definition of the diagonalizable tensors [17, 18] is presented as follows.

**Definition 2.5** Suppose that $\mathscr{A} \in ST_{m,n}$. $\mathscr{A}$ is called diagonalizable iff $\mathscr{A}$ can be represented as

$$\{\mathscr{A} \in T_{m,n} | \mathscr{A} = \mathscr{D} \times_1 B \times_2 B \cdots \times_m B\},$$

where $B \in \mathbb{R}^{n \times n}$ with $\det(B) \neq 0$ and $\mathscr{D}$ is a diagonal tensor. Denote all diagonalizable tensors by $D_{m,n}$.

It is obvious that $D_{m,n} \subseteq ST_{m,n}$ and $\mathscr{A}$ is congruent to $\mathscr{D}$ when $m = 2$.

## 3 Lemmas and Problem Description

Let $F$ be a mapping from $\mathbb{R}^n$ into itself. The nonlinear complementarity problem, denoted by NCP$(F)$, is to find a vector $\mathbf{x}^* \in \mathbb{R}^n_+$ such that

$$F(\mathbf{x}^*) \in \mathbb{R}^n_+, \quad \langle F(\mathbf{x}^*), \mathbf{x}^* \rangle = 0.$$

When $F(\mathbf{x})$ is an affine function of $\mathbf{x}$, say $F(\mathbf{x}) = \mathbf{q} + M\mathbf{x}$ for some given vectors $\mathbf{q} \in \mathbb{R}^n$ and matrix $M \in \mathbb{R}^{n \times n}$, the problem NCP$(F)$ reduces to the linear complementarity problem, which is denoted by $\text{LCP}(\mathbf{q},M)$. The results of the linear complementarity problem can be found in the references [11,19].

A further generalization of the $\text{NCP}(F)$ is the variational inequality: given a mapping $F:\mathbb{R}^n\to\mathbb{R}^n$, and $\emptyset\neq\mathfrak{K}\subseteq\mathbb{R}^n$, find a $\mathbf{x}^*\in\mathfrak{K}$ satisfying

$$\langle F(\mathbf{x}^*),\mathbf{y}-\mathbf{x}^*\rangle\geq 0,\quad \text{for all } \mathbf{y}\in\mathfrak{K},$$

denoted by $\text{VI}(\mathfrak{K},F)$.

If $\mathfrak{K}=\{\mathbf{x}:\mathbf{x}\geq\mathbf{0}\}$, then a $\mathbf{x}^*$ is a solution of $\text{VI}(\mathfrak{K},F)$, solves the $\text{NCP}(F)$.

It is well-known that $A$ is a $P$-matrix [11] if and only if the linear complementarity problem

$$\text{find } \mathbf{z}\in\mathbb{R}^n \text{ such that } \mathbf{z}\geq\mathbf{0},\quad \mathbf{q}+\text{A}\mathbf{z}\geq\mathbf{0},\ \text{and } \langle\mathbf{q}+\text{A}\mathbf{z},\mathbf{z}\rangle=0.$$

has a unique solution for all $\mathbf{q}\in\mathbb{R}^n$. Then for a $P$-tensor [20] $\mathscr{A}\in T_{m,n},(m>2)$, does a similar property hold for the following nonlinear complementarity problem

$$\text{find } \mathbf{x}\in\mathbb{R}^n \text{ such that } \mathbf{x}\geq\mathbf{0},\quad \mathbf{q}+\mathscr{A}\mathbf{x}^{m-1}\geq\mathbf{0},\ \text{and } \langle\mathbf{q}+\mathscr{A}\mathbf{x}^{m-1},\mathbf{x}\rangle=0?$$

In this paper, we consider a special kind of $\text{NCP}(F)$, that is, $F_i(\mathbf{x})$ is a multivariate polynomial and the degree of $F_i(\mathbf{x})$ is $k_i$, then $F(\mathbf{x})$ can be expressed by,

$$F(\mathbf{x})=\sum_{i=1}^{k}\mathscr{A}_i\mathbf{x}^{i-1},$$

where $\mathscr{A}_i\in T_{i-1,n}$, $\mathscr{A}_i\mathbf{x}^{i-1}$ means the tensor-vector product given in Definition 2.1, $k=\max\limits_{1\leq i\leq n}k_i$. Particularly, $\mathscr{A}_1$ is a vector and $\mathscr{A}_2$ is a matrix.

### 3.1 Lemmas

The following lemma is an existence and uniqueness theorem by Cottle [1]. It involves the notion of positively bounded Jacobians, and the original proof was constructive in the sense that an algorithm was employed to actually compute the unique solution.

**Lemma 3.1 ([1,3])** *Let $F:\mathbb{R}^n_+\to\mathbb{R}^n$ be continuously differentiable and suppose that there exists one $\delta\in(0,1)$, such that all principal minors of the Jacobian matrix $\nabla F(\mathbf{x})$ are bounded between $\delta$ and $\delta^{-1}$, for all $\mathbf{x}\in\mathbb{R}^n_+$. Then the* $\text{NCP}(F)$ *has a unique solution.*

If mapping $F$ is strictly copositive, then the following result holds.

**Lemma 3.2 ([5])** *Let $F:\mathbb{R}^n_+\to\mathbb{R}^n$ be continuous and strictly copositive with respect to $\mathbb{R}^n_+$. If there exists a mapping $c:\mathbb{R}_+\to\mathbb{R}$ such that $c(\lambda)\to\infty$ as $\lambda\to\infty$, and for all $\lambda\geq 1$, $\mathbf{x}\geq\mathbf{0}$,*

$$\langle F(\lambda\mathbf{x})-F(\mathbf{0}),\mathbf{x}\rangle\geq c(\lambda)\langle F(\mathbf{x})-F(\mathbf{0}),\mathbf{x}\rangle,\tag{3}$$

*then the problem* $\text{NCP}(F)$ *has a nonempty, compact solution set.*

For the linear complementarity problem, the mapping $G$, given in formula (1), is obviously linear and thus, condition (3) is satisfied with $c(\lambda)=\lambda$. More generally, the same condition will hold with $c(\lambda)=\lambda^{\alpha}$, if $G$ is positively homogeneous of degree $\alpha>0$; i.e., if $G(\lambda\mathbf{x})=\lambda^{\alpha}G(\mathbf{x})$ for $\lambda>0$.

If $F$ is strictly copositive with respect to $\mathbb{R}^n_+$, then the mapping $G$ in (1) is $\mathbf{d}$-regular for any $\mathbf{d}>\mathbf{0}$. The following lemma presents an existence result for the nonlinear complementarity problem with $\mathbf{d}$-regular mapping.

**Lemma 3.3 ([4])** *Let $F$ be a continuous mapping from $\mathbb{R}^n$ into itself and $G$ defined by* (1). *Suppose that $G$ is positively homogeneous of degree $\alpha>0$ and that $G$ is $\mathbf{d}$-regular for some $\mathbf{d}>\mathbf{0}$. Then the problem* NCP($F$) *has a nonempty, compact solution set.*

The main characterization theorem for copositive tensors can be summarized as follows.

**Lemma 3.4 ([14, Theorem 5])** *Let $\mathscr{A}\in T_{m,n}$ be a symmetric tensor. Then, $\mathscr{A}$ is copositive of and only if*

$$\min\left\{\mathscr{A}\mathbf{x}^m:\mathbf{x}\in\mathbb{R}^n_+,\ \sum_{i=1}^n x_i^m=1\right\}\geq 0.$$

*$\mathscr{A}$ is strictly copositive if and only if*

$$\min\left\{\mathscr{A}\mathbf{x}^m:\mathbf{x}\in\mathbb{R}^n_+,\ \sum_{i=1}^n x_i^m=1\right\}>0.$$

### 3.2 Problem Description

In this subsection, we present two problems which we shall discuss in this paper.

**Problem 3.1 ([20])** Given $\mathscr{A}\in T_{m,n}$ and $\mathbf{q}\in\mathbb{R}^n$. The NCP($\mathbf{q},\mathscr{A}$) is to find a vector $\mathbf{x}\in\mathbb{R}^n_+$ such that

$$F(\mathbf{x})=\mathscr{A}\mathbf{x}^{m-1}+\mathbf{q}\in\mathbb{R}^n_+,\quad \mathscr{A}\mathbf{x}^m+\langle\mathbf{q},\mathbf{x}\rangle=0.$$

**Problem 3.2** Given $\mathscr{A}_k\in T_{m-(2k-2),n}$ and $\mathbf{q}\in\mathbb{R}^n$ with $k\in[m/2]$. The NCP($\mathbf{q},\{\mathscr{A}_k\}$) is to find a vector $\mathbf{x}\in\mathbb{R}^n_+$ such that

$$F(\mathbf{x})=\sum_{k=1}^{m/2}\mathscr{A}_k\mathbf{x}^{m-(2k-1)}+\mathbf{q}\in\mathbb{R}^n_+,\quad \sum_{k=1}^{m/2}\mathscr{A}_k\mathbf{x}^{m-2k+2}+\langle\mathbf{q},\mathbf{x}\rangle=0,$$

where $\mathscr{A}_{m/2}$ is a square matrix.

Let FEA($\mathbf{q},\mathscr{A}$) $=\{\mathbf{x}\in\mathbb{R}^n_+:\mathscr{A}\mathbf{x}^{m-1}+\mathbf{q}\in\mathbb{R}^n_+\}$. If FEA($\mathbf{q},\mathscr{A}$) $\neq\emptyset$, then we see that NCP($\mathbf{q},\mathscr{A}$) is feasible. It is obvious that Problem 3.1 is a special case of Question 3.2. However, for simplicity, we only consider the solvability of Problem 3.1, in detail, and make use of the results obtained by solving Problem 3.1, we then consider the solvability of Problem 3.2.

## 4 Main Results

Without loss of generality, suppose that $\mathbf{q} \in \mathbb{R}^n$ in Problems 3.1 and 3.2 is nonzero. For example, let $\mathscr{A} \in ST_{m,n}$ be positive definite. If $\mathbf{q}$ is zero, then the solution of Problem 3.1 is zero. This situation is extraordinary, in order to avoid this situation, let $\mathbf{q} \in \mathbb{R}^n$ in Problems 3.1 and 3.2 be nonzero.

If zero vector $\mathbf{0}$ solves Problems 3.1 and 3.2, we derive that $\mathbf{q}$ is a nonnegative vector. Hence, in this paper, we only consider a nonzero solution $\mathbf{x}$ of Problems 3.1 and 3.2.

### 4.1 Necessary Conditions for Solving Problem 3.1

The cornerstone for the *necessary* conditions to be presented is the nonlinear programming formulation of the Problem 3.1,

$$\begin{aligned} \min \quad & \mathscr{A}\mathbf{x}^m + \langle \mathbf{q}, \mathbf{x} \rangle \\ s.t. \quad & \mathscr{A}\mathbf{x}^{m-1} + \mathbf{q} \in \mathbb{R}^n_+, \ \mathbf{x} \in \mathbb{R}^n_+. \end{aligned} \tag{4}$$

Because $\mathrm{FEA}(\mathbf{q}, \mathscr{A})$ is also the feasible set of (4), if $\mathbf{x}_*$ minimizes the nonlinear programming given in (4) and $\mathscr{A}\mathbf{x}_*^m + \langle \mathbf{q}, \mathbf{x}_* \rangle = 0$, then $\mathbf{x}_*$ is a solution of Problem 3.1. According to first-order necessary conditions given in [21], we obtain the following theorem.

**Theorem 4.1** *If* $\mathrm{FEA}(\mathbf{q}, \mathscr{A}) \neq \emptyset$ *and* $\mathbf{x}_*$ *is a local solution of* (4)*. Then, there exists a vector* $\mathbf{u}_*$ *of multipliers satisfying the conditions,*

$$\begin{aligned} \mathbf{q} + m\mathscr{A}\mathbf{x}_*^{m-1} - (m-1)\mathscr{A}\mathbf{x}_*^{m-2}\mathbf{u}_* &\geq \mathbf{0} \\ \langle \mathbf{x}_*, \mathbf{q} + m\mathscr{A}\mathbf{x}_*^{m-1} - (m-1)\mathscr{A}\mathbf{x}_*^{m-2}\mathbf{u}_* \rangle &= 0 \\ \mathbf{u}_* &\geq \mathbf{0} \\ \langle \mathbf{u}_*, \mathbf{q} + \mathscr{A}\mathbf{x}_*^{m-1} \rangle &= 0. \end{aligned} \tag{5}$$

*Finally, the vectors* $\mathbf{x}_*$ *and* $\mathbf{u}_*$ *satisfy*

$$(m-1)(\mathbf{x}_* - \mathbf{u}_*)_i (\mathscr{A}\mathbf{x}_*^{m-2}(\mathbf{x}_* - \mathbf{u}_*))_i \leq 0, \quad i \in [n]. \tag{6}$$

*Proof* Since $\mathrm{FEA}(\mathbf{q}, \mathscr{A}) \neq \emptyset$, the nonlinear programming (4) is feasible. Such an optimal solution $\mathbf{x}_*$ and a suitable vector $\mathbf{u}_*$ of multipliers will satisfy the Karush-Kuhn-Tucker conditions (5). To prove (6), we examine the inner product

$$\langle \mathbf{x}_*, \mathbf{q} + m\mathscr{A}\mathbf{x}_*^{m-1} - (m-1)\mathscr{A}\mathbf{x}_*^{m-2}\mathbf{u}_* \rangle = 0,$$

at the componentwise level and deduce that for all $i \in [n]$,

$$(m-1)(\mathbf{x}_*)_i (\mathscr{A}\mathbf{x}_*^{m-2}(\mathbf{x}_* - \mathbf{u}_*))_i \leq 0, \tag{7}$$

using the fact that $\mathbf{x}_* \in \mathrm{FEA}(\mathbf{q}, \mathscr{A})$. Similarly, multiplying the $i$th component in

$$\mathbf{q} + m\mathscr{A}\mathbf{x}_*^{m-1} - (m-1)\mathscr{A}\mathbf{x}_*^{m-2}\mathbf{u}_* \geq \mathbf{0},$$

by $\mathbf{u}_*$ and then invoking the complementarity condition

$$(\mathbf{u}_*)_i(\mathbf{q}+\mathscr{A}\mathbf{x}_*^{m-1})_i = 0,$$

which is implied by $\mathbf{u}_* \geq \mathbf{0}$, $\langle \mathbf{u}_*, \mathbf{q}+\mathscr{A}\mathbf{x}_*^{m-1}\rangle = 0$, and the feasibility of of $\mathbf{x}_*$, we obtain

$$-(m-1)(\mathbf{u}_*)_i(\mathscr{A}\mathbf{x}_*^{m-2}(\mathbf{x}_*-\mathbf{u}_*))_i \leq 0. \tag{8}$$

Now, (6) follows by adding (7) and (8). □

*Remark 4.1* Theorem 4.1 is the special case of the result given in Cottle [1, Theorem 3].

With Theorem 4.1, we prove the following existence result for the NCP$(\mathbf{q},\mathscr{A})$.

**Theorem 4.2** *Let nonzero* $\mathbf{x}_*$ *be a local solution of* (4)*. If* $\mathscr{A}\mathbf{x}^{m-2}$ *is positive definite for all* $\mathbf{x}\in\mathbb{R}^n$*, then* $\mathbf{x}_*$ *solves* NCP$(\mathbf{q},\mathscr{A})$.

*Proof* According to Theorem 4.1, there exists a nonnegative vector $\mathbf{u}_*$ such that

$$(m-1)(\mathbf{x}_*-\mathbf{u}_*)_i(\mathscr{A}\mathbf{x}_*^{m-2}(\mathbf{x}_*-\mathbf{u}_*))_i \leq 0, \quad i\in[n],$$

that is,

$$\langle \mathbf{x}_*-\mathbf{u}_*, \mathscr{A}\mathbf{x}_*^{m-2}(\mathbf{x}_*-\mathbf{u}_*)\rangle \leq 0.$$

According to proposition assumptions, we know that $\mathbf{x}_* = \mathbf{u}_*$. Based on (5), then, $\mathbf{x}_*$ solves NCP$(\mathbf{q},\mathscr{A})$. □

*Remark 4.2* If $\mathbf{x}_* = \mathbf{0}$ is a local solution of (4), then $\mathbf{x}_*$ solves NCP$(\mathbf{q},\mathscr{A})$ for all vectors $\mathbf{q}\in\mathbb{R}^n_+$.

Moreover, we can derive some results about Problem 3.2, similar to Theorems 4.1 and 4.2. Here, we do not list them out.

## 4.2 Solving Problem 3.1

In Probelm 3.1, let $F(\mathbf{x}) = \mathscr{A}\mathbf{x}^{m-1}+\mathbf{q}$. We first consider some properties of $F(\mathbf{x})$ when $\mathscr{A}$ is selected from sets of structured tensors.

**Theorem 4.3** *Suppose* $\mathscr{A}\in ST_{m,n}$ *and* $\mathbf{x}\in\mathbb{R}^n_+$.

(a) *If* $\mathscr{A}$ *is* (*strictly*) *copositive, then the mapping* $F(\mathbf{x})$ *is* (*strictly*) *copositive with respect to* $\mathbb{R}^n_+$.
(b) *If* $\mathscr{A}$ *is positive definite, then the mapping* $F(\mathbf{x})$ *is strongly copositive with respect to* $\mathbb{R}^n_+$ *when* $\alpha \leq \lambda_{\min}$ $(\leq \lambda_{\min}\frac{\|\mathbf{x}\|_2^2}{\|\mathbf{x}\|_m^m})$*, where* $\lambda_{\min}$ *is the smallest Z-eigenvalue* (*H-eigenvalue*) *of* $\mathscr{A}$.

*Proof* According to Definition 2.2, because of $F(\mathbf{x}) = \mathscr{A}\mathbf{x}^{m-1} + \mathbf{q}$, so $\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x}\rangle = \mathscr{A}\mathbf{x}^m$. Since $\mathscr{A}$ is (strictly) copositive, that is, $\mathscr{A}\mathbf{x}^m \ (>) \geq 0$ for all (nonzero) $\mathbf{x} \in \mathbb{R}^n_+$ Then, part $(a)$ is proved.

We now prove part $(b)$. Since $\mathscr{A}$ is positive definite, according to [7, Theorem 5], we know that the smallest Z-eigenvalue (H-eigenvalue) of $\mathscr{A}$, denoted by $\lambda_{\min}$, is greater than zero, that is $\lambda_{\min} > 0$.

When $\lambda_{\min}$ is the smallest Z-eigenvalue of $\mathscr{A}$, then

$$\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x}\rangle = \mathscr{A}\mathbf{x}^m \geq \lambda_{\min}\|\mathbf{x}\|_2^2.$$

Hence, under this case, part $(b)$ is proved.

It is obvious to prove part $(b)$ in the case of $\lambda_{\min}$ is the smallest H-eigenvalue of $\mathscr{A}$. □

When $\mathscr{A} \in D_{m,n}$ is positive semi-definite, the following theorem will give a property of the Jacobian matrix $\nabla F(\mathbf{x})$, where $\mathbf{x}$ is nonzero vector.

**Theorem 4.4** *Let $\mathscr{A} \in D_{m,n}$ be positive semi-definite. Then the Jacobian matrix $\nabla F(\mathbf{x})$ is positive semi-definite with nonzero vectors $\mathbf{x} \in \mathbb{R}^n$.*

*Proof* As $\mathscr{A}$ is diagonalizable, for a vector $\mathbf{x}$, according to Definition 2.1, we have

$$\begin{aligned}\mathscr{A}\mathbf{x}^m &= (\mathscr{D} \times_1 B \times_2 B \cdots \times_m B)\mathbf{x}^m = \mathscr{D}(B^\top \mathbf{x})^m \\ &= \mathscr{D}\mathbf{y}^m \ (\mathbf{y} \stackrel{\triangle}{=} B^\top \mathbf{x}) = \sum_{i=1}^n d_i y_i^m,\end{aligned}$$

where $d_i$ is the $i$th diagonal entry of $\mathscr{D}$. According to the proposition assumption, $d_i \geq 0$, we have $\mathscr{A}\mathbf{x}^m \geq 0$ for all nonzero vectors $\mathbf{x}$.

Since the Jacobian matrix $\nabla F(\mathbf{x})$ is $(m-1)\mathscr{A}\mathbf{x}^{m-2}$, for any vector $\mathbf{z} \in \mathbb{R}^n$, $\langle \mathbf{z}, \nabla F(\mathbf{x})\mathbf{z}\rangle$ can be expressed by

$$\langle \mathbf{z}, \nabla F(\mathbf{x})\mathbf{z}\rangle = (m-1)\sum_{i=1}^n d_i y_i^{m-2}\tilde{z}_i^2 \geq 0,$$

where $\tilde{\mathbf{z}} = B^\top \mathbf{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)^\top$.

Hence, the Jacobian matrix $\nabla F(\mathbf{x})$ is positive semi-definite with $\mathbf{x} \in \mathbb{R}^n$. □

For $\mathscr{A} \in ST_{m,n}$, the existence theorems on solutions of Problem 3.1 are given in the following theorem.

**Theorem 4.5** *Suppose that $\mathscr{A} \in ST_{m,n}$. For Problem* 3.1*, the following results hold.*

(a) *If $\mathscr{A}$ is positive definite, then the* NCP$(\mathbf{q}, \mathscr{A})$ *has a nonempty, compact solution set.*
(b) *If $\mathscr{A}$ is strictly copositive with respect to $\mathbb{R}^n_+$, then the* NCP$(\mathbf{q}, \mathscr{A})$ *has a nonempty, compact solution set.*

*Proof* Since $\mathscr{A}$ is positive definite, according to Theorem 4.3, we have $F(\mathbf{x}) = \mathscr{A}\mathbf{x}^{m-1}a$ $+\mathbf{q}$ is strictly copositive. Let $c(\lambda) = \lambda^{\alpha}$ with $0 < \alpha \leq m-1$ and $\lambda \geq 1$, we know that $c(\lambda) \to \infty$ as $\lambda \to \infty$ and $\langle F(\lambda\mathbf{x}) - F(\mathbf{0}), \mathbf{x}\rangle \geq c(\lambda)\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x}\rangle$. Based on Lemma 3.2, we know that if $\mathscr{A}$ is positive definite, then the NCP$(\mathbf{q}, \mathscr{A})$ has a nonempty, compact solution set.

The rest is to prove part $(b)$. By Theorem 4.3, we obtain that $F(\mathbf{x}) = \mathscr{A}\mathbf{x}^{m-1} + \mathbf{q}$ is strictly copositive. In [4], we have that if $F(\mathbf{x})$ is strictly copositive with respect to $\mathbb{R}^n_+$, then the mapping $G$ in (1) is $\mathbf{d}$-regular for any $\mathbf{d} > \mathbf{0}$ and $G(\lambda\mathbf{x}) = \lambda^{m-1}G(\mathbf{x})$ with $\lambda > 0$. Hence, according to Lemma 3.3, if $\mathscr{A}$ is strictly copositive with respect to $\mathbb{R}^n_+$, then the NCP$(\mathbf{q}, \mathscr{A})$ has a nonempty, compact solution set. □

## 4.3 Solving Problem 3.2

In the above subsection, we have considered the solvability of Problem 3.1. Analogously, The following theorems have been described by the solvability of Problem 3.2.

**Theorem 4.6** *Suppose that $\mathscr{A}_k \in ST_{m-(2k-2),n}$, with $k \in [m/2]$. For Question 3.2, the following results hold.*

(a) *If $\mathscr{A}_k$ $(k \in [m/2-1])$ are diagonalizable and positive semi-definite and $\mathscr{A}_{m/2}$ is positive definite, then the* NCP$(\mathbf{q}, \{\mathscr{A}_k\})$ *has a unique solution;*
(b) *if $\mathscr{A}_k$ are positive semi-definite and there exists at least $k_0 \in [m/2]$ such that $\mathscr{A}_{k_0}$ is positive definite, then the* NCP$(\mathbf{q}, \{\mathscr{A}_k\})$ *has a nonempty, compact solution set;*
(c) *if $\mathscr{A}_k$ are strictly copositive with respect to $\mathbb{R}^n_+$, then the* NCP$(\mathbf{q}, \{\mathscr{A}_k\})$ *has a nonempty, compact solution set;*

*where $\mathscr{A}_{m/2}$ is a square matrix.*

*Proof* For part $(a)$, according to the assumption, we can derive that $\mathscr{A}_k\mathbf{x}^{m-2k}$ $(k \in [m/2-1])$ are symmetric and positive semi-definite, with $\mathbf{x} \in \mathbb{R}^n$. When $\mathscr{A}_{m/2}$ is symmetric and positive definite, then, we obtain that $\nabla F(\mathbf{x})$ is symmetric and positive definite, where $F(\mathbf{x})$ is defined in Question 3.2. Then, according to Lemma 3.1, NCP$(\mathbf{q}, \{\mathscr{A}_k\})$ has a unique solution.

We will prove part $(b)$ as follows. According to these assumptions, if $k_0 = m/2$, then,

$$\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x}\rangle = \sum_{k=1}^{m/2} \mathscr{A}_k\mathbf{x}^{m-(2k)} \geq \lambda\|\mathbf{x}\|_2^2 > 0,$$

where $\lambda$ is the smallest eigenvalue of $\mathscr{A}_{m/2}$ for all nonzero vectors $\mathbf{x} \in \mathbb{R}^n_+$; and if $k_0 \in [m/2-1]$, then,

$$\langle F(\mathbf{x}) - F(\mathbf{0}), \mathbf{x}\rangle = \sum_{k=1}^{m/2} \mathscr{A}_k\mathbf{x}^{m-(2k)} \geq \lambda\|\mathbf{x}\|_2^2 > 0,$$

where $\lambda$ is the smallest Z-eigenvalue of $\mathscr{A}_{k_0}$ for all nonzero vectors $\mathbf{x} \in \mathbb{R}^n_+$ (Meanwhile, we can also consider the case when $\lambda$ is the smallest H-eigenvalue of $\mathscr{A}_{k_0}$).

Then, $F(\mathbf{x})$ is strictly copositive. Let $c(\lambda)=\lambda$ with $\alpha=1$ and $\lambda\geq 1$, we know that $c(\lambda)\to\infty$ as $\lambda\to\infty$ and $\langle F(\lambda\mathbf{x})-F(\mathbf{0}),\mathbf{x}\rangle\geq c(\lambda)\langle F(\mathbf{x})-F(\mathbf{0}),\mathbf{x}\rangle$. Based on Lemma 3.2, we derive that if $\mathscr{A}_k$ are positive semi-definite and there exists at least $k_0\in[m/2]$ such that $\mathscr{A}_{k_0}$ is positive definite, then the NCP$(\mathbf{q},\{\mathscr{A}_k\})$ has a nonempty, compact solution set.

The rest is to prove part $(c)$. By Theorem 4.3, we obtain that

$$F(\mathbf{x})=\sum_{k=1}^{m/2}\mathscr{A}_k\mathbf{x}^{m-(2k-1)}+\mathbf{q}$$

is strictly copositive. Let $c(\lambda)=\lambda$ with $\alpha=1$ and $\lambda\geq 1$, we know that $c(\lambda)\to\infty$ as $\lambda\to\infty$ and $\langle F(\lambda\mathbf{x})-F(\mathbf{0}),\mathbf{x}\rangle\geq c(\lambda)\langle F(\mathbf{x})-F(\mathbf{0}),\mathbf{x}\rangle$. Then, according to Lemma 3.2, if $\mathscr{A}$ is strictly copositive with respect to $\mathbb{R}^n_+$, then the NCP$(\mathbf{q},\{\mathscr{A}_k\})$ has a nonempty, compact solution set. □

These constraints of tensors $\mathscr{A}_k$ given in part $(a)$ of Theorem 4.6 can be weakened. Hence, a more general result is given as follows.

**Theorem 4.7** *Suppose that $\mathscr{A}_k\in ST_{m-(2k-2),n}$, with $k\in[m/2-1]$ and $\mathscr{A}_{m/2}$ is a square matrix. For Problem 3.2, the following result holds.*

*If $\mathscr{A}_k$ are diagonalizable and positive semi-definite and there exists one $\delta\in(0,1)$, such that all principal minors of $\mathscr{A}_{m/2}$ are bounded between $\delta$ and $\delta^{-1}$, then the* NCP$(\mathbf{q},\{\mathscr{A}_k\})$ *has a unique solution.*

*Proof* Since there exists one $\delta\in(0,1)$, such that all principal minors of $\mathscr{A}_{m/2}$ are bounded between $\delta$ and $\delta^{-1}$, then, the real part of every eigenvalue of $\mathscr{A}_{m/2}$ is positive. Hence, for all nonzero vector $\mathbf{x}$, we can derive $\langle\mathbf{x},\mathscr{A}_{m/2}\mathbf{x}\rangle>0$. Meanwhile, according to the assumption, we can obtain that the Jacobian matrix $\Delta F(\mathbf{x})$ of $F(\mathbf{x})$ given in Problem 3.2 is positive definite. Hence, the NCP$(\mathbf{q},\{\mathscr{A}_k\})$ has a unique solution. □

*Remark 4.3* In the above two theorems, the assumptions of $\mathscr{A}_k\in ST_{m-(2k-2),n}$, with $k\in[m/2]$ can be appropriately reduced. However, we do not here consider these situations.

## 5 Perspectives

In this paper, by structured tensors, the main task is to consider the existence and uniqueness about solutions of Problems 3.1 and 3.2. However, we do not completely solve Problems 3.1 and 3.2. Now we present two conjectures about Question 3.1.

*Conjecture 5.1* If $\mathscr{A}\in D_{m,n}$ is positive definite, then $NCP(\mathbf{q},\mathscr{A})$ has a unique solution.

*Conjecture 5.2* If FEA$(\mathbf{q},\mathscr{A})\neq\emptyset$, then the nonlinear programming (4) has an optimal solution, $\mathbf{x}_*$. Moreover, there exists a vector $\mathbf{u}_*$ of multipliers satisfying the conditions,

$$\begin{aligned} \mathbf{q}+m\mathscr{A}\mathbf{x}_*^{m-1}-(m-1)\mathscr{A}\mathbf{x}_*^{m-2}\mathbf{u}_* &\geq \mathbf{0} \\ \langle \mathbf{x}_*, \mathbf{q}+m\mathscr{A}\mathbf{x}_*^{m-1}-(m-1)\mathscr{A}\mathbf{x}_*^{m-2}\mathbf{u}_*\rangle &= 0 \\ \mathbf{u}_* &\geq \mathbf{0} \\ \langle \mathbf{u}_*, \mathbf{q}+\mathscr{A}\mathbf{x}_*^{m-1}\rangle &= 0. \end{aligned}$$

Finally, the vectors $\mathbf{x}_*$ and $\mathbf{u}_*$ satisfy

$$(m-1)(\mathbf{x}_*-\mathbf{u}_*)_i(\mathscr{A}\mathbf{x}_*^{m-2}(\mathbf{x}_*-\mathbf{u}_*))_i \leq 0, \quad i \in [n].$$

When $m=2$, this is the theorem about the existence result for a solution of the quadratic programming associated to the linear complementarity problem given in [19]. Unfortunately, Cottle [1] presented some counter examples to explain that this conjecture is not true for the general nonlinear programming.

Finally, for the existence and uniqueness about solutions of Problem 3.1, we have an open question given as follows.

*Question 5.1* Suppose $\mathscr{A} \in T_{m,n}$ and nonzero $\mathbf{x} \in \mathbb{R}^n$. What conditions of $\mathscr{A}$ will make sure that there exists one $\delta \in (0,1)$, such that all principal minors of matrix $\mathscr{A}\mathbf{x}^{m-2}$ are bounded between $\delta$ and $\delta^{-1}$, for all $\mathbf{x} \in \mathbb{R}^n_+$?

When $m$ is odd, another open question will be listed below.

*Question 5.2* Given $\mathscr{A} \in T_{m,n}$ and $\mathbf{q} \in \mathbb{R}^n$. The NCP$(\mathbf{q}, \mathscr{A})$ is to find a vector $\mathbf{x} \in \mathbb{R}^n_+$ such that

$$F(\mathbf{x}) = \mathscr{A}\mathbf{x}^{m-1}+\mathbf{q} \in \mathbb{R}^n_+, \quad \mathscr{A}\mathbf{x}^m + \langle \mathbf{q}, \mathbf{x}\rangle = 0.$$

## 6 Conclusions

In this paper, we consider a special case of the nonlinear complementarity problem based on structured tensors.

For Problem 3.1, we prove that the Jacobian matrix of the multilinear map is positive semi-definite under mild conditions. Based on Lemmas 3.2, 3.3 and 3.4 and Definitions 2.2 and 2.4, we derive some results of KKT condition and the existence about solutions of Problem 3.1. Meanwhile, we first give two conditions to ensure Problem 3.2 has a unique solution and then we derive the existence theorem on solutions of Problem 3.2.

Finally, we present some open problems about this topic that we will investigate in the near future.

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