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A NECESSARY AND SUFFICIENT CONDITION FOR EXISTENCE OF A POSITIVE PERRON VECTOR*

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Abstract. In 1907, Perron showed that a positive square matrix has a unique largest positive eigenvalue with a positive eigenvector. This result was extended to irreducible nonnegative matrices by Frobenius in 1912, and to irreducible nonnegative tensors and weakly irreducible nonnegative tensors recently. This result is a fundamental result in matrix theory and has found wide applications in probability theory, internet search engines, spectral graph and hypergraph theory, etc. In this paper, we give a necessary and sufficient condition for the existence of such a positive eigenvector, i.e., a positive Perron vector, for a nonnegative tensor. We show that every nonnegative tensor has a canonical nonnegative partition form, from which we introduce strongly nonnegative tensors. A tensor is called strongly nonnegative if the spectral radius of each genuine weakly irreducible block is equal to the spectral radius of the tensor, which is strictly larger than the spectral radius of any other block. We prove that a nonnegative tensor has a positive Perron vector if and only if it is strongly nonnegative. The proof is nontrivial. Numerical results for finding a positive Perron vector are reported.

Key words. nonnegative tensor, tensor eigenvalue, Perron–Frobenius theorem, spectral radius, positive eigenvector

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1. Introduction.

More than one century ago, in 1907, Perron showed that a positive square matrix has a unique largest positive eigenvalue with a positive eigenvector (Perron vector) [19]. This result was extended to irreducible nonnegative matrices by Frobenius [10] in 1912, and to irreducible nonnegative tensors by Chang, Pearson, and Zhang [3] in 2008, and weakly irreducible nonnegative tensors by Friedland, Gaubert, and Han [9] in 2013. This result is a fundamental result in matrix theory [2, 12] and has found wide applications in probability theory [2, 12], the Google PageRank [17], spectral graph and hypergraph theory [5, 21], etc.

Perhaps the most important part of the Perron–Frobenius theorem, as well as a key feature ensuring Google PageRank's success, is the assertion that *the nonnegative Perron vector is unique for an irreducible nonnegative matrix and it is the positive Perron vector.* Thus, irreducibility is a sufficient condition to guarantee the existence of a positive Perron vector. In this paper, we will study *the necessary and sufficient condition for the existence of a positive Perron vector.* We will study this problem in a general setting for all nonnegative tensors of orders higher than or equal to two, which includes the nonnegative matrix case since matrices are second order tensors.

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In the matrix case, this result is known [22, pp. 10–11]. However, in the case of orders higher than two, the proof is nontrivial.

The following theorem will be proved.

THEOREM 1. *A nonnegative tensor has a positive Perron vector if and only if it is strongly nonnegative.*

Strongly nonnegative tensors will be defined in Definition 15, whereas the matrix counterpart will be presented in section 1.1 for illustration. Actually, Theorem 1 can be used as the definition for strongly nonnegative tensors.

1.1. Strongly nonnegative matrix. Given an $n \times n$ nonnegative matrix A, we can always partition A (up to some permutations) into the following upper triangular block form (known as the Frobenius normal form)

$$A = \begin{bmatrix} A_1 & A_{12} & \dots & \dots & \dots & \dots & \dots & A_{1r} \\ & A_2 & A_{23} & \dots & \dots & \dots & \dots & A_{2r} \\ & & \ddots & \ddots & & & & \vdots \\ & & & A_s & A_{ss+1} & \dots & \dots & A_{sr} \\ & & & & A_{s+1} & 0 & \dots & 0 \\ & & & & & \ddots & & \vdots \\ & & & & & & \ddots & \\ & & & & & & & A_r \end{bmatrix} \tag{1}$$

such that

1. each diagonal block matrix A_i is irreducible for $i \in [r] := \{1, \dots, r\}$. Here we regard a scalar (zero or not) as a one dimensional irreducible matrix for convenience;
2. for each $i \in [s]$, at least one of the matrices A_{ij} is not zero for $j = i+1, \dots, r$.

Then the matrix A is *strongly nonnegative* if

$$\rho(A_i) < \rho(A) \text{ for all } i \in [s] \text{ and } \rho(A_i) = \rho(A) \text{ for all } i = s+1, \dots, r.$$

It can be shown that a nonnegative matrix A is strongly nonnegative if and only if

$$A = \rho(A) DSD^{-1},$$

where S is a stochastic matrix, and D is a positive definite diagonal matrix. This has a tensorial analogue, which will be given in section 5.

1.2. Outline. In section 2, we will present some basic definitions and results on nonnegative tensors. In section 3 we will first review the nonnegative tensor partition result from [13], and then refine the partition by introducing *genuine weakly irreducible principal subtensors*. In section 4, we will give the necessary and sufficient condition for a nonnegative tensor possessing a positive spectral radius. This class of nonnegative tensors is called *nontrivially nonnegative*. In section 5, we will first introduce *strongly nonnegative tensors*, and then prove that a tensor being strongly nonnegative is both necessary and sufficient for it to have a positive Perron vector. The proofs in both sections 4 and 5 are based on the nonnegative tensor partition developed in section 3. In section 6, we will propose an algorithm to determine whether a nonnegative tensor is strongly nonnegative or not, and find a positive Perron vector when it is.

2. Preliminaries. Eigenvalues of tensors, independently proposed by Qi [20] and Lim [15] in 2005, have become active research topics in numerical multilinear algebra and beyond. We refer to [3, 4, 5, 7, 8, 11, 14, 16, 21, 23, 24] and references therein for some recent developments and applications. Among others, nonnegative tensors have wide investigations; see [3, 9, 13, 23, 24] for the Perron–Frobenius-type theorems, and [5, 6, 8, 11, 21, 26] for some applications; and the survey [4] for more connections to other problems in hypergraphs, quantum entanglement, higher order Markov chains, etc.

An mth order n-dimensional tensor $\mathcal{A}$ is a multiway array $\mathcal{A} = (a_{i_1\dots i_m})$ indexed by m indices i_j for $j \in [m]$ with each i_j being within the set $[n]$. Usually, the entries $a_{i_1\dots i_m}$ can be elements in any prefixed set S, not necessarily scalars; and the set of tensors of order m and dimension n with entries in S is denoted by $T_{m,n}(S)$. The space of tensors of order m and dimension n with entries in the field $\mathbb{C}$ of complex numbers is denoted simply by $T_{m,n}$. In this article, we will focus on the case when S is the nonnegative orthant $\mathbb{R}_+$. The interior of $\mathbb{R}_+$ is denoted by $\mathbb{R}_{++}$. A tensor $\mathcal{A} = (a_{i_1\dots i_m})$ with $a_{i_1\dots i_m} \in \mathbb{R}_+$ for all $i_j \in [n]$ and $j \in [m]$ is called a *nonnegative tensor*. Let $N_{m,n} \subset T_{m,n}$ be the set of all nonnegative tensors with order m and dimension n. Therefore, $N_{2,n}$ represents the set of all nonnegative $n \times n$ matrices.

For any $j \in [n]$, let

$$I(j) := \{(i_2, \dots, i_m) \in [n]^{m-1} : j \in \{i_2, \dots, i_m\}\}. \tag{2}$$

Obviously, $I(j)$ depends on m. For example, $I(j) = \{j\}$ when $m = 2$. We omit this dependence notationally for simplicity, since m is always clear from the content. For each $\mathcal{A} \in N_{m,n}$, we associate it with a nonnegative matrix $M_{\mathcal{A}} = (m_{ij}) \in N_{2,n}$ with

$$m_{ij} = \sum_{(i_2,\dots,i_m)\in I(j)} a_{ii_2\dots i_m}.$$

The matrix $M_{\mathcal{A}}$ is called the *majorization matrix* of $\mathcal{A}$ (cf. [13, 18]).

Definition 2 (weakly irreducible nonnegative tensor [9, 13]). *A nonnegative tensor $\mathcal{A} \in N_{m,n}$ is called weakly irreducible if the majorization $M_{\mathcal{A}}$ is irreducible. $\mathcal{A}$ is weakly reducible if it is not weakly irreducible.*

For convenience, tensors in $N_{m,1} = \mathbb{R}_+$ are always regarded as weakly irreducible. Note that the weak irreducibility for tensors in $N_{2,n}$ (i.e., matrices) reduces to the classical irreducibility for nonnegative matrices (cf. [2, 12]).

For any nonnegative matrix $A \in N_{2,n}$, we associate it with a directed graph $G = (V, E)$ as $V = \{1, \dots, n\}$ and

$$(i, j) \in E \text{ if and only if } a_{ij} > 0.$$

It is known that the irreducibility of the matrix A is equivalent to the strong connectedness of the corresponding directed graph defined as above [12].

Definition 3 (eigenvalues and eigenvectors [15, 20]). *Let tensor $\mathcal{A} = (a_{i_1\dots i_m}) \in T_{m,n}$. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{A}$, if there exists a vector $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ which is called an eigenvector such that*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{3}$$

where $\mathbf{x}^{[m-1]} \in \mathbb{C}^n$ is an n-dimensional vector with its ith component being x_i^{m-1}, and $\mathcal{A}\mathbf{x}^{m-1} \in \mathbb{C}^n$ with

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i := \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_m} \text{ for all } i = 1, \dots, n.$$

The number of eigenvalues of a tensor is always finite (cf. [20]). The spectral radius $\rho(\mathcal{A})$ of a tensor $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

In general $\rho(\mathcal{A})$ is not an eigenvalue of $\mathcal{A}$; however, it is when $\mathcal{A}$ is nonnegative [24]. There is extensive research on Perron–Frobenius-type-theorems for nonnegative tensors; see [3, 4, 9, 13, 23, 24] and references therein.

In the following, we summarize the Perron-Frobenius theorem for nonnegative tensors which will be used in this article.

THEOREM 4 (Perron–Frobenius theorem [9, 12, 24]). *Suppose that $\mathcal{A} \in N_{m,n} \setminus \{0\}$. Then, the following results hold.*

(i) *$\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with an eigenvector in $\mathbb{R}^n_+$.*

(ii) *If $\mathcal{A}$ has a positive eigenvector $\mathbf{y} \in \mathbb{R}^n_{++}$ associated with an eigenvalue λ, then $\lambda = \rho(\mathcal{A})$ and*

$$\min_{\mathbf{x}\in\mathbb{R}^n_{++}} \max_{1\leq i\leq n} \frac{\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i}{x_i^{m-1}} = \rho(\mathcal{A}) = \max_{\mathbf{x}\in\mathbb{R}^n_{++}} \min_{1\leq i\leq n} \frac{\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i}{x_i^{m-1}}.$$

(iii) *If $\mathcal{A}$ is weakly irreducible, then $\mathcal{A}$ has a positive eigenpair $(\lambda = \rho(\mathcal{A}), \mathbf{y})$, and $\mathbf{y}$ is unique up to a multiplicative constant.*

It follows from Theorem 4 that for any eigenpair $(\lambda, \mathbf{x})$ of $\mathcal{A} \in N_{m,n}$ (i.e., $\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}^{[m-1]}$), whenever $\mathbf{x} \in \mathbb{R}^n_{++}$ we have $\lambda = \rho(\mathcal{A})$.

We will call a nonnegative eigenvector of $\mathcal{A} \in N_{m,n}$ corresponding to $\rho(\mathcal{A})$ as a *nonnegative Perron vector*, and a positive eigenvector of $\mathcal{A} \in N_{m,n}$ as a *positive Perron vector*. Thus, each nonnegative tensor has a nonnegative Perron vector, and a weakly irreducible nonnegative tensor has a unique positive Perron vector.

The next theorem will help us to prove our main theorem.

THEOREM 5. *Let integers $m \geq 2$ and $n \geq 2$. Let $g_i \in \mathbb{R}_+[\mathbf{x}]$ be polynomials in $\mathbf{x}$ with nonnegative coefficients for all $i \in [n]$. If there are two positive vectors $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n_{++}$ such that*

$$\mathbf{y} \leq \mathbf{z},\ g_i(\mathbf{y}) \geq y_i^{m-1} \text{ and } g_i(\mathbf{z}) \leq z_i^{m-1} \text{ for all } i \in [n],$$

then there exists a vector $\mathbf{w} \in [\mathbf{y}, \mathbf{z}] := \{\mathbf{x} : y_i \leq x_i \leq z_i \text{ for all } i \in [n]\}$ such that

$$g_i(\mathbf{w}) = w_i^{m-1} \text{ for all } i \in [n].$$

Moreover, for any initial point $\mathbf{x}_0 \in [\mathbf{y}, \mathbf{z}]$, the iteration

$$(\mathbf{x}_{k+1})_i := [g_i(\mathbf{x}_k)]^{\frac{1}{m-1}} \text{ for all } i \in [n]$$

satisfies

1. *$\mathbf{x}_{k+1} \geq \mathbf{x}_k$, and*
2. *$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_*$ with $\mathbf{x}_* \in \mathbb{R}^n_{++}$ such that $g_i(\mathbf{x}_*) = (\mathbf{x}_*)_i^{m-1}$ for all $i \in [n]$.*

Proof. Define $f_i : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ as

$$f_i(\mathbf{x}) := [g_i(\mathbf{x})]^{\frac{1}{m-1}} \text{ for all } i \in [n].$$

It follows from $g_i(\mathbf{y}) \geq y_i^{m-1} > 0$ that the mapping $f := (f_1, \ldots, f_n)^\mathsf{T} : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is well-defined. Since g_i's are polynomials with nonnegative coefficients, the mapping

f is clearly increasing in the interval $[\mathbf{y}, \mathbf{z}]$, i.e., $f(\mathbf{x}_1) \geq f(\mathbf{x}_2)$ whenever $\mathbf{x}_1 - \mathbf{x}_2 \in \mathbb{R}^n_+$ and $\mathbf{x}_1, \mathbf{x}_2 \in [\mathbf{y}, \mathbf{z}]$, and compact on every subinterval of $[\mathbf{y}, \mathbf{z}]$, i.e., f is continuous and maps subintervals into compact sets. Note that $f(\mathbf{y}) \geq \mathbf{y}$ and $f(\mathbf{z}) \leq \mathbf{z}$. It follows from [1, Theorem 6.1] that there exists $\mathbf{w} \in [\mathbf{y}, \mathbf{z}]$ such that $f(\mathbf{w}) = \mathbf{w}$, which is exactly the first half of the desired result.

With the established results, the convergence of the iteration follows also from [1, Theorem 6.1]. ∎

3. Nonnegative tensor partition. Given a tensor $\mathcal{A} \in N_{m,n}$ and an index subset $I = \{j_1, \ldots, j_{|I|}\} \subseteq \{1, \ldots, n\}$, $\mathcal{A}_I \in N_{m,|I|}$ is the mth order $|I|$-dimensional *principal subtensor* of $\mathcal{A}$ defined as

$$(\mathcal{A}_I)_{i_1 \ldots i_m} = a_{j_{i_1} \ldots j_{i_m}} \text{ for all } i_j \in [|I|] \text{ and } j \in [m].$$

In particular, $\mathbf{x}_I \in \mathbb{R}^{|I|}$ is the *subvector* of $\mathbf{x}$ indexed by I.

Let $\mathfrak{S}(n)$ be the group of permutations on n elements, also called the symmetric group on the set $[n]$. We can define a group action on $N_{m,n}$ by $\mathfrak{S}(n)$ as

$$(\sigma \cdot \mathcal{A})_{i_1 \ldots i_m} = a_{\sigma(i_1) \ldots \sigma(i_m)} \text{ for } \sigma \in \mathfrak{S}(n) \text{ and } \mathcal{A} \in N_{m,n}.$$

Let $\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$. For any $\sigma \in \mathfrak{S}(n)$, define $\mathbf{y} \in \mathbb{C}^n$ with $y_i = x_{\sigma(i)}$. We have $(\sigma \cdot \mathcal{A})\mathbf{y}^{m-1} = \lambda \mathbf{y}^{[m-1]}$. Therefore, for any $\mathcal{A} \in N_{m,n}$, tensors in the orbit $\{\sigma \cdot \mathcal{A} : \sigma \in \mathfrak{S}(n)\}$ have the same set of eigenvalues. In particular,

$$\rho(\sigma \cdot \mathcal{A}) = \rho(\mathcal{A}) \text{ for all } \sigma \in \mathfrak{S}(n). \tag{4}$$

The main result in [13] can be stated as follows.

PROPOSITION 6 (nonnegative tensor partition). *For any $\mathcal{A} \in N_{m,n}$, there exists a partition of the index set $[n]$,*

$$I_1 \cup \cdots \cup I_r = [n],$$

such that for all $j = 1, \ldots, r$

$$\begin{aligned} &\mathcal{A}_{I_j} \text{ is weakly irreducible, } a_{si_2 \ldots i_m} = 0 \text{ for all } s \in I_j, \\ &\qquad \text{and } (i_2, \ldots, i_m) \in I(t) \cap \left(\cup_{k=1}^{j} I_k \right)^{m-1} \text{ for all } t \in I_1 \cup \cdots \cup I_{j-1}. \end{aligned} \tag{5}$$

Note that each $\mathcal{A}_{I_j}$ is a weakly irreducible principal subtensor of $\mathcal{A}$. We can assume that

$$I_j = [s_j] \setminus [s_{j-1}]$$

with $s_0 = 0$, $s_0 < s_1 < \cdots < s_r = n$, and $[0] := \emptyset$. In general, it should be there exists a $\sigma \in \mathfrak{S}(n)$ such that $\sigma \cdot \mathcal{A}$ has such a partition, while, in view of (4), we may assume throughout this paper, without loss of generality, that $\sigma = \mathrm{id}$, the multiplicative identity of the group $\mathfrak{S}(n)$.

The next result is proved in [13].

PROPOSITION 7. *Let $\mathcal{A} \in N_{m,n}$ be partitioned as* (5). *Then*

$$\rho(\mathcal{A}) = \max\{\rho(\mathcal{A}_{I_j}) : j \in [r]\}.$$

DEFINITION 8 (genuine weakly irreducible). *A weakly irreducible principal subtensor $\mathcal{A}_{I_j}$ of $\mathcal{A}$ is genuine if*

$$a_{si_2 \ldots i_m} = 0 \text{ for all } s \in I_j \text{ and } (i_2, \ldots, i_m) \in I(t) \text{ for all } t \in [n] \setminus I_j. \tag{6}$$

In the matrix case, genuine weakly irreducible subtensors correspond to basic classes of [22]. Note that for each tensor $\mathcal{A} \in N_{m,n}$, it always has one genuine weakly irreducible principal subtensor, namely, the principal subtensor $\mathcal{A}_{I_r}$ in (5).

With Definition 8, we can further rearrange $I_1, \dots, I_r$ to get a partition as follows.

PROPOSITION 9 (canonical nonnegative tensor partition). *Let $\mathcal{A} \in N_{m,n}$. The index set $[n]$ can be partitioned into $R \cup I_{s+1} \cup \cdots \cup I_r$ with $R = I_1 \cup \cdots \cup I_s$ such that, in addition to* (5)*,*

1. *$\mathcal{A}_{I_j}$ is a genuine weakly irreducible principal subtensor for all $j \in \{s+1, \dots, r\}$, and*
2. *for each $t \in [s]$ there exist $p_t \in I_t$ and $q_t \in I_{t+1} \cup \cdots \cup I_r$ such that*

$$a_{p_t i_2 \dots i_m} > 0 \textit{ for some } (i_2, \dots, i_m) \in I(q_t).$$

Moreover, the partition for the genuine principal subtensor blocks $\mathcal{A}_{I_{s+1}}, \dots, \mathcal{A}_{I_r}$ is unique up to permutation on the index sets $\{I_{s+1}, \dots, I_r\}$.

Proof. Suppose that we have the tensor $\mathcal{A}$ with a partition as in Proposition 6. It then follows from Definition 8 that a weakly irreducible principal subtensor $\mathcal{A}_{I_j}$ is genuine if and only if

$$(M_{\mathcal{A}})_{I_j} = M_{\mathcal{A}_{I_j}}, \text{ i.e., } a_{i_1 i_2 \dots i_m} = 0 \text{ whenever } i_1 \in I_j \text{ and } \{i_2 \dots, i_m\} \cap I_j^{\complement} \neq \emptyset.$$

Therefore, the genuine weakly irreducible principal subtensors are uniquely determined, and we can group the genuine weakly irreducible principal subtensors together, say $I_{s+1}, \dots, I_r$, without loss of generality. This sorting can be done without destroying the relative back and forth orders of the blocks which are not genuine.

For any $j \in [s]$, since $\mathcal{A}_{I_j}$ is not a genuine weakly irreducible principal subtensor, there exists a $p_j \in I_j$ and $q_j \notin I_j$ such that

$$a_{p_j i'_2 \dots i'_m} > 0 \text{ for some } (i'_2, \dots, i'_m) \in I(q_j).$$

However, from (5) for all $t \in I_1 \cup \cdots \cup I_{j-1}$,

$$a_{s i_2 \dots i_m} = 0 \text{ for all } s \in I_j, \text{ and } (i_2, \dots, i_m) \in I(t) \cap \left(\cup_{k=1}^{j} I_k \right)^{m-1},$$

we must have that

$$\{i'_2, \dots, i'_m\} \cap (I_{j+1} \cup \cdots \cup I_r) \neq \emptyset,$$

since otherwise $q_j \in I_1 \cup \cdots \cup I_{j-1}$ and $(i'_2, \dots, i'_m) \in (\cup_{k=1}^{j} I_k)^{m-1}$. Thus, q_j can be chosen in $I_{j+1} \cup \cdots \cup I_r$. The result then follows. ∎

It is easy to see that $r \geq 1$ and $s \leq r - 1$, since there always is a genuine weakly irreducible principal subtensor. A tensor $\mathcal{A} \in N_{m,n}$ in the form described in Proposition 9 is in a *canonical nonnegative partition form.* It follows that each nonnegative tensor (up to a group action by $\mathfrak{S}(n)$) can be written in a canonical nonnegative partition form.

4. Nontrivially nonnegative tensors.

Let $\mathbf{e} \in \mathbb{R}^n$ be the vector of all ones. We will write $\mathbf{x} \geq \mathbf{0}$ ($> \mathbf{0}$, respectively) whenever $\mathbf{x} \in \mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$, respectively).

DEFINITION 10 (see [13]). *A tensor $\mathcal{A} \in N_{m,n}$ is strictly nonnegative, if $\mathcal{A}\mathbf{e}^{m-1} > \mathbf{0}$.*

It is easy to see that in Definition 10, $\mathcal{A} \in N_{m,n}$ being strictly nonnegative is equivalent to $\mathcal{A}\mathbf{x}^{m-1} > \mathbf{0}$ for any positive vector $\mathbf{x}$.

The next observation follows from Definition 10 and Theorem 4.

PROPOSITION 11. *If $\mathcal{A} \in N_{m,n}$ is weakly irreducible and $\rho(\mathcal{A}) > 0$, then $\mathcal{A}$ is strictly nonnegative.*

DEFINITION 12 (nontrivially nonnegative tensor). *A tensor $\mathcal{A} \in N_{m,n}$ is nontrivially nonnegative, if there exists a nonempty subset $I \subseteq \{1, \ldots, n\}$ such that $\mathcal{A}_I$ is strictly nonnegative.*

THEOREM 13 (positive spectral radius). *Given any $\mathcal{A} \in N_{m,n}$, $\rho(\mathcal{A}) > 0$ if and only if $\mathcal{A}$ is nontrivially nonnegative.*

Proof. Necessity: Suppose that $\mathbf{x} \geq \mathbf{0}$ is an eigenvector of $\mathcal{A}$ such that (cf. Theorem 4)

$$\mathcal{A}\mathbf{x}^{m-1} = \rho(\mathcal{A})\mathbf{x}^{[m-1]}$$

and $\rho(\mathcal{A}) > 0$. From section 3, $[n]$ can be partitioned into $\{I_1, \ldots, I_r\}$ such that each $\mathcal{A}_{I_j}$ is weakly irreducible. By Proposition 7, $\rho(\mathcal{A}_{I_j}) = \rho(\mathcal{A}) > 0$ for some $j \in [r]$. Therefore, the principal subtensor $\mathcal{A}_{I_j}$ is strictly nonnegative by Proposition 11. By Definition 12, $\mathcal{A}$ is nontrivially nonnegative.

Sufficiency: Suppose that $\mathcal{A}_I$ is strictly nonnegative for some index subset $I \subseteq \{1, \ldots, n\}$. It follows from Definition 10 that

$$\mathcal{A}_I \mathbf{e}_I^{m-1} > \mathbf{0},$$

which implies that

$$(\mathcal{A}\mathbf{y}^{m-1})_I \geq \mathcal{A}_I \mathbf{e}_I^{m-1} > \mathbf{0},$$

where $\mathbf{y}_I = \mathbf{e}_I$ and $y_i = 0$ for $i \notin I$. By [24, Theorem 5.5], we have

$$\rho(\mathcal{A}) = \max_{\mathbf{x} \geq 0} \min_{x_i > 0} \frac{(\mathcal{A}\mathbf{x}^{m-1})_i}{x_i^{m-1}} \geq \min_{i \in I} (\mathcal{A}\mathbf{y}^{m-1})_i \geq \min_{i \in I} (\mathcal{A}_I \mathbf{e}_I^{m-1})_i > 0.$$

Therefore, we have $\rho(\mathcal{A}) > 0$. ∎

5. Positive Perron vector.

PROPOSITION 14. *Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m}) \in N_{m,n}$ has a positive eigenvector. Then it is a positive Perron vector and $\mathcal{A}$ is either the zero tensor or a strictly nonnegative tensor.*

Proof. Suppose that $\mathcal{A}$ has a positive eigenvector $\mathbf{x}$, corresponding to an eigenvalue λ. Then $\lambda = \rho(\mathcal{A})$ by Theorem 4 as $\mathcal{A}$ is nonnegative. If $\lambda = 0$, then by the eigenvalue equations, $\mathcal{A}$ must be the zero tensor. Suppose that $\lambda > 0$. Then by the fact that $\mathcal{A}\mathbf{x}^{m-1} > \mathbf{0}$ is a positive vector, the result follows from Definition 10. ∎

DEFINITION 15 (strongly nonnegative tensor). *Let $\mathcal{A} \in N_{m,n}$ have a canonical nonnegative partition as in Proposition 9. $\mathcal{A}$ is called strongly nonnegative, if*

$$\begin{cases} \rho(A_{I_j}) = \rho(\mathcal{A}) & \textit{if } \mathcal{A}_{I_j} \textit{ is genuine}, \\ \rho(A_{I_j}) < \rho(\mathcal{A}) & \textit{otherwise}. \end{cases} \tag{7}$$

With Definition 15, we present our main theorem.

THEOREM 16 (positive perron vector). *Let $\mathcal{A} \in N_{m,n}$. Then $\mathcal{A}$ has a positive Perron vector if and only if $\mathcal{A}$ is strongly nonnegative.*

Theorem 16 will be proved in section 5.3, after the preparation sections 5.1 and 5.2.

By Theorems 4 and 16 and Proposition 14, we see that a weakly irreducible nonnegative tensor is a strongly nonnegative tensor, and a nonzero strongly nonnegative tensor is a strictly nonnegative tensor.

Before traveling to the proof for Theorem 16, we give a connection between strongly nonnegative tensors and stochastic tensors.

DEFINITION 17 (stochastic tensor). *A nonnegative tensor $\mathcal{A} \in N_{m,n}$ is a stochastic tensor if $\mathcal{A}\mathbf{e}^{m-1} = \mathbf{e}$.*

If $D = (d_{ij}) \in N_{2,n}$ is a positive definite diagonal matrix, then we define $D^{m-1} \cdot \mathcal{A} \cdot D^{1-m} \in N_{m,n}$ as

$$\left(D^{m-1} \cdot \mathcal{A} \cdot D^{1-m}\right)_{i_1 i_2 \dots i_m} := d_{i_1 i_1}^{m-1} a_{i_1 i_2 \dots i_m} d_{i_2 i_2}^{-1} \dots d_{i_m i_m}^{-1} \quad \text{for all } i_1, \dots, i_m \in \{1, \dots, n\}.$$

$\mathcal{A}$ and $D^{m-1} \cdot \mathcal{A} \cdot D^{1-m}$ can be regarded as *diagonally similar*, as it is the diagonal similarity of two matrices when $m = 2$.

PROPOSITION 18 (diagonal similarity). *Let $\mathcal{A} \in N_{m,n}$. Then $\mathcal{A}$ is strongly nonnegative if and only if $\mathcal{A}$ is diagonally similar to $\rho(\mathcal{A})\mathcal{S}$ for a stochastic tensor $\mathcal{S} \in N_{m,n}$, i.e.,*

$$\mathcal{A} = \rho(\mathcal{A}) D^{m-1} \cdot \mathcal{S} \cdot D^{1-m} \tag{8}$$

for a positive definite diagonal matrix D.

Proof. Suppose that $\mathcal{A} = \rho(\mathcal{A}) D^{m-1} \cdot \mathcal{S} \cdot D^{1-m}$ holds as (8). Obviously,

$$\left(\mathcal{A}(D\mathbf{e})^{m-1}\right)_i = \rho(\mathcal{A}) d_{ii}^{m-1} \text{ for all } i \in \{1, \dots, n\}.$$

This is the same as $\mathcal{A}\mathbf{x}^{m-1} = \rho(\mathcal{A})\mathbf{x}^{[m-1]}$ with $\mathbf{x} = D\mathbf{e} > \mathbf{0}$. Therefore, $\mathcal{A}$ is strongly nonnegative by Theorem 16.

For the other implication, suppose that $\mathcal{A}\mathbf{x}^{m-1} = \rho(\mathcal{A})\mathbf{x}^{[m-1]}$ with some positive Perron vector $\mathbf{x} > \mathbf{0}$. With diagonal matrix $D \in N_{2,n}$ given by $d_{ii} := x_i$ for $i \in \{1, \dots, n\}$, and $\mathcal{S} := \frac{1}{\rho(\mathcal{A})} \left(D^{-1}\right)^{m-1} \cdot \mathcal{A} \cdot \left(D^{-1}\right)^{1-m}$ when $\rho(\mathcal{A}) > 0$ and $\mathcal{S} = \mathcal{I}$ (the identity tensor in $N_{m,n}$) when $\rho(\mathcal{A}) = 0$, it is easy to check that $\mathcal{S}$ is a stochastic tensor. If $\rho(\mathcal{A}) > 0$, then (8) is obviously fulfilled. If $\rho(\mathcal{A}) = 0$, then $\mathcal{A}$ is the zero tensor, and (8) holds as well. ∎

5.1. Systems of eigenvalue equations. In the following, we will always assume that a given tensor $\mathcal{A} \in N_{m,n}$ is in a canonical nonnegative partition form (cf. Proposition 9). Recall that $[n] = R \cup I_{s+1} \cdots \cup I_r$ with $R = I_1 \cup \cdots \cup I_s$.

For any $j \in [r]$, if we let $K_j := [n] \setminus I_j$, then

$$(\mathcal{A}\mathbf{x}^{m-1})_{I_j} = \mathcal{A}_{I_j} \mathbf{x}_{I_j}^{m-1} + \sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{x}_{K_j}) \mathbf{x}_{I_j}^{u-1} \tag{9}$$

for some tensors $\mathcal{A}_{j,u}(\mathbf{x}_{K_j}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{K_j}])$ for all $u = 1, \dots, m-1$. Namely, $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ is a tensor of order u and dimension $|I_j|$ with the entries being polynomials in the variables $\mathbf{x}_{K_j}$ with coefficients in the set $\mathbb{R}_+$. Moreover, it follows from (9) that each entry of $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ is either zero or a homogeneous polynomial of degree $m - u$.

We note that there can be many choices of tensors $\mathcal{A}_{j,u}(\mathbf{x}_{K_j}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{K_j}])$ to fulfill the system (9), similar to the rationale that there are many tensors $\mathcal{T} \in T_{m,n}$ which can result in the same polynomial system $\mathcal{T}\mathbf{x}^{m-1}$. However, it is well-defined in the sense that the polynomial systems $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})\mathbf{x}_{I_j}^{u-1}$'s are all uniquely determined by $\mathcal{A}$. We note that when the system of polynomials $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})\mathbf{x}_{I_j}^{u-1} = \mathbf{0}$, the tensor $\mathcal{A}_{j,u}(\mathbf{x}_{K_j}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{K_j}])$ is uniquely determined as the zero tensor.

Lemma 19. *Suppose the notation is adopted as above. Then a weakly irreducible principal subtensor $\mathcal{A}_{I_j}$ of $\mathcal{A}$ is genuine if and only if*

$$\sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{e}_{K_j})\mathbf{e}_{I_j}^{u-1} = \mathbf{0}, \tag{10}$$

which is further equivalent to each tensor $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ is the zero tensor for all $u \in [m-1]$.

Proof. It follows from (6) that a weakly irreducible principal subtensor $\mathcal{A}_{I_j}$ is genuine if and only if the right polynomials of $\mathbf{x}$ in (9) involve variables $\{x_t : t \in I_j\}$ only. This is equivalent to $\mathcal{A}_{j,u}(\mathbf{x}_{K_j}) = 0$ for all $u = 1, \dots, m-1$ for any choice of $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ in (9).

From the facts

- the tensors $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ all take polynomials with nonnegative coefficients as entries, and
- $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})\mathbf{x}_{I_j}^{u-1}$'s are uniquely determined,

we have that the above zero polynomials condition is equivalent to that each entry of all the tensors $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ is zero. This is further equivalent to

$$\sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{e}_{K_j})\mathbf{e}_{I_j}^{u-1} = \mathbf{0}.$$

∎

For all $j \in [s-1]$, let $L_j = R \setminus I_j$. From (5), we can further partition $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ for all $j \in [s]$ into two parts,

$$\mathcal{A}_{j,u}(\mathbf{x}_{K_j}) = \mathcal{H}_{j,u}(\mathbf{x}_{L_j}) + \mathcal{B}_{j,u}(\mathbf{x}_{K_j}) \text{ for all } u \in [m-1], \tag{11}$$

where $\mathcal{H}_{j,u}(\mathbf{x}_{L_j}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{L_j}])$ with each entry being either zero or a homogeneous polynomial of degree $m-u$ in the variables $\mathbf{x}_{L_j}$, and

$$\mathcal{B}_{j,u}(\mathbf{x}_{K_j}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{K_j}][\mathbf{x}_{L_j}])$$

with each entry being either zero or a polynomial of degree in the variables $\mathbf{x}_{L_j}$ strictly smaller than $m-u$.

Proposition 20. *Suppose the notation is adopted as above. Then*

(a) *for $j \in [s]$ it follows $\sum_{u=1}^{m-1} \mathcal{H}_{j,u}(\mathbf{y}_{L_j})\mathbf{e}_{I_j}^{u-1} = \mathbf{0}$ with $\mathbf{y}_{I_t} = \mathbf{e}_{I_t}$ for $t \in [j-1]$ and $\mathbf{y}_{I_t} = \mathbf{0}$ for the other $t \in [s] \setminus \{j\}$;*

(b) *$\sum_{u=1}^{m-1} \mathcal{B}_{s,u}(\mathbf{e}_{K_s})\mathbf{e}_{I_s}^{u-1} \neq \mathbf{0}$; and*

(c) *for $j \in [s-1]$, either $\sum_{u=1}^{m-1} \mathcal{B}_{s,u}(\mathbf{e}_{K_s})\mathbf{e}_{I_s}^{u-1} \neq \mathbf{0}$, or $\sum_{u=1}^{m-1} \mathcal{H}_{j,u}(\mathbf{e}_{L_j})\mathbf{e}_{I_j}^{u-1} \neq \mathbf{0}$.*

Proof. Note that item (a) follows from (5), and from which $\mathcal{H}_{s,u}(\mathbf{y}_{L_s}) = \mathbf{0}$ for all $u \in [m-1]$. Items (b) and (c) then follow from Proposition 9 that for each $j \in [s]$ there exist $p_j \in I_j$ and $q_j \in I_{j+1} \cup \cdots \cup I_r$ such that

$$a_{p_j i_2 \dots i_m} > 0 \text{ for some } (i_2, \dots, i_m) \in I(q_j),$$

which, together with Lemma 19, implies that

$$\sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{e}_{K_j})\mathbf{e}_{I_j}^{u-1} \neq \mathbf{0}.$$

The results now follow. ∎

5.2. Solvability of polynomial systems. The notation in the following is independent of the previous one.

LEMMA 21. *Let $\mathcal{A} \in N_{m,n}$. For arbitrary $\epsilon > 0$, there exists a positive vector $\mathbf{x} \in \mathbb{R}^n_{++}$ such that*

$$\mathcal{A}\mathbf{x}^{m-1} \leq (\rho(\mathcal{A}) + \epsilon)\mathbf{x}^{[m-1]}.$$

Proof. Suppose that $\mathcal{A}$ is weakly reducible, since otherwise the conclusion follows from Theorem 4. Thus, we assume that $I_1 \cup \cdots \cup I_r = [n]$ is a partition of $\mathcal{A}$ as shown in Proposition 6 (cf. 5).

The proof is by induction on the block number r. The case when $r = 1$ follows from Theorem 4 as we showed. Suppose that the conclusion is true when $r = s - 1$ for some $s \geq 2$. In the following, we assume that $r = s$. Let $\kappa = \frac{1}{2}\epsilon$. Denote by $\mathcal{C} = \mathcal{A}_{I_1\cup\cdots\cup I_{r-1}} \in N_{m,n-|I_r|}$ the principal subtensor of $\mathcal{A}$. It is easy to see that $I_1 \cup \cdots \cup I_{r-1}$ is a partition of $\mathcal{C}$ as shown in Proposition 6. Therefore, by the inductive hypothesis, we can find a vector $\mathbf{y} \in \mathbb{R}^{|I_1|+\cdots+|I_{r-1}|}_{++}$ such that

$$\mathcal{C}\mathbf{y}^{m-1} \leq (\rho(\mathcal{C}) + \kappa)\mathbf{y}^{[m-1]} \leq (\rho(\mathcal{A}) + \kappa)\mathbf{y}^{[m-1]},$$

where we have the last inequality following from $\rho(\mathcal{C}) = \max\{\rho(\mathcal{A}_{I_j}) : j \in [r-1]\} \leq \max\{\rho(\mathcal{A}_{I_j}) : j \in [r]\} = \rho(\mathcal{A})$ by Proposition 7. We also have that there exists $\mathbf{z} \in \mathbb{R}^{|I_r|}_{++}$ such that

$$\mathcal{A}_{I_r}\mathbf{z}^{m-1} \leq (\rho(\mathcal{A}_{I_r}) + \kappa)\mathbf{z}^{[m-1]}.$$

It follows from Proposition 6 that there are some tensors $\mathcal{C}_u(\mathbf{z}) \in T_{u,n-|I_r|}(\mathbb{R}_+[\mathbf{z}])$ for $u \in [m-1]$ such that with $\mathbf{w} := (\beta\mathbf{y}^\mathsf{T}, \mathbf{z}^\mathsf{T})^\mathsf{T} \in \mathbb{R}^n$

$$(\mathcal{A}\mathbf{w}^{m-1})_{I_1\cup\cdots\cup I_{r-1}} = \beta^{m-1}\mathcal{C}\mathbf{y}^{m-1} + \sum_{u=1}^{m-1} \beta^{u-1}\mathcal{C}_u(\mathbf{z})\mathbf{y}^{u-1}$$
$$\text{and } (\mathcal{A}\mathbf{w}^{m-1})_{I_r} = \mathcal{A}_{I_r}\mathbf{z}^{m-1}.$$

Thus, when $\beta > 0$ is sufficiently large we have

$$\begin{aligned}(\mathcal{A}\mathbf{w}^{m-1})_{I_1\cup\cdots\cup I_{r-1}} &\leq \beta^{m-1}(\rho(\mathcal{A}) + 2\kappa)\mathbf{y}^{[m-1]}\\ &= (\rho(\mathcal{A}) + 2\kappa)\mathbf{w}^{[m-1]}_{I_1\cup\cdots\cup I_{r-1}}\\ &= (\rho(\mathcal{A}) + \epsilon)\mathbf{w}^{[m-1]}_{I_1\cup\cdots\cup I_{r-1}}\end{aligned}$$

as well as

$$
\begin{aligned}
(\mathcal{A}\mathbf{w}^{m-1})_{I_r} &= \mathcal{A}_{I_r}\mathbf{z}^{m-1} \le (\rho(\mathcal{A}_{I_r})+\kappa)\mathbf{z}^{[m-1]}\\
&\le (\rho(\mathcal{A}_{I_r})+\epsilon)\mathbf{w}_{I_r}^{[m-1]} \le (\rho(\mathcal{A})+\epsilon)\mathbf{w}_{I_r}^{[m-1]}.
\end{aligned}
$$

The result then follows. ∎

LEMMA 22. *Let $\lambda > 0$, integers $n, s > 0$, and partition $I_1 \cup \cdots \cup I_s = [n]$. Suppose that for all $j \in [s]$, $\mathcal{A}_{I_j} \in N_{m,|I_j|}$ is weakly irreducible with $\rho(\mathcal{A}_{I_j}) < \lambda$, and $\mathcal{A}_{j,u}(\mathbf{x}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{[n]\setminus I_j}])$ for $u = 1, \ldots, m-1$ are such that*

1. *the degree of each entry of $\mathcal{A}_{j,u}(\mathbf{x})$ is not greater than $m-u$;*
2. *if we let $\mathcal{B}_{j,u}(\mathbf{x}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{[n]\setminus I_j}])$ by deleting polynomials of degree $m-u$ in each entry of $\mathcal{A}_{j,u}(\mathbf{x})$, and $\mathcal{H}_{j,u}(\mathbf{x}) = \mathcal{A}_{j,u}(\mathbf{x}) - \mathcal{B}_{j,u}(\mathbf{x})$ for all $u \in [m-1]$ and $j \in [s]$, then $\mathcal{H}_{j,u}(\mathbf{w}^{(j)})\mathbf{e}_{I_j}^{u-1} = \mathbf{0}$ with $\mathbf{w}^{(j)}_{I_1\cup\cdots\cup I_{j-1}} = \mathbf{e}_{I_1\cup\cdots\cup I_{j-1}}$ and $\mathbf{w}^{(j)}_{I_j\cup\cdots\cup I_s} = \mathbf{0}$ for all $j \in [s]$; and*
3. *it holds*

$$
\sum_{u=1}^{m-1} \mathcal{B}_{s,u}(\mathbf{e})\mathbf{e}_{I_s}^{u-1} \neq \mathbf{0}, \tag{12}
$$

and, with $\mathbf{y} = \mathbf{e}_{I_1\cup\cdots\cup I_j} + t\mathbf{e}_{I_{j+1}\cup\cdots\cup I_s}$,

$$
\sum_{u=1}^{m-1} \mathcal{B}_{j,u}(\mathbf{e})\mathbf{e}_{I_j}^{u-1} \neq \mathbf{0} \ \text{or} \ \lim_{t\to\infty}\left\| \sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{y})\mathbf{e}_{I_j}^{u-1} \right\| \to \infty \tag{13}
$$

for all $j \in [s-1]$.

Then we have that there is a positive solution $\mathbf{x} \in \mathbb{R}^n_{++}$ for the following system,

$$
\mathcal{A}_{I_j}\mathbf{x}_{I_j}^{m-1} + \sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{x})\mathbf{x}_{I_j}^{u-1} = \lambda \mathbf{x}_{I_j}^{[m-1]} \ \text{for all } j \in [s]. \tag{14}
$$

Proof. We divide the proof into three parts.

Part I. Let $f := (f_{I_1}, \ldots, f_{I_s}) : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ with

$$
f_{I_j}(\mathbf{x}) := \left[\frac{1}{\lambda}\left(\mathcal{A}_{I_j}\mathbf{x}_{I_j}^{m-1} + \sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{x})\mathbf{x}_{I_j}^{u-1}\right)\right]^{\frac{1}{[m-1]}}.
$$

Since $\mathcal{A}_{I_j}$ is weakly irreducible and $\mathcal{A}_{j,u}(\mathbf{x})$'s are tensors with entries being nonnegative polynomials, $f_{I_j} : \mathbb{R}^n_{++} \to \mathbb{R}^{|I_j|}_{++}$ is well-defined when either $|I_j| > 1$ or $\mathcal{A}_{I_j} > 0$ when $|I_j| = 1$. The case when $|I_j| = 1$ and $\mathcal{A}_{I_j} = 0$ is also well-defined, since (13) implies the existence of a positive entry. Therefore, the map $f : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is well-defined.

Part II. Let $\mathcal{A} \in N_{m,n}$ be the tensor with the principal subtensors $\mathcal{A}_{I_j}$ for $j \in [s]$ and such that it satisfies the polynomial systems

$$
(\mathcal{A}\mathbf{x}^{m-1})_{I_j} = \mathcal{A}_{I_j}\mathbf{x}_{I_j}^{m-1} + \sum_{u=1}^{m-1} \mathcal{H}_{j,u}(\mathbf{x})\mathbf{x}_{I_j}^{u-1} \ \text{for all } j \in [s].
$$

It follows from the second listed hypothesis and Proposition 6 that $I_1 \cup \cdots \cup I_s = [n]$ forms a partition for the tensor $\mathcal{A}$. By Proposition 7, $\rho(\mathcal{A}) = \max\{\rho(\mathcal{A}_{I_j}) : j \in [s]\} < \lambda$.

Since $\lambda > \rho(\mathcal{A})$, it follows from Lemma 21 that there exists a vector $\mathbf{y} > \mathbf{0}$ such that

$$\mathcal{A}\mathbf{y}^{m-1} < \lambda \mathbf{y}^{[m-1]}. \tag{15}$$

So, with $\beta > 0$, we have $\beta\mathbf{y} > \mathbf{0}$ and

$$\begin{aligned} f_{I_j}(\beta\mathbf{y}) &= \beta \left[\frac{1}{\lambda} \left(\mathcal{A}_{I_j}\mathbf{y}_{I_j}^{m-1} + \sum_{u=1}^{m-1} \beta^{u-m}\mathcal{A}_{j,u}(\beta\mathbf{y})\mathbf{y}_{I_j}^{u-1} \right) \right]^{\frac{1}{[m-1]}} \\ &= \beta \left[\frac{1}{\lambda} \left(\mathcal{A}_{I_j}\mathbf{y}_{I_j}^{m-1} + \sum_{u=1}^{m-1} \beta^{u-m}\mathcal{H}_{j,u}(\beta\mathbf{y})\mathbf{y}_{I_j}^{u-1} + \sum_{u=1}^{m-1} \beta^{u-m}\mathcal{B}_{j,u}(\beta\mathbf{y})\mathbf{y}_{I_j}^{u-1} \right) \right]^{\frac{1}{[m-1]}} \\ &= \beta \left[\frac{1}{\lambda} \left((\mathcal{A}\mathbf{y}^{m-1})_{I_j} + \sum_{u=1}^{m-1} \beta^{u-m}\mathcal{B}_{j,u}(\beta\mathbf{y})\mathbf{y}_{I_j}^{u-1} \right) \right]^{\frac{1}{[m-1]}} \\ &\le \beta\mathbf{y}_{I_j} \qquad (16) \end{aligned}$$

for sufficiently large $\beta > 0$. Here, the inequality follows from (15) and the fact that the maximal possible degree for the polynomials in the entries of each tensor $\mathcal{B}_{j,u}(\mathbf{y})$ is $m-u-1$ for all $u \in [m-1]$. Since there are finite j's, $f(\beta\mathbf{y}) \le \beta\mathbf{y}$ for some sufficiently large β.

Part III. Recall that $\mathcal{B}_{j,u}(\mathbf{x}) \in T_{u,|I_j|}(\mathbb{R}_+[\mathbf{x}_{[n]\setminus I_j}])$ is obtained by deleting polynomials of degree $m-u$ in each entry of $\mathcal{A}_{j,u}(\mathbf{x})$ for all $u \in [m-1]$ and $j \in [s]$. Let

$$P_j := \operatorname{supp}\left(\sum_{u=1}^{m-1} \mathcal{B}_{j,u}(\mathbf{e})\mathbf{e}_{I_j}^{i-1} \right) \subseteq I_j$$

for all $j \in [s]$, and for $j \in [s-1]$

$$Q_j := \left\{ z \in I_j : \lim_{t\to\infty} \sum_{w \in I_{j+1}\cup\cdots\cup I_s} \left(\sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{x}_w)\mathbf{e}_{I_j}^{u-1} \right)_z \to \infty, \right. \\ \left. \text{with } (\mathbf{x}_w)_w = t \text{ and } (\mathbf{x}_w)_v = 1 \text{ for the others} \right\}. \tag{17}$$

It follows from (12) that $P_s \neq \emptyset$, and (13) that $P_j \cup Q_j \neq \emptyset$ for $j \in [s-1]$. Let $Q_s = \emptyset$. Let $W_j := Q_j \setminus P_j$ for $j \in [s]$.

For each $j \in [s]$, let the majorization matrix for $\mathcal{A}_{I_j}$ be $M_j \in \mathbb{R}_+^{|I_j|\times|I_j|}$. It follows from the weak irreducibility that the directed graph $G_j = (V_j = I_j, E_j)$ associated with M_j is strongly connected for every $j \in [s]$. Therefore, for any nonempty proper subset $K_j \subset I_j$ and $t \in I_j \setminus K_j$, there should be a directed path from t to some $w \in K_j$, and the intermediate vertices in this path all come from the set $I_j \setminus K_j$. We will generate a forest (a union of trees) $T = (I_1 \cup \cdots \cup I_s, F)$ through the following procedure: Algorithm 1.

We take a short break to show that the procedure is well-defined.

- Step 1 is well-defined, since $W_j \subseteq Q_j$ and Q_j is defined as (17).
- Note that $P_j \cup W_j = P_j \cup Q_j \neq \emptyset$ for all $j \in [s]$.
- Step 3 is well-defined from the words before the procedure as well as Step 5.

Algorithm 1. Forest Generating Algorithm.

The inputs are the directed graphs G_j and the sets P_j and W_j for $j \in [s]$.

Step 0: Set $F = \emptyset$, $j = s$.

Step 1: If $j = 0$, stop; otherwise set $J_j = P_j \cup W_j$. For each $v \in W_j$, pick a $w \in I_{j+1} \cup \cdots \cup I_s$ such that $\lim_{t\to\infty}(\sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{x}_w)\mathbf{e}_{I_j}^{u-1})_v \to \infty$, and add (v, w) into F. Go to Step 2.

Step 2: Let $K_j = I_j \setminus J_j$, $S_j = \emptyset$. If $K_j = \emptyset$, go to Step 4; otherwise, go to Step 3.

Step 3: Pick a vertex $v \in K_j \setminus S_j$, add a directed path in G_j from v to some $w \in J_j$ with all intermediate vertices being distinct and in K_j into T, add all the vertices in this path from K_j into S_j, go to Step 4.

Step 4: If $S_j = K_j$, go to Step 6; otherwise, go to Step 5.

Step 5: If there is $v \in K_j \setminus S_j$ such that $(v, w) \in E_j$ for some $w \in S_j$, put v into S_j and (v, w) into F, go to Step 4; otherwise go to Step 3.

Step 6: Set $j = j - 1$, go to Step 1.

Since G_j is strongly connected for all $j \in [s]$, the procedure should terminate in finitely many steps. We note that the generated forest may not be unique. For every edge $(v, w) \in F$, the vertex v is a *child* of the vertex w, which is the *parent* of the vertex v. A vertex with no child is a *leaf*, and a vertex with no parent is a *root*. An isolated vertex is both a leaf and a root. It is easy to see from the above procedure that every root is a vertex in $\cup_{j=1}^s P_j$, and vice versa. It is also a fact that from every vertex we can get a unique root along the directed edges. Therefore, we can define the *height* of a vertex unambiguously as the length of the unique directed path from it to the root. Thus, a root has height 1. The maximum height of the vertices in a tree is the *height of the tree*, and the maximum height of the trees in a forest is the *height of the forest*. We denote by $h(T)$ the height of the forest $T = (I_1 \cup \cdots \cup I_s, F)$ generated by Algorithm 1.

Let $\mathbf{x}$ be a positive vector, $\gamma > 0$, and $(v, w) \in F$. Obviously, v is not a root. Suppose that $v \in I_j$. If $w \in I_j$, we have $(v, w) \in E_j$ and

$$(f_{I_j}(\gamma\mathbf{x}))_v = \gamma\left[\frac{1}{\lambda}\left(\sum_{i_2,\dots,i_m\in I_j} a_{vi_2\dots i_m}x_{i_2}\dots x_{i_m} + \left(\sum_{p=1}^{m-1}\gamma^{p-m}\mathcal{A}_{j,p}(\gamma\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_v\right)\right]^{\frac{1}{m-1}}$$

$$\geq \gamma\left(\frac{1}{\lambda}a_{vi_2'\dots i_m'}x_{i_2'}\dots x_{i_m'}\right)^{\frac{1}{m-1}} \tag{18}$$

for some $\{i_2', \dots, i_m'\} \in I_j^{m-1}$ such that

$$a_{vi_2'\dots i_m'} > 0 \text{ and } w \in \{i_2', \dots, i_m'\},$$

since $(v, w) \in F \cap I_j^2 \subseteq E_j$. Therefore, when x_w is sufficiently large $(f_{I_j}(\gamma\mathbf{x}))_v \geq \gamma x_v$. If $w \notin I_j$, then $w \in I_{j+1} \cup \cdots \cup I_s$ is such that (cf. Algorithm 1)

$$\lim_{t\to\infty}\left(\sum_{u=1}^{m-1}\mathcal{A}_{j,u}(\mathbf{y})\mathbf{e}_{I_j}^{u-1}\right)_v \to \infty \text{ with } y_w = t \text{ and } y_p = 1 \text{ for the other } p \in [n] \setminus \{w\}.$$

Moreover, we should have that $v \in W_j$ (cf. Algorithm 1) and, therefore,

$$\left(\sum_{u=1}^{m-1}\mathcal{B}_{j,u}(\mathbf{e})\mathbf{e}_{I_j}^{u-1}\right)_v = 0.$$

Thus,

$$\left(\sum_{u=1}^{m-1}\mathcal{A}_{j,u}(\mathbf{x})\mathbf{x}_{I_j}^{u-1}\right)_v=\left(\sum_{u=1}^{m-1}\mathcal{H}_{j,u}(\mathbf{x})\mathbf{x}_{I_j}^{u-1}\right)_v$$

is a homogeneous polynomial of degree $m-1$. Henceforth, if x_w is sufficiently large, we have

$$\begin{aligned}(f_{I_j}(\gamma\mathbf{x}))_v&=\gamma\left[\frac{1}{\lambda}\left(\sum_{i_2,\dots,i_m\in I_j}a_{vi_2\dots i_m}x_{i_2}\dots x_{i_m}+\left(\sum_{p=1}^{m-1}\gamma^{p-m}\mathcal{A}_{j,p}(\gamma\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_v\right)\right]^{\frac{1}{m-1}}\\&=\gamma\left[\frac{1}{\lambda}\left(\sum_{i_2,\dots,i_m\in I_j}a_{vi_2\dots i_m}x_{i_2}\dots x_{i_m}+\left(\sum_{p=1}^{m-1}\mathcal{H}_{j,p}(\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_v\right)\right]^{\frac{1}{m-1}}\\&\geq\gamma\left(\frac{1}{\lambda}\left(\sum_{p=1}^{m-1}\mathcal{H}_{j,p}(\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_v\right)^{\frac{1}{m-1}}\\&\geq\gamma x_v.\end{aligned}$$

To get a desired vector $\mathbf{x}\in\mathbb{R}^n_{++}$ such that $f(\mathbf{x})\geq\mathbf{x}$, we can start with $\mathbf{x}=\mathbf{e}$ and leaves with height $h(T)$. The case $h(T)=1$ is trivial. Suppose that $h(T)\geq 2$ and $L\subset(I_1\cup\dots\cup I_s)\setminus(P_1\cup\dots\cup P_s)$ is the set of leaf vertices of height $h(T)$. Then, we can set the parents of these leaves sufficiently large such that

$$(f(\gamma\mathbf{x}))_v\geq\gamma x_v \text{ for all } v\in L. \tag{19}$$

Second, let us consider the set L' of vertices with height $h(T)-1$ if $h(T)>2$, which includes the parents of L. Vertices in L' are not roots. If we set the set P' of the parents of vertices in L' sufficiently large, we can get

$$(f(\gamma\mathbf{x}))_p\geq\gamma x_p \text{ for all } p\in L'.$$

It follows from the above analysis that we still withhold (19) when we increase $x_{p'}$ for $p'\in P'$ if necessary. The next step is to consider the set L'' of vertices with height $h(T)-2$ if $h(T)>3$, which includes the parents of L'. In this way, $(f(\gamma\mathbf{x}))_v\geq\gamma x_v$ for all child vertices $v\in(I_1\cup\dots\cup I_s)\setminus(P_1\cup\dots\cup P_s)$ by increasing their parents sufficiently large successively from vertices of height $h(T)$ to vertices of height 2. Since we have the constructed forest structure and any $v\in(I_1\cup\dots\cup I_s)\setminus(P_1\cup\dots\cup P_s)$ is a child of some parent $w\in I_1\cup\dots\cup I_s$, we can terminate the procedure in $h(T)-1$ steps, and therefore get that

$$(f(\gamma\mathbf{x}))_v\geq\gamma x_v \text{ for all } v\in(I_1\cup\dots\cup I_s)\setminus(P_1\cup\dots\cup P_s)$$

for some positive $\mathbf{x}$. Note that, we still have the freedom to choose $\gamma>0$.

If $w\in P_j\subset P_1\cup\dots\cup P_s$ is a root, then

$$\left(\sum_{u=1}^{m-1}\mathcal{B}_{j,u}(\mathbf{e})\mathbf{e}_{I_j}^{i-1}\right)_w>0 \tag{20}$$

by definition. We have

$$
\begin{aligned}
(f_{I_j}(\gamma\mathbf{x}))_w &= \gamma\left[\frac{1}{\lambda}\left(\sum_{i_2,\dots,i_m\in I_j} a_{wi_2\dots i_m}x_{i_2}\dots x_{i_m}+\left(\sum_{p=1}^{m-1}\gamma^{p-m}\mathcal{A}_{j,p}(\gamma\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_w\right)\right]^{\frac{1}{m-1}}\\
&= \gamma\left[\frac{1}{\lambda}\left(\sum_{i_2,\dots,i_m\in I_j} a_{wi_2\dots i_m}x_{i_2}\dots x_{i_m}\right.\right.\\
&\qquad\left.\left.+\left(\sum_{p=1}^{m-1}\mathcal{H}_{j,p}(\mathbf{x})\mathbf{x}_{I_j}^{p-1}+\sum_{p=1}^{m-1}\gamma^{p-m}\mathcal{B}_{j,p}(\gamma\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_w\right)\right]^{\frac{1}{m-1}}\\
&\geq \gamma\left(\frac{1}{\lambda}\left(\sum_{p=1}^{m-1}\gamma^{p-m}\mathcal{B}_{j,p}(\gamma\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_w\right)^{\frac{1}{m-1}}. \qquad (21)
\end{aligned}
$$

Note that the highest degree of entries in $\mathcal{B}_{j,p}(\mathbf{x})$ is smaller than $m-p-1$ for all $p \in [m-1]$. This, together with (20), implies that the leading term of

$$\left(\sum_{p=1}^{m-1}\gamma^{p-m}\mathcal{B}_{j,p}(\gamma\mathbf{x})\mathbf{x}_{I_j}^{p-1}\right)_w$$

is a term of $\frac{1}{\gamma^u}$ with positive coefficient for some integer $u>0$. Therefore, if $\gamma>0$ is sufficiently small, we definitely have

$$(f_{I_j}(\gamma\mathbf{x}))_w \geq \gamma x_w.$$

Since there are only finitely many roots, we have

$$(f(\gamma\mathbf{x}))_w \geq \gamma x_w \text{ for all } w \in P_1\cup\dots\cup P_s.$$

Therefore, we can find a $\mathbf{x}$ with $\gamma>0$ such that $f(\gamma\mathbf{x}) \geq \gamma\mathbf{x}$ and $\gamma\mathbf{x} \leq \beta\mathbf{y}$ (cf. $\beta\mathbf{y}$ from *Part II.*).

In summary,

$$f(\gamma\mathbf{x}) \geq \gamma\mathbf{x} \text{ and } f(\beta\mathbf{y}) \leq \beta\mathbf{y}.$$

It then follows from Theorem 5 that there is a positive $\mathbf{w} \in [\gamma\mathbf{x}, \beta\mathbf{y}]$ such that

$$f(\mathbf{w}) = \mathbf{w}.$$

It is nothing but a positive solution $\mathbf{w}$ to (14). ∎

LEMMA 23. *Suppose that $\mathcal{A} \in N_{m,n}$ is weakly irreducible, and $\mathcal{A}_i \in N_{i,n}$ for $i = 1,\dots,m-1$ are such that*

$$\sum_{i=1}^{m-1}\mathcal{A}_i\mathbf{e}^{i-1} \neq \mathbf{0}.$$

If for some $\lambda>0$, there is a positive solution $\mathbf{x} \in \mathbb{R}^n_{++}$ for the following system,

$$\mathcal{A}\mathbf{x}^{m-1}+\sum_{i=1}^{m-1}\mathcal{A}_i\mathbf{x}^{i-1} = \lambda\mathbf{x}^{[m-1]},$$

then $\rho(\mathcal{A}) < \lambda$.

Proof. Suppose, without loss of generality, that

$$I := \{1, \dots, r\} := \operatorname{supp}\left(\sum_{i=1}^{m-1} \mathcal{A}_i \mathbf{e}^{i-1}\right)$$

for some $r \leq n$. It follows from the hypothesis that

$$(\mathcal{A}\mathbf{x}^{m-1})_i < \lambda x_i^{m-1} \text{ for all } i = 1, \dots, r. \tag{22}$$

If $r = n$, then the result follows from Theorem 4 directly.

In the following, we assume that $r < n$. Note that $r > 0$ by the assumption on $\mathcal{A}_i$'s. We have

$$(\mathcal{A}\mathbf{x}^{m-1})_j = \lambda x_j^{m-1} \text{ for all } j = r+1, \dots, n.$$

By the weak irreducibility, there should be a $j \in J := \{1, \dots, n\} \setminus I$ and an $i \in I$ such that

$$a_{ji_2\dots i_m} > 0 \text{ for some multiset } \{i_2, \dots, i_m\} \ni i.$$

Therefore, there is a nonzero term in $\sum_{i_2,\dots,i_m=1}^n a_{ji_2\dots i_m} x_{i_2} \dots x_{i_m}$ involving the variable x_i. We can define a new positive vector, denoted also by $\mathbf{x}$, through decreasing x_i a little bit. It follows from the nonnegativity of $\mathcal{A}$ that

$$(\mathcal{A}\mathbf{x}^{m-1})_j < \lambda x_j^{m-1}. \tag{23}$$

By the continuity, we can still withhold (22) for a sufficiently small decrease of x_i, as well as getting (23), while, as we can see, we get at least $r+1$ strict inequalities now. Inductively in this way, we can find a positive vector $\mathbf{x}$ such that

$$\mathcal{A}\mathbf{x}^{m-1} < \lambda \mathbf{x}^{[m-1]}.$$

The result then follows from Theorem 4. ∎

5.3. Proof of Theorem 16.

Proof. In the proof, we assume all the notation in section 5.1. We prove the sufficiency first.

For any $j = s+1, \dots, r$, we see that

$$(\mathcal{A}\mathbf{x}^{m-1})_{I_j} = \mathcal{A}_{I_j} \mathbf{x}_{I_j}^{m-1} \text{ for all } \mathbf{x} \in \mathbb{C}^n. \tag{24}$$

It follows from Theorem 4 that there exists a positive vector $\mathbf{y}_j \in \mathbb{R}_{++}^{|I_j|}$ such that $\mathcal{A}_{I_j}\mathbf{y}_j^{m-1} = \rho(\mathcal{A}_{I_j})\mathbf{y}_j^{[m-1]} = \rho(\mathcal{A})\mathbf{y}_j^{[m-1]}$ for all $j = s+1, \dots, r$.

Let $\mathbf{x}$ be an n-dimensional vector with $\mathbf{x}_{I_j} = \mathbf{y}_j$ for $j = s+1, \dots, r$ and $\mathbf{x}_{I_1 \cup \dots \cup I_s}$ indeterminant to be determined. It is sufficient to show that the following system of polynomials has a positive solution in $\mathbb{R}_{++}^{|I_1|+\dots+|I_s|}$:

$$\mathcal{A}_{I_j}\mathbf{x}_{I_j}^{m-1} + \sum_{u=1}^{m-1} \mathcal{A}_{j,u}(\mathbf{x}_{K_j})\mathbf{x}_{I_j}^{u-1} = \rho(\mathcal{A})\mathbf{x}_{I_j}^{[m-1]} \text{ for all } j \in [s], \tag{25}$$

where $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ has the partition (11).

Note that $\mathbf{x}_{I_j}$'s are given positive vectors for $j = s+1, \dots, r$. The indeterminant variables are $\mathbf{x}_{I_j}$ for $j \in [s]$. Therefore, the tensor formed by the polynomials of degree $m-u$ in $\mathcal{A}_{j,u}(\mathbf{x}_{K_j})$ is $\mathcal{H}_{j,u}(\mathbf{x}_{L_j})$ for any $u \in [m-1]$ and $j \in [s]$. If for some $j \in [m-1]$, we have $\sum_{u=1}^{m-1}\mathcal{B}_{j,u}(\mathbf{e}_{K_j})\mathbf{e}_{I_j}^{u-1}=\mathbf{0}$, then $\sum_{u=1}^{m-1}\mathcal{H}_{j,u}(\mathbf{e}_{L_j})\mathbf{e}_{I_j}^{u-1}\neq\mathbf{0}$ by item (c) in Proposition 20. It follows from (a) in Proposition 20 that $\sum_{u=1}^{m-1}\mathcal{H}_{j,u}(\mathbf{h}_{L_j})\mathbf{e}_{I_j}^{u-1}=\mathbf{0}$ with $\mathbf{h}_{I_t}=\mathbf{e}_{I_t}$ for $t\in[j-1]$ and $\mathbf{h}_{I_t}=\mathbf{0}$ for the others. Therefore, a nonzero term involving variables from $I_{j+1}\cup\cdots\cup I_s$ occurs in some entry of one tensor $\mathcal{H}_{j,u}(\mathbf{x}_{L_j})$ for some $u\in[m-1]$.

In summary, it follows from Proposition 20 that (12) and (13), as well as the second hypothesis in Lemma 22, are fulfilled for the system (25). Therefore, by Lemma 22, we can find a solution $\mathbf{z}\in\mathbb{R}_{++}^{|I_1|+\cdots+|I_s|}$ for (25). Therefore, $\mathbf{x}$ with $\mathbf{x}_{I_j}=\mathbf{z}_{I_j}$ for $j\in[s]$ and $\mathbf{x}_{I_j}=\mathbf{y}_j$ for $j=s+1,\dots,r$ is a positive Perron vector of $\mathcal{A}$.

For necessity, suppose that $\mathcal{A}\mathbf{x}^{m-1}=\lambda\mathbf{x}^{[m-1]}$ with $\mathbf{x}\in\mathbb{R}_{++}^n$ being a positive eigenvector for some $\lambda\geq 0$. By Theorem 4, $\lambda=\rho(\mathcal{A})$. The case for $\rho(\mathcal{A})=0$ is trivial. In the following, we assume $\rho(\mathcal{A})>0$. We have that for each I_j,

$$(\mathcal{A}\mathbf{x}^{m-1})_{I_j}=\rho(\mathcal{A})\mathbf{x}_{I_j}^{[m-1]}.$$

If $\mathcal{A}_{I_j}$ is a genuine weakly irreducible principal subtensor of $\mathcal{A}$, it follows from Proposition 9 that

$$\mathcal{A}_{I_j}\mathbf{x}_{I_j}^{m-1}=\rho(\mathcal{A})\mathbf{x}_{I_j}^{[m-1]}$$

and Theorem 4 that $\rho(\mathcal{A}_{I_j})=\rho(\mathcal{A})$.

If $\mathcal{A}_{I_j}$ is not a genuine weakly irreducible principal subtensor of $\mathcal{A}$, it follows from Proposition 9 that

$$\mathcal{A}_{I_j}\mathbf{x}_{I_j}^{m-1}+\sum_{u=1}^{m-1}\mathcal{A}_{j,u}(\mathbf{x}_{K_j})\mathbf{x}_{I_j}^{u-1}=\rho(\mathcal{A})\mathbf{x}_{I_j}^{[m-1]}$$

and $\sum_{u=1}^{m-1}\mathcal{A}_{j,u}(\mathbf{e}_{K_j})\mathbf{e}_{I_j}^{u-1}\neq\mathbf{0}$, and from Lemma 23 that $\rho(\mathcal{A}_{I_j})<\rho(\mathcal{A})$. The proof is thus complete. ∎

6. Algorithmic aspects.

In order to get a canonical nonnegative partition for a nonnegative tensor as in Proposition 9, we have to recursively partition the majorization matrix of the nonnegative tensor and marjorization matrices of some induced principal sub-tensors (cf. [13]).

6.1. Majorization matrix partition.

LEMMA 24. *Let $M_{\mathcal{A}}=(m_{ij})$ be the majorization matrix of $\mathcal{A}\in N_{m,n}$. If $m_{ij}=0$ for all $i\in I$ and $j\in I^{\complement}$ for some nonempty proper subset $I\subset[n]$, then*

$$M_{\mathcal{A}_I}=\big(M_{\mathcal{A}}\big)_I. \tag{26}$$

Proof. Without loss of generality, we can assume that $I^{\complement}=[p]$ for some positive $p<n$. It follows from $m_{ij}=0$ for all $i>p$ and $j\in[p]$ that

$$a_{ii_2\dots i_m}=0 \text{ for all } i>p \text{ and } (i_2,\dots,i_m)\in I(1)\cup\cdots\cup I(p).$$

Note that for any $i'\in I$

$$m_{ii'}=\sum_{(i_2,\dots,i_m)\in I(i')}a_{ii_2\dots i_m}=\sum_{\substack{(i_2,\dots,i_m)\in I(i')\\ \{i_2,\dots,i_m\}\cap[p]=\emptyset}}a_{ii_2\dots i_m},$$

where the rightmost summation only involves indices $\{i_2,\dots,i_m\}\subseteq I$. With the definition for the majorization matrices for nonnegative tensors, we immediately get (26). ∎

In general, we do not simultaneously have both $M_{\mathcal{A}_I} = \big(M_{\mathcal{A}}\big)_I$ and $M_{\mathcal{A}_{I^{\complement}}} = \big(M_{\mathcal{A}}\big)_{I^{\complement}}$.

Example 1. Let $\mathcal{A} \in N_{3,3}$ with entries

$$a_{123} = a_{213} = a_{333} = 1 \text{ and } a_{ijk} = 0 \text{ for the others.}$$

Letting $I = \{3\} = \{1,2\}^{\complement}$, we have

$$M_{\mathcal{A}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ M_{\mathcal{A}_{\{3\}}} = [1] = \big(M_{\mathcal{A}}\big)_{\{3\}},\ M_{\mathcal{A}_{\{1,2\}}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \big(M_{\mathcal{A}}\big)_{\{1,2\}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A partition $I_1, \dots, I_r$ of $[n]$ is a *refined partition* of a partition $J_1, \dots, J_s$ of $[n]$ if

$$J_j = I_{j_1} \cup \cdots \cup I_{j_i} \text{ for some } j_1, \dots, j_i \in [r] \text{ for all } j \in [s].$$

COROLLARY 25. *Let $M_{\mathcal{A}} = (m_{ij})$ be the majorization matrix of $\mathcal{A} \in N_{m,n}$. If $\mathcal{A}$ has a canonical nonnegative partition $\{I_1, \dots, I_r\}$, then $M_{\mathcal{A}}$ has an upper triangular block structure with diagonal blocks being $\{J_1, \dots, J_s\}$ for which $\{I_1, \dots, I_r\}$ is a refined partition.*

With Corollary 25, we can get a canonical nonnegative partition for $\mathcal{A} \in N_{m,n}$ by first partitioning its majorization matrix $M_{\mathcal{A}}$ into (up to permutation) an upper triangular block form with each diagonal block being irreducible, and then recursively perform the partition to each principal subtensor induced by these irreducible diagonal blocks. This improves the partition method proposed in [13].

6.2. Algorithms. Given a nonnegative tensor $\mathcal{A} \in N_{m,n}$, one way to find a canonical nonnegative partition as in Proposition 9 is by recursively partitioning the majorization matrices of the induced principal subtensors (cf. Corollary 25). We will denote by *Algorithm P* an algorithm which is able to find a canonical nonnegative partition for any given nonnegative tensor. There is such an algorithm (cf. [13, section 6]), which can be improved using section 6.1. Therefore, in the following, we will assume that we have already computed a partition $I_1 \cup \cdots \cup I_r = [n]$ with properties described in Proposition 9. Let $I_{s+1}, \dots, I_r$ be the genuine weakly irreducible blocks, and $R = I_1 \cup \cdots \cup I_s$.

If $\mathcal{A}$ is weakly irreducible, we can use the following algorithm to find the spectral radius, together with the positive Perron vector in the 1-norm being one.

Note that when $\mathcal{A} \in N_{m,n}$ is weakly irreducible, then $\mathcal{A} + \mathcal{I}$ is weakly primitive (cf. [13, 25]), where $\mathcal{I} \in N_{m,n}$ is the identity tensor. It follows that $\rho(\mathcal{A})+1 = \rho(\mathcal{A}+\mathcal{I})$.

PROPOSITION 26 (see [13, Theorem 4.1(iv)]). *If $\mathcal{B} \in N_{m,n}$ is weakly irreducible, then the sequence $\{\mathbf{x}^{(k)}\}$ generated by Algorithm 2 with $\mathcal{A} := \mathcal{B} + \mathcal{I}$ converges globally R-linearly to the positive Perron vector $\mathbf{x}^*$ of $\mathcal{B}$ corresponding to $\rho(\mathcal{B})$ satisfying $\mathbf{e}^{\mathsf{T}}\mathbf{x}^* = 1$.*

Next, we present an algorithm for determining whether a nonnegative tensor $\mathcal{A} \in N_{m,n}$ is strongly nonnegative or not, and finding a positive Perron vector when it is.

It is easy to see that $\lambda = \rho(\mathcal{A})$ by Proposition 7.

PROPOSITION 27. *For any given $\mathcal{A} \in N_{m,n}$, if γ is sufficiently small, then Algorithm* 3

Algorithm 2. A Higher Order Power Method [13].

The input is a strictly nonnegative tensor $\mathcal{A} \in N_{m,n}$.

Step 0: Initialization: choose $\mathbf{x}^{(0)} \in \mathbb{R}^n_{++}$. Let $k := 0$.

Step 1: Compute

$$\bar{\mathbf{x}}^{(k+1)} := \mathcal{A}(\mathbf{x}^{(k)})^{m-1}, \quad \mathbf{x}^{(k+1)} := \frac{\left(\bar{\mathbf{x}}^{(k+1)}\right)^{[\frac{1}{m-1}]}}{\mathbf{e}^T\left[\left(\bar{\mathbf{x}}^{(k+1)}\right)^{[\frac{1}{m-1}]}\right]},$$
$$\alpha\left(\mathbf{x}^{(k+1)}\right) := \max_{1\le i\le n} \frac{\left(\mathcal{A}(\mathbf{x}^{(k)})^{m-1}\right)_i}{\left(\mathbf{x}^{(k)}\right)_i^{m-1}},$$
$$\beta\left(\mathbf{x}^{(k+1)}\right) := \min_{1\le i\le n} \frac{\left(\mathcal{A}(\mathbf{x}^{(k)})^{m-1}\right)_i}{\left(\mathbf{x}^{(k)}\right)_i^{m-1}}.$$

Step 2: If $\alpha\left(\mathbf{x}^{(k+1)}\right) = \beta\left(\mathbf{x}^{(k+1)}\right)$ or a tolerance for $\alpha\left(\mathbf{x}^{(k+1)}\right) - \beta\left(\mathbf{x}^{(k+1)}\right)$ is satisfied, stop. Otherwise, let $k := k+1$, go to Step 1.

1. *either terminates in Step* 2 *or Step* 3*, which concludes that* $\mathcal{A}$ *is not strongly nonnegative and there does not exist a positive Perron vector for* $\mathcal{A}$*,*
2. *or generates a sequence* $\{\mathbf{z}_k\}$ *such that* $\mathbf{z}_{k+1} \ge \mathbf{z}_k$ *and* $\lim_{k\to\infty} \mathbf{z}_k = \mathbf{z}_*$ *with* $\mathbf{z}_*$ *being a positive Perron vector of* $\mathcal{A}$ *(i.e.,* $\mathcal{A}\mathbf{z}_*^{m-1} = \rho(\mathcal{A})\mathbf{z}_*^{[m-1]}$*).*

Algorithm 3. Positive Perron Vector Algorithm.

The input is a nonnegative tensor $\mathcal{A} \in N_{m,n}$.

Step 0: Let γ be a given small positive scalar. Find a canonical nonnegative partition of $\mathcal{A}$ by Algorithm P with genuine weakly irreducible blocks $I_{s+1}, \dots, I_r$ and $R := I_1 \cup \dots \cup I_s$.

Step 1: For each $j = 1, \dots, r$, find the positive eigenvector $\mathbf{x}_j \in \mathbb{R}^{|I_j|}_{++}$ such that $\mathcal{A}_{I_j}\mathbf{x}_j^{m-1} = \rho(\mathcal{A}_{I_j})\mathbf{x}_j^{[m-1]}$ by Algorithm 2. If $|I_j| = 1$ and $\mathcal{A}_{I_j} = 0$, then simply set $\rho(\mathcal{A}_{I_j}) = 0$ and $\mathbf{x}_j = 1$.

Step 2: If $\max\{\rho(\mathcal{A}_{I_j}) : j = s+1, \dots, r\} \ne \min\{\rho(\mathcal{A}_{I_j}) : j = s+1, \dots, r\}$, we claim that $\mathcal{A}$ is not strongly nonnegative and no positive Perron vector exists; stop. Otherwise, let $\lambda := \max\{\rho(\mathcal{A}_{I_j}) : j = s+1, \dots, r\}$.

Step 3: If $\max\{\rho(\mathcal{A}_{I_j}) : j \in [s]\} \ge \lambda$, we claim that $\mathcal{A}$ is not strongly nonnegative and no positive Perron vector exists; stop. Otherwise, let $\mathbf{y} \in \mathbb{R}^n_{++}$ with

$$\mathbf{y}_{I_j} = \mathbf{x}_j \text{ for all } j = s+1, \dots, r.$$

Step 4: Let $\mathbf{w}_0 := \gamma(\mathbf{x}_1^\top, \dots, \mathbf{x}_s^\top)^\top$, $\mathbf{z}_0 := (\mathbf{w}_0^\top, \mathbf{y}_{R^\complement}^\top)^\top$, and $k = 1$.

Step 5: Compute

$$\mathbf{w}_k := \left(\frac{(\mathcal{A}\mathbf{z}_{k-1}^{m-1})_R}{\lambda}\right)^{\frac{1}{[m-1]}}$$

and $\mathbf{z}_k := (\mathbf{w}_k^\top, \mathbf{y}_{R^\complement}^\top)^\top$.

Step 6: If $\mathbf{w}_k = \mathbf{w}_{k-1}$ or a tolerance for $\|\mathbf{w}_k - \mathbf{w}_{k-1}\|_2$ is satisfied, stop. Otherwise, let $k := k+1$, go to Step 5.

Proof. The conclusion for termination in either Step 2 or Step 3 follows from Theorem 16.

Suppose $\rho(\mathcal{A}_{I_j}) < \rho(\mathcal{A})$ for all $j \in [s]$ and $\rho(\mathcal{A}_{I_j}) = \rho(\mathcal{A})$ for $j = s+1, \dots, r$. Then Algorithm 3 will execute Steps 4–6. It follows from the proof of Lemma 22 that we can find positive vectors $\mathbf{x}$ and $\mathbf{z}$, and sufficiently small κ and sufficiently large β such that

$$\left(\frac{(\mathcal{A}\mathbf{u}^{m-1})_R}{\lambda}\right)^{\frac{1}{[m-1]}} \geq \kappa\mathbf{x} \text{ and } \left(\frac{(\mathcal{A}\mathbf{v}^{m-1})_R}{\lambda}\right)^{\frac{1}{[m-1]}} \leq \beta\mathbf{z},$$

where $\mathbf{u} := (\kappa\mathbf{x}^\mathsf{T}, \mathbf{y}_{R^\complement}^\mathsf{T})^\mathsf{T}$, $\mathbf{v} = (\beta\mathbf{z}^\mathsf{T}, \mathbf{y}_{R^\complement}^\mathsf{T})^\mathsf{T}$, and $\mathbf{y}$ is defined as in Step 3. It also follows from the proof of Lemma 22 that if κ fulfills the above inequality, then each positive $\tau \leq \kappa$ does also. Thus, for every sufficiently small $\gamma > 0$, we can choose κ to make sure that $\mathbf{w}_0 \in [\kappa\mathbf{x}, \beta\mathbf{z}]$. The convergent result then follows from Theorem 5. ◻

6.3. Numerical computation. All the experiments were conducted by MatLab on a laptop with 2.5 GH Intel processor with 4 GB memory. We only tested third order strongly nonnegative tensors for Algorithm 3. Details of the problems we tested are given in Examples 2, 3, and 4. Algorithm 3 is terminated whenever

$$\|\mathcal{A}\mathbf{z}_k^2 - \lambda\mathbf{z}_k^{[2]}\|_2 < 10^{-6} \text{ and } \|\mathbf{w}_k - \mathbf{w}_{k-1}\|_2 < 10^{-6}. \tag{27}$$

Algorithm 3 successfully computed out a positive vector to satisfy the criteria (27) for every tested case. Algorithm 2 was terminated when $\alpha(\mathbf{x}^{(k)}) - \beta(\mathbf{x}^{(k)}) < 10^{-6}$.

Example 2. The tensor $\mathcal{A} \in N_{8,2}$ with a partition

$$I_1 := \{1,2\},\ I_2 := \{3,4\},\ I_3 := \{5,6\},\ I_4 := \{7,8\}$$

and a unique genuine block $\mathcal{A}_4$. Note that we will simplify $\mathcal{A}_{I_j}$ as $\mathcal{A}_j$ for $j \in [4]$. The weakly irreducible principal subtensors are as follows:

$$\mathcal{A}_1(:,:,1) = \begin{bmatrix} 0.4423 & 0.3309 \\ 0.0196 & 0.4243 \end{bmatrix} \text{ and } \mathcal{A}_1(:,:,2) = \begin{bmatrix} 0.2703 & 0.8217 \\ 0.1971 & 0.4299 \end{bmatrix},$$

$$\mathcal{A}_2(:,:,1) = \begin{bmatrix} 0.3185 & 0.0900 \\ 0.5341 & 0.1117 \end{bmatrix} \text{ and } \mathcal{A}_2(:,:,2) = \begin{bmatrix} 0.1363 & 0.4952 \\ 0.6787 & 0.1897 \end{bmatrix},$$

$$\mathcal{A}_3(:,:,1) = \begin{bmatrix} 0.6664 & 0.6260 \\ 0.0835 & 0.6609 \end{bmatrix} \text{ and } \mathcal{A}_3(:,:,2) = \begin{bmatrix} 0.7298 & 0.9823 \\ 0.8908 & 0.7690 \end{bmatrix},$$

$$\mathcal{A}_4(:,:,1) = \begin{bmatrix} 0.3642 & 1.0317 \\ 0.6636 & 0.5388 \end{bmatrix} \text{ and } \mathcal{A}_4(:,:,2) = \begin{bmatrix} 1.1045 & 1.0251 \\ 0.5921 & 1.0561 \end{bmatrix}.$$

The other nonzero components of the tensor $\mathcal{A}$ are expressed by the following equations:

$$
\begin{aligned}
(\mathcal{A}\mathbf{x})_{I_1} &= \mathcal{A}_1\mathbf{x}_{I_1}^2 + \begin{pmatrix} 0.8085x_1x_7 \\ 0.7551x_2x_6 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0.5880x_6x_7 + 0.1548x_3x_8 + 0.1999x_4x_8 + 0.4070x_8^2 \\ 0.7487x_3x_4 + 0.8256x_3x_6 + x_4x_8 \end{pmatrix}, \\
(\mathcal{A}\mathbf{x})_{I_2} &= \mathcal{A}_2\mathbf{x}_{I_2}^2 + \begin{pmatrix} 0.5606x_3x_5 + 0.9296x_4x_7 \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0.9009x_2x_5 \\ 0.5747x_5x_8 + 0.8452x_1x_6 + 0.7386x_6x_7 + 0.5860x_7^2 + 0.2467x_8^2 \end{pmatrix}, \\
(\mathcal{A}\mathbf{x})_{I_3} &= \mathcal{A}_3\mathbf{x}_{I_3}^2 + \begin{pmatrix} 0 \\ 0.5801x_6x_8 \end{pmatrix} + \begin{pmatrix} 0.1465x_3x_8 + 0.1891x_4x_7 \\ 0.2819x_4x_7 \end{pmatrix}.
\end{aligned}
$$

The spectral radii of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ are, respectively (via Algorithm 2),

$$\rho(\mathcal{A}_1) = 1.3183,\ \rho(\mathcal{A}_2) = 1.2581,\ \rho(\mathcal{A}_3) = 2.6317,\ \text{and } \rho(\mathcal{A}_4) = 3.1253.$$

Therefore, $\rho(\mathcal{A}) = \rho(\mathcal{A}_4) = 3.1253$ by Proposition 7. By Algorithm 2, we first computed out the positive Perron vectors of $\mathcal{A}_i$ with the 1-norm being one as $\mathbf{x}_i$, then we use $\mathbf{w}_0 := 0.5(\mathbf{x}_1^\mathsf{T}, \mathbf{x}_2^\mathsf{T}, \mathbf{x}_3^\mathsf{T})^\mathsf{T}$ (i.e., $\gamma = 0.5$ in Algorithm 3) as the initial point for our fixed point iteration (i.e., Steps 5 and 6 in Algorithm 3). It took 0.65 seconds with 52 iterations to compute out a positive Perron vector of $\mathcal{A}$:

$$\mathbf{x} = (0.4462, 0.4143, 0.3808, 0.4446, 0.2943, 0.3055, 0.5257, 0.4743)^\mathsf{T}$$

with the corresponding eigenvalue being $\rho(\mathcal{A}) = 3.1253$. The final residue of the eigenvalue equations is $\|\mathcal{A}\mathbf{x}^2 - 3.1253\mathbf{x}^{[2]}\|_2 = 9.2323 \times 10^{-7}$. Figure 1 gives information on the residue of the eigenvalue equations and the distance between two successive iterations $\mathbf{w}_k$ and $\mathbf{w}_{k-1}$ in the iteration process. The magnitude is of logarithmic order. It shows that the convergence is sublinear, as expected for the fixed point iteration.

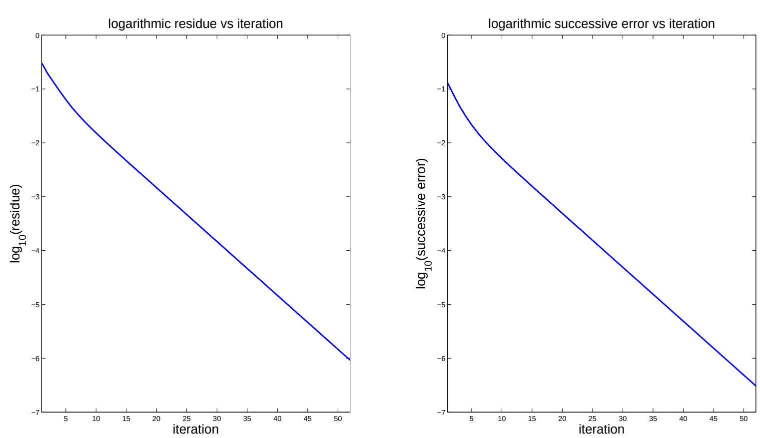

FIG. 1. *The logarithmic residue of the eigenvalue equations* $\|\mathcal{A}\mathbf{x}_k^2 - \rho(\mathcal{A})\mathbf{x}_k^{[2]}\|_2$ *(left), and the logarithmic successive error* $\|\mathbf{w}_k - \mathbf{w}_{k-1}\|_2$ *(right) along with the iteration.*

Example 3. In this example, we tested randomly generated nonnegative tensors. Each tested tensor has the following canonical nonnegative partition with

$$I_1 \cup I_2 \cup I_3 \cup I_4$$

and a unique genuine weakly irreducible block I_4. The cardinality of $|I_4|$ is 10. We tested two sets of I_1, I_2, and I_3.

Case I: $|I_1| = 8, |I_2| = 9, |I_3| = 10$, and

Case II: $|I_1| = 30, |I_2| = 30, |I_3| = 30$.

The tensors were generated as follows:

1. Randomly generate $\mathcal{A}_i \in N_{3,|I_i|}$ for $i \in [4]$. The generated tensors all have positive components, and, hence, are weakly irreducible.
2. Using Algorithm 2 to compute out the spectral radii of the above generated tensors, denote the maximum of them by λ_0.
3. Let $\mathrm{Rt} > 1$ be a parameter and define $\lambda := \lambda_0 \mathrm{Rt}$.
4. Generate the remaining components, satisfying conditions in Proposition 9, of each system of polynomials $(\mathcal{A}\mathbf{x}^2)_{I_j}$ for $j \in [3]$ randomly with sparsity $\mathrm{den} = 10\%$.

Then, we implemented Algorithm 3 to compute a positive Perron vector of the generated tensor $\mathcal{A}$ with the new $\mathcal{A}_{I_4}$ being $\frac{\lambda}{\rho(\mathcal{A}_4)}\mathcal{A}_4$. Thus, $\mathcal{A}$ is strongly nonnegative, and the hypothesis in Theorem 16 is satisfied. We tested various Rt according to the following rule:

Case I: $\mathrm{Rt}(i) := 1.1 + 0.2i$ for all $i \in [50]$, and

Case II: $\mathrm{Rt}(i) := 2 + 0.2i$ for all $i \in [50]$.

We will refer to Rt as a measure of the ratio of the spectral radii. For both the cases, the parameter γ in Algorithm 3 was chosen as

$$\gamma := 10^{-5}(1/\mathrm{Rt}(i)) \text{ for } i \in [50].$$

For each case and each $i \in [50]$, we made ten simulations, and recorded the means and standard variances of the numbers of iterations and the CPU time. Figure 2 contains the information for Case I, and Figure 3 for Case II. In each figure, we have a subwindow for an enlarged part of the whole curve. We see from the figures that it

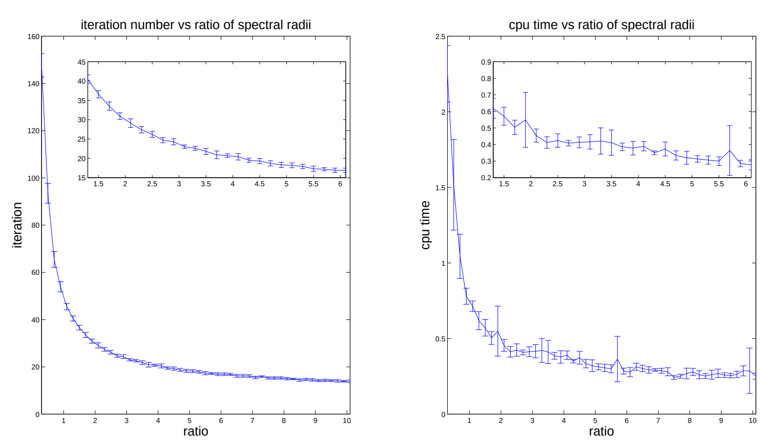

FIG. 2. *The number of iterations (left), and the CPU time (right) along with the ratios of the spectral radii (Case* I*).*

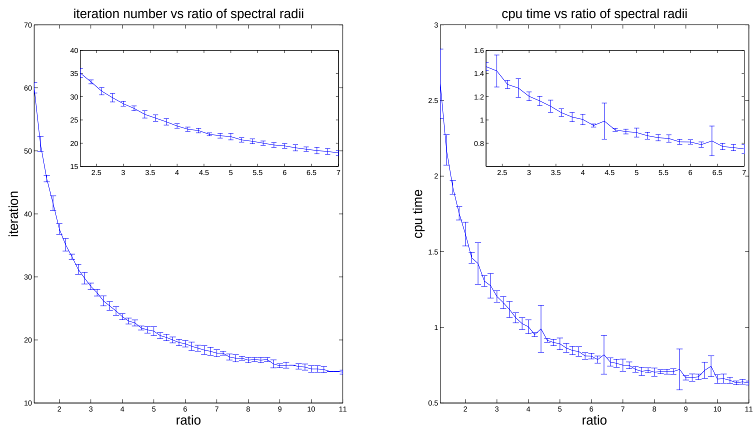

FIG. 3. *The number of iterations (left), and the CPU time (right) along with the ratios of the spectral radii (Case* II*).*

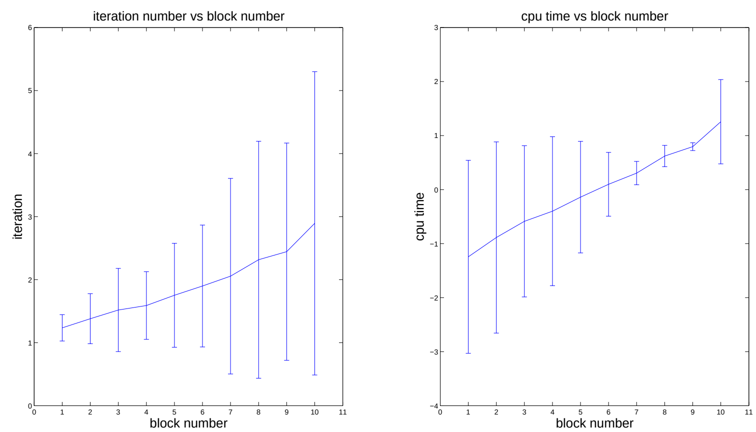

FIG. 4. *The number of iterations (left), and the CPU time (right) along with the number of blocks.*

is more efficient to compute a positive Perron vector $\mathbf{x}$ for a larger ratio of the spectral radii. This is reasonable, for with larger Rt, $\max\{|x_i| : i \in I_1 \cup I_2 \cup I_3\}$ is smaller. Thus, $\mathbf{x}$ is closer to the initial point.

Example 4. This example tested tensors generated similarly to Example 3, with the only difference being that the partitions are

$$|I_1| = \cdots = |I_i| = 2 \text{ and } |I_{i+1}| = 10 \text{ for } i \in [10]$$

with again a unique genuine weakly irreducible block I_{i+1}. The parameters were chosen as follows:

$$\mathrm{Rt} = 2 \text{ and } \gamma = 0.5 \times 10^{-4} i \text{ for all } i \in [10].$$

The information was recorded in Figure 4. The magnitude is of logarithmic order. We see that the computational effort increases exponentially along with the increase of the number of blocks.

In conclusion, we see that Algorithm 3 works very well. The choice for the initial point for the fixed iteration in Algorithm 3 should affect the performance dramatically.

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REFERENCES

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., 18 (1976), pp. 620–709.

[2] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics. Appl. Math. 9, SIAM, Philadelphia, 1994.

[3] K. C. Chang, K. Pearson, and T. Zhang, *Perron-Frobenius theorem for nonnegative tensors*, Commun. Math. Sci., 6 (2008), pp. 507–520.

[4] K. C. Chang, L. Qi, and T. Zhang, *A survey on the spectral theory of nonnegative tensors*, Numer. Linear Algebra Appl., 20 (2013), pp. 891–912.

[5] J. Cooper and A. Dutle, *Spectra of uniform hypergraphs*, Linear Algebra Appl., 436 (2012), pp. 3268–3292.

[6] W. Ding, L. Qi, and Y. Wei, *M-Tensors and nonsingular M-tensors*, Linear Algebra Appl., 439 (2013), pp. 3264–3278.

[7] W. Ding and Y. Wei, *Generalized tensor eigenvalue problems*, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 1073–1099.

[8] W. Ding and Y. Wei, *Solving multi-linear systems with M-tensors*, J. Sci. Comput., 68 (2016), pp. 689–715, doi:10.1007/s10915-015-0156-7.

[9] S. Friedland, S. Gaubert, and L. Han, *Perron-Frobenius theorem for nonnegative multilinear forms and extensions*, Linear Algebra Appl., 438 (2013), pp. 738–749.

[10] G. Frobenius, *Üeber Matrizen aus nicht negativen Elementen*, Kónigliche Akademiefder Wissenschaften, Berlin, 1912, pp. 456–477.

[11] D. F. Gleich, L.-H. Lim, and Y. Yu, *Multilinear PageRank*, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 1507–1541.

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[13] S. Hu, Z.H. Huang, and L. Qi, *Strictly nonnegative tensors and nonnegative tensor partition*, Sci. China Ser. A Math., 57 (2014), pp. 181–195.

[14] R. Kannan, N. Shaked-Monderer, and A. Berman, *Some properties of strong H-tensors and general H-tensors*, Linear Algebra Appl., 476 (2015), pp. 42–55.

[15] L.-H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, in Computational Advances in Multi-Sensor Adaptive Processing, 2005 1st IEEE International Workshop on Computational Advances in Multi-Sensor Addaptive Processing (CAMSAP-05), IEEE, Piscataway, NJ, 2005, pp. 129–132.

[16] L. Oeding and G. Ottaviani, *Eigenvectors of tensors and algorithms for Waring decomposition*, J. Symbolic Comput., 64 (2013), pp. 9–35.

[17] L. Page, S. Brin, R. Motwani, and T. Winograd, *The PageRank Citation Ranking: Bringing Order to the Web*, http://ilpubs.stanford.edu:8090/422/.

[18] K. J. Pearson, *Essentially positive tensors*, Internat. J. Algebra, 4 (2010), pp. 421–427.

[19] O. Perron, *Zur Theorie der Matrices*, Math. Ann., 64 (1907), pp. 248–263.

[20] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 40 (2005), pp. 1302–1324.

[21] L. Qi, *H^+-eigenvalues of Laplacian and signless Lapaclian tensors*, Commun. Math. Sci., 12 (2014), pp. 1045–1064.

[22] U. G. Rothblum, *Nonnegative matrices and stochastic matrices*, in Handbook of Linear Algebra, 2nd Ed., L. Hogben ed., Chapman and Hall/CRC, London, 2014, pp. 137–162.

[23] Q. Yang and Y. Yang, *Further results for Perron–Frobenius theorem for nonnegative tensors* II, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 1236–1250.

[24] Y. Yang and Q. Yang, *Further results for Perron–Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2517–2530.

[25] L. Zhang, L. Qi, and Y. Xu, *Linear convergence of the LZI algorithm for weakly positive tensors*, J. Comput. Math., 20 (2012), pp. 24–33.

[26] L. Zhang, L. Qi, and G. Zhou, *M-tensors and some applications*, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 437–452.