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# COMPLETELY POSITIVE TENSORS: PROPERTIES, EASILY CHECKABLE SUBCLASSES, AND TRACTABLE RELAXATIONS*

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**Abstract.** The completely positive (CP) tensor verification and decomposition are essential in tensor analysis and computation due to the wide applications in statistics, computer vision, exploratory multiway data analysis, blind source separation, and polynomial optimization. However, it is generally NP-hard as we know from its matrix case. To facilitate the CP tensor verification and decomposition, more properties for the CP tensor are further studied, and a great variety of its easily checkable subclasses such as the positive Cauchy tensors, the symmetric Pascal tensors, the Lehmer tensors, the power mean tensors, and all of their nonnegative fractional Hadamard powers and Hadamard products are exploited in this paper. Particularly, a so-called CP-Vandermonde decomposition for positive Cauchy–Hankel tensors is established and a numerical algorithm is proposed to obtain such a special type of CP decomposition. The doubly nonnegative (DNN) matrix is generalized to higher-order tensors as well. Based on the DNN tensors, a series of tractable outer approximations are characterized to approximate the CP tensor cone, which serve as potential useful surrogates in the corresponding CP tensor cone programming arising from polynomial programming problems.

**Key words.** completely positive tensors, sum-of-squares tensors, doubly nonnegative tensors, positive Cauchy tensors, Lehmer tensors, completely positive Vandermonde decomposition

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## 1. Introduction.

Completely positive (CP) matrices have attracted considerable attention due to their applications in optimization, especially in creating convex formulations of NP-hard problems, such as the quadratic assignment problem in combinatorial optimization and the polynomial optimization problems (see [2, 21, 3, 7] and references therein). In recent years, an emerging interest in the assets of multilinear algebra has been concentrated on the higher-order tensors, which serve as a numerical tool, complementary to the arsenal of existing matrix techniques. In this vein, the concept of CP matrices has been extended to higher-order CP tensors, which are connected with nonnegative tensor factorization and have wide applications in statistics, computer vision, exploratory multiway data analysis, blind source separation, and higher degree polynomial optimization [14, 19, 33, 35]. As an extension of the CP matrix, a CP tensor admits its definition in a pretty natural way as initiated by Qi, Xu, and Xu in [33] and recalled below.

DEFINITION 1.1. *A real tensor $\mathcal{A}$ of order $m$ and dimension $n$ is said to be a* CP tensor *if there exist an integer $r$ and some $n$-dimensional nonnegative real vector $u^{(k)}$, $k = 1, \ldots, r$, such that $\mathcal{A} = \sum_{k=1}^{r}(u^{(k)})^m$, where $(u^{(k)})^m = (u_{i_1}^{(k)} \cdots u_{i_m}^{(k)})$ is rank-one*

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*tensor generated by* $u^{(k)}$*. If further all the involved vectors* $u^{(k)}$*'s can span the entire* $n$*-dimensional Euclidean space, then* $\mathcal{A}$ *is said to be a* strongly completely positive tensor (SCP tensor)*.*

The new defined concept in the second part of Definition 1.1 can be regarded as a counterpart tailored for all order tensors comparing to the positive definiteness solely customized for even-order tensors.

Practical applications have triggered the research on CP tensors both on theoretical analysis and numerical computations. From [28, 30, 33], it has been known that all CP tensors contribute a closed pointed convex cone in symmetric tensor space, associated with the copositive tensor cone as its dual. Due to the NP-hardness of the CP matrix verification, it is apparent that to check the membership of the CP tensor cone is a hard task. Spectral properties, dominance properties, the Hadamard product preservation property, and even a special structured subclass were proposed for CP tensors [30, 33], which can be served as necessary or sufficient conditions for CP tensor verification. An optimization algorithm based on semidefinite relaxation was even proposed by Fan and Zhou in their recent work [19], from which either a certificate would be provided for non-CP tensors or a numerical CP decomposition would be obtained. Additionally, numerical optimization for the best fit of CP tensors with given length of decomposition was formulated as a nonnegative constrained least-squares problem in Kolda's more recent paper [22].

In this paper, more properties will be emphasized and exploited for CP tensors, and a series of easily checkable structured CP tensors will be discussed as subclasses of CP tensors. All these will to some extent facilitate the CP tensor verification and decomposition. For example, as a noteworthy observation, the zero-entry dominance property turns out to be a very powerful tool to exclude some higher-order tensors, such as the well-known signless Laplacian tensors of nonempty $m$-uniform hypergraphs with $m \geq 3$, from the class of CP tensors. More importantly, besides the subclass of strongly symmetric hierarchically dominated nonnegative tensors as introduced in [33], more subclasses of CP tensors of any order (even or odd) are provided, which include positive Cauchy tensors, (generalized) symmetric Pascal tensors, (generalized) Lehmer tensors, power mean tensors, and their nonnegative fractional Hadamard powers and Hadamard products. These easily checkable subclasses will certainly provide checkable sufficient conditions for the CP tensor verification and hence serve as a rich variety of testing instances for CP decomposition methods. As a more special type of structured CP tensor, the positive Cauchy–Hankel tensor is proved to admit a CP decomposition in a nonnegative Vandermonde manner. An algorithm is then proposed to pursue this special CP decomposition for low-order low-dimensional cases.

It is known that optimization programming over CP tensor cones was employed to reformulate polynomial optimization problems which are not necessarily quadratic [28]. In spite of the better tightness of CP cone relaxation comparing to the well-known positive semidefinite relaxation, the former one is still not efficiently tractable especially for large scale cases. As a popular relaxation strategy, the doubly nonnegative (DNN) matrix cone is always treated as a surrogate to the CP matrix cone [39], since testing the membership of the DNN matrix cone can be done in $O(n^3)$ for $n \times n$ matrices. Inspired by this relaxation scheme, a tensor counterpart for the DNN matrix is introduced based on sum-of-squares (SOS) tensors which can be verified in polynomial time via semidefinite programming [23, 24], and a series of tractable outer approximations for CP tensor cones are proposed by employing a similar idea from [17], for potential useful surrogates of CP tensor cone programming arising from polynomial programming problems.

The rest of the paper is organized as follows. In section 2, we briefly review some related concepts and properties on symmetric tensors. Basic properties on general CP tensors and SCP tensors will be presented in section 3. Several easily checkable subclasses of CP tensors are provided and discussed in section 4. Section 5 is devoted to the CP decomposition with rank-one Vandermonde tensor terms for positive Cauchy–Hankel tensors, supplied with a numerical algorithm for achieving such a special CP decomposition. Tractable relaxations for the CP tensor cone are developed in section 6 based on the DNN tensors, and concluding remarks are drawn in section 7.

Some notation that will be used throughout the paper is listed here. Denote $[n] := \{1, 2, \ldots, n\}$. The $n$-dimensional real Euclidean space is denoted by $\mathbb{R}^n$, where $n$ is a given natural number. The nonnegative orthant in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, with the interior $\mathbb{R}^n_{++}$ consisting of all positive vectors. The $n$-by-$l$ real matrix space is denoted by $\mathbb{R}^{n\times l}$. Vectors are denoted by lowercase letters such as $x$, $u$, matrices are denoted by capital letters such as $A$, $P$, and tensors are written as calligraphic capital letters such as $\mathcal{A}$, $\mathcal{B}$. For any $i \in [n]$, $e^{(i)}$ denotes the $i$th column vector of the identity matrix. The space of all real $m$th order $n$-dimensional tensors is denoted by $\mathbb{T}_{m,n}$, and the space of all symmetric tensors in $\mathbb{T}_{m,n}$ is denoted by $\mathbb{S}_{m,n}$. For a subset $\Gamma \subseteq [n]$, $|\Gamma|$ stands for its cardinality. Adopting the notation in the literature (see, e.g., [33]), $CP_{m,n}$ and $COP_{m,n}$ are used to denote the sets of all CP tensors and all copositive tensors of order $m$ and dimension $n$, respectively. In addition, $SCP_{m,n}$ and $SCOP_{m,n}$ are used to stand for the set of all strongly CP tensors and the strictly copositive tensors, respectively. The set of all symmetric positive semidefinite (positive definite) tensors is denoted by $PSD_{m,n}$ ($PD_{m,n}$) for convenience. The sets of all DNN tensors and SOS tensors of order $m$ and dimension $n$ are denoted by $DNN_{m,n}$ and $SOS_{m,n}$. $m$ is restricted to be even in the cases of $PSD_{m,n}$, $PD_{m,n}$, and $SOS_{m,n}$.

## 2. Preliminaries.

Some basic concepts for symmetric tensors are recalled in this section. Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$ be an $m$th order $n$-dimensional real tensor. $\mathcal{A}$ is called a *symmetric tensor* if the entries $a_{i_1\ldots i_m}$ are invariant under any permutation of their indices for all $i_j \in [n]$ and $j \in [m]$. A symmetric tensor $\mathcal{A}$ is said to be *positive semidefinite (definite)* if $\mathcal{A}x^m := \sum_{i_1,\ldots,i_m\in[n]} a_{i_1\ldots i_m} x_{i_1}\cdots x_{i_m} \geq 0 (> 0)$ for any $x \in \mathbb{R}^n \backslash \{0\}$. Here, $x^m$ is a rank-one tensor in $\mathbb{S}_{m,n}$ defined as $(x^m)_{i_1\ldots i_m} := x_{i_1}\cdots x_{i_m}$ for all $i_1, \ldots, i_m \in [n]$. Evidently, when $m$ is odd, $\mathcal{A}$ could not be positive definite and $\mathcal{A}$ is positive semidefinite if and only if $\mathcal{A} = O$, where $O$ stands for the zero tensor. A tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is said to be (strictly) copositive if $\mathcal{A}x^m \geq 0$ $(> 0)$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$. The definitions on eigenvalues of symmetric tensors are recalled as follows.

Definition 2.1 (see [29]). *Let $\mathcal{A} \in \mathbb{S}_{m,n}$ and $\mathbb{C}$ be the complex field. We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenvalue-eigenvector pair of $\mathcal{A}$ if $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$, where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are all $n$-dimensional column vectors given by*

$$(2.1)\qquad \left(\mathcal{A}x^{m-1}\right)_i := \sum_{i_2,\ldots,i_m\in[n]} a_{ii_2\ldots i_m} x_{i_2}\cdots x_{i_m}, \quad \left(x^{[m-1]}\right)_i = x_i^{m-1}, \quad \forall i \in [n].$$

*If the eigenvalue $\lambda$ and the eigenvector $x$ are real, then $\lambda$ is called an $H$-eigenvalue of $\mathcal{A}$ and $x$ an $H$-eigenvector of $\mathcal{A}$ associated with $\lambda$.*

Definition 2.2 (see [29]). *Let $\mathcal{A} \in \mathbb{S}_{m,n}$ and $\mathbb{C}$ be the complex field. We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an $E$ eigenvalue-eigenvector pair of $\mathcal{A}$ if $\mathcal{A}x^{m-1} = \lambda x$*

*and* $x^T x = 1$, *where* $\mathcal{A}x^{m-1}$ *is defined as in* (2.1). *If the E-eigenvalue* $\lambda$ *and the E-eigenvector* $x$ *are real, then* $\lambda$ *is called a Z-eigenvalue of* $\mathcal{A}$ *and* $x$ *a Z-eigenvector of* $\mathcal{A}$ *associated with* $\lambda$.

Some related algebraic operations for tensors are reviewed to close this section. For any given $\mathcal{A} = (a_{i_1\ldots i_m})$, $\mathcal{B} = (b_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$, the *Hadamard product* of $\mathcal{A}$ and $\mathcal{B}$ is defined as $(a_{i_1\ldots i_m} b_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$, termed as $\mathcal{A} \circ \mathcal{B}$. For any given nonnegative $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$ and any given nonnegative real scalar $\alpha$, we can also define the corresponding *nonnegative fractional Hadamard power* as $\mathcal{A}^{\circ\alpha} := (a^{\alpha}_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$. For any given nonnegative matrix $P = (p_{ij}) \in \mathbb{R}^{l\times n}$, we can define a linear transformation as follows:

$$P^m(\mathcal{A}) := \left( \sum_{j_1,\ldots,j_m \in [n]} a_{j_1\cdots j_m} p_{i_1 j_1} \cdots p_{i_m j_m} \right) \in \mathbb{S}_{m,l} \ \forall \mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{S}_{m,n}. \tag{2.2}$$

**3. Basic properties of completely positive tensors.** CP tensors play an important role in combinatorial optimization and polynomial optimization problems [28]. It has been shown that the set of all CP tensors form a closed, pointed, convex, full-dimensional cone $CP_{m,n}$ [28, Proposition 1] with its dual cone $COP_{m,n}$ consisting of all copositive tensors [28, 33]. Obviously, these two cones are dual to each other and are both actually closed, convex, pointed, and full-dimensional. To test the membership of $CP_{m,n}$, Fan and Zhou [19] provided an optimization algorithm based on semidefinite relaxation. Besides the Fan–Zhou algorithmic verification, properties for CP-tensors can sometimes help us to verify the complete positivity of tensors more directly, such as to exclude the tensor in question from $CP_{m,n}$, or to ensure the membership under certain algebraic operations that preserve the complete positivity for tensors. This section mainly focuses on this type of verification issues.

Definition 3.1. *A tensor* $\mathcal{A} \in \mathbb{S}_{m,n}$ *is said to have the* zero-entry dominance property *if* $a_{i_1\ldots i_m} = 0$ *implies that* $a_{j_1\ldots j_m} = 0$ *for any* $(j_1, \ldots, j_m)$ *satisfying* $\{j_1, \ldots, j_m\} \supseteq \{i_1, \ldots, i_m\}$.

Proposition 3.2 (see [33, Theorem 3]). *If* $\mathcal{A}$ *is a CP tensor, then* $\mathcal{A}$ *has the zero-entry dominance property.*

Utilizing the zero-entry dominance property, we can exclude some special symmetric nonnegative tensors from the class of CP tensors very efficiently. A typical example is the following signless Laplacian tensor of a uniform hypergraph.

Definition 3.3 (see [31]). *Let* $G = (V, E)$ *be an* $m$*-uniform hypergraph. The* adjacency tensor *of* $G$ *is defined as the* $m$*th order* $n$*-dimensional tensor* $\mathcal{A}$ *whose* $(i_1, \ldots, i_m)$*th entry is*

$$a_{i_1\ldots i_m} = \begin{cases} \frac{1}{(m-1)!} & \text{if } \{i_1, \ldots, i_m\} \in E; \\ 0 & \text{otherwise.} \end{cases}$$

*Let* $\mathcal{D}$ *be an* $m$*th order* $n$*-dimensional diagonal tensor with its diagonal element* $d_{i\ldots i}$ *being* $d_i$*, the degree of vertex* $i$*, for any* $i \in [n]$*. Then* $\mathcal{Q} := \mathcal{D} + \mathcal{A}$ *is called the* signless Laplacian tensor *of the hypergraph* $G$.

Note that signless Laplacian tensors are symmetric nonnegative tensors.

Proposition 3.4. *The signless Laplacian tensor of a nonempty uniform* $m$*-hypergraph for* $m \geq 3$ *is not CP.*

*Proof.* Suppose that $m \geq 3$ and $G$ is a nonempty uniform $m$-hypergraph. Suppose that $(j_1, \ldots, j_m)$ is an edge of $G$. Let $\mathcal{Q} = (q_{i_1 \cdots i_m}) \in \mathbb{S}_{m,n}$ be the signless Laplacian tensor of $G$. By definition, $q_{j_1 \ldots j_m} = \frac{1}{(m-1)!} \neq 0$. Note that $q_{j_1 j_1 \ldots j_1 j_2} = 0$ by Definition 3.3. Obviously, the zero-entry dominance property fails and hence $\mathcal{Q}$ is not CP. ∎

The zero-entry dominance works well for the CP verification of some Hankel tensors, whose definition is recalled here.

Definition 3.5 (see [32]). *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$. If there is a vector $v = (v_0, \ldots, v_{(n-1)m})^T \in \mathbb{R}^{(n-1)m+1}$ such that*

$$a_{i_1 \ldots i_m} = v_{i_1 + \cdots + i_m - m}, \ \forall i_j \in [n], \ j \in [m],$$

*then we say that $\mathcal{A}$ is an $m$th order $n$-dimensional* Hankel tensor*. Let $t := \lfloor \frac{(n-1)m+2}{2} \rfloor + 1$ and $A = (a_{ij})$ be a $t \times t$ Hankel matrix with $a_{ij} := v_{i+j-2}$, where $v_{2t}$ is an additional number when $(n-1)m$ is odd. If $A$ is positive semidefinite, then $\mathcal{A}$ is called a* strong Hankel tensor*. Suppose $\mathcal{A}$ is a Hankel tensor with its Vandermonde decomposition $\mathcal{A} = \sum_{k=1}^r \alpha_k (u^{(k)})^m$, where $u^{(k)} := (1, \xi_k, \ldots, \xi_k^{n-1})^T$, $\xi_k \in \mathbb{R}$, for all $k \in [r]$. If $\alpha_k > 0$ for all $k \in [r]$, then $\mathcal{A}$ is called a* complete Hankel tensor*.*

Proposition 3.6. *Let $\mathcal{A} \in \mathbb{S}_{m,n}$ be a Hankel tensor, and $v = (v_0, \ldots, v_{(n-1)m})^T \in \mathbb{R}^{(n-1)m+1}$ be its generating vector.*

(i) *If $v_0 = v_{(n-1)m} = 0$, then $\mathcal{A} \in CP_{m,n}$ if and only if $\mathcal{A} = O$.*
(ii) *If $\mathcal{A} \in CP_{m,n}$ and $v_{(i-1)m} = 0$ for some $2 \leq i \leq n-1$, then $v_0 \geq 0$, $v_{(n-1)m} \geq 0$, and $\mathcal{A} = v_0 e_1^m + v_{(n-1)m} e_n^m$.*
(iii) *If $v_0 = 0$ and $v_j \neq 0$ for some $j \in [m-1]$, then $\mathcal{A}$ is not CP.*

*Proof.* (i) It is trivial that $O$ is a CP tensor. If $\mathcal{A}$ is CP and $v_0 = v_{(n-1)m} = 0$, then $a_{1 i_2 \ldots i_m} = 0$ for all $i_2, \ldots, i_m \in [n]$. Thus $v_j = 0$ for any $j \in [(m-1)n + 1 - m]$. When $n = 2$, then $\mathcal{A} = O$. When $n \geq 3$, $n \geq 2 + \frac{1}{m-1}$, which implies that $m \leq (m-1)n + 1 - m$. Thus $a_{2 \ldots 2} = 0$. Using the zero-entry dominance property again, we get $v_j = 0$ for all $j \in [(m-1)n + 2 - m]$. Keeping on doing this, we can find that for any given $k \in [n-1]$, $v_j = 0$ for all $j \in [(m-1)n + k - 1 - m]$; then $a_{k \ldots k} = 0$ since $n \geq k + \frac{1}{m-1}$. The zero-entry dominance property and the fact $v_{(n-1)m} = 0$ finally give us $\mathcal{A} = O$. Thus (i) is obtained. Using a similar proof as in (i), we can prove that $a_{k \ldots k} = 0$ for all $k = 2, \ldots, n-1$. Thus, the zero-entry dominance property shows $\mathcal{A} = v_0 e_1^m + v_{(n-1)m} e_n^m$. By the nonnegativity of $\mathcal{A}$, $v_0$ and $v_{(n-1)m}$ are nonnegative. This implies the assertion in (ii). By Definition 3.5, we know that $a_{1 \ldots 1} = 0$. The hypothesis $v_j \neq 0$ for some $j \in [m-1]$ tells us that $a_{i_1 \ldots i_m} \neq 0$ for all $i_1, \ldots, i_m \in [n]$ satisfying that $i_1 + \cdots + i_m = j + m \leq 2m - 1$. In all these cases, $1 \in \{i_1, \ldots, i_m\}$. Thus, the zero-entry dominance property fails and hence $\mathcal{A}$ is not a CP tensor from Proposition 3.2. This leads to (iii). ∎

In [13], the Toeplitz matrix has been generalized to high-order tensors, where a tensor $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ is called an $m$th order $n$-dimensional *Toeplitz tensor* if for all $i_j \in [n-1]$ and all $j \in [m]$, $a_{i_1 \ldots i_m} = a_{i_1 + 1 \ldots i_m + 1}$.

Proposition 3.7. *Let $\mathcal{A}$ be a Toeplitz tensor with its diagonal entry* 0*. Then $\mathcal{A}$ is CP if and only if $\mathcal{A} = O$.*

*Proof.* The sufficiency is trivial. If $\mathcal{A}$ is CP and $a_{1 \ldots 1} = 0$, by the definition of Toeplitz tensors, we have $a_{i \ldots i} = 0$ for all $i \in [n]$. Invoking the zero-entry dominance property in Proposition 3.2, it follows that $\mathcal{A} = O$. ∎

The Hadamard product preserves the complete positivity as shown in [33, Proposition 1]. It also preserves the strong complete positivity as stated below. The Hadamard product preservation property for CP tensors plays an important role in identifying some easily checkable subclasses of CP tensors as we will see in section 4. Additionally, it is worth pointing out that the Hadamard product does not preserve positive semidefiniteness and the SOS property, as shown by examples in [31] and [26]. Thus, it is an important feature of CP and SCP tensors.

PROPOSITION 3.8. *For any given two SCP tensors $\mathcal{A}$, $\mathcal{B} \in \mathbb{S}_{m,n}$, $\mathcal{A} \circ \mathcal{B}$ is also an SCP tensor.*

*Proof.* We first claim that if $U = (u^{(1)}\ u^{(2)} \dots u^{(n)})$, $V = (v^{(1)}\ v^{(2)} \dots v^{(n)})$ are any two nonsingular matrices in $\mathbb{R}^{n\times n}$, then

$$span\{u^{(1)} \circ v^{(1)}, u^{(1)} \circ v^{(2)}, \dots, u^{(1)} \circ v^{(n)}, u^{(2)} \circ v^{(1)}, u^{(2)} \circ v^{(2)}, \dots, u^{(n)} \circ u^{(n)}\} = \mathbb{R}^n. \tag{3.1}$$

The nonsingularity of $U$ indicates that $u^{(1)}, u^{(2)}, \dots, u^{(n)}$ can form a basis for $\mathbb{R}^n$. Thus, we can find $a_{ik}$, $i$, $k \in [n]$, such that

$$e^{(i)} = \sum_{k=1}^{n} a_{ik} u^{(k)} \ \forall i \in [n], \tag{3.2}$$

where there exists at least one nonzero element among $a_{i1}, \dots, a_{in}$ for any $i \in [n]$. The equalities in (3.2) derive that

$$e^{(i)} \circ v^{(j)} = \sum_{k=1}^{n} a_{ik} \left( u^{(k)} \circ v^{(j)} \right) \ \forall i,\ j \in [n]. \tag{3.3}$$

By the nonsingularity of $V$, it follows that $0 \neq \det(V) = \sum_{\sigma \in S_n} sgn(\sigma) \Pi_{i=1}^n v_{\sigma_i}^{(i)}$, where $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of $[n]$, $sgn(\sigma)$ is the signature of $\sigma$, and $S_n$ is the set of all permutations of $[n]$. Thus, there always exists some permutation $\sigma$ such that $\Pi_{i=1}^n v_{\sigma_i}^{(i)} \neq 0$. Without loss of generality, we assume that $\sigma = (1, 2, \dots, n)$, that is, $v_i^{(i)} \neq 0$ for all $i \in [n]$. This together with (3.3) yields that

$$v_i^{(i)} e^{(i)} = \sum_{k=1}^{n} a_{ik} \left( u^{(k)} \circ v^{(i)} \right) \ \forall i \in [n],$$

which indicates that $e^{(i)} = \sum_{k=1}^n \frac{a_{ik}}{v_i^{(i)}} (u^{(k)} \circ v^{(i)})$ for all $i \in [n]$. Thus, $\{e^{(1)}, \dots, e^{(n)}\}$ can be linearly expressed by $\{u^{(i)} \circ v^{(j)}\}_{i,j=1,\dots,n}$, and our claim is proved. Now we consider any two SCP tensors $\mathcal{A}$ and $\mathcal{B}$ with their corresponding nonnegative rank-one decompositions $\mathcal{A} = \sum_{i=1}^r (u^{(i)})^m$, and $\mathcal{B} = \sum_{j=1}^{r'} (v^{(j)})^m$, where $span\{u^{(1)}, \dots, u^{(r)}\} = span\{v^{(1)}, \dots, v^{(r')}\} = \mathbb{R}^n$. Easily we can verify that $\mathcal{A} \circ \mathcal{B} = \sum_{i=1}^r \sum_{j=1}^{r'} (u^{(i)} \circ v^{(j)})^m$. The involved $u^{(i)} \circ v^{(j)}$ is certainly nonnegative by the nonnegativity of $u^{(i)}$ and $v^{(j)}$ for all $i$, $j \in [n]$. Note that $r$ and $r'$ should be no less than $n$. Therefore, we can always pick up $n$ vectors from $\{u^{(1)}, \dots, u^{(r)}\}$ to form a basis of $\mathbb{R}^n$. Let's simply say they are $u^{(1)}, \dots, u^{(n)}$. Similarly, we can do this to $v^{(1)}, \dots, v^{(r')}$ and get $n$ linearly independent vectors, namely, $v^{(1)}, \dots, v^{(n)}$. The aforementioned claim tells us that all involved vectors $u^{(i)} \circ v^{(j)}$, $i$, $j \in [n]$, can span the whole space $\mathbb{R}^n$, which means $\mathcal{A} \circ \mathcal{B}$ is strongly CP. ∎

The spectral property on SCP tensors is proposed which reveals the phenomenon that among CP tensors, the SCP tensors serve as the counterpart of PD tensors among PSD tensors. This is the primary motivation to introduce this new concept. By adopting the notation $\bar{E}_{m,n}(H) := \{\mathcal{A} \in \mathbb{S}_{m,n} : \text{all } H\text{-eigenvalues of } \mathcal{A} \text{ are nonzero}\}$ and $\bar{E}_{m,n}(Z) := \{\mathcal{A} \in \mathbb{S}_{m,n} : \text{all } Z\text{-eigenvalues of } \mathcal{A} \text{ are nonzero}\}$, we know that for any even integer $m \geq 2$, $PD_{m,n} = PSD_{m,n} \cap \bar{E}_{m,n}(H)$, and $PD_{m,n} = PSD_{m,n} \cap \bar{E}_{m,n}(Z)$. The relation between $CP_{m,n}$ and $SCP_{m,n}$ are exactly the same, which is even valid for both even and odd-order cases.

THEOREM 3.9. *$CP_{m,n} \cap \bar{E}_{m,n}(H) = SCP_{m,n}$, $CP_{m,n} \cap \bar{E}_{m,n}(Z) = SCP_{m,n}$.*

*Proof.* Suppose that $\mathcal{A}$ is an SCP tensor. Write $\mathcal{A}$ as $\mathcal{A} = \sum_{k=1}^r (u^{(k)})^m$, where $u^{(k)} \in \mathbb{R}_+^n$ and

$$span\{u^{(1)}, \ldots, u^{(r)}\} = \mathbb{R}^n. \tag{3.4}$$

Assume on the contrary that $\mathcal{A}$ has $\lambda = 0$ as one of its $H$-eigenvalues, and the corresponding $H$-eigenvector is $x$. Certainly $x \neq 0$. When $m$ is even, we have

$$0 = \lambda \sum_{i=1}^n x_i^m = \mathcal{A}x^m = \sum_{k=1}^r \left(x^T u^{(k)}\right)^m.$$

The nonnegativity of each term in the summation on the right-hand side immediately leads to $x^T u^{(k)} = 0$ for all $k \in [r]$. Invoking the condition in (3.4), $x$ has no choice but 0, which comes to a contradiction since $x$ is an $H$-eigenvector. Thus, all $H$-eigenvalues of $\mathcal{A}$ are nonzero when the order is even. When $m$ is odd, we have

$$0 = \lambda x_i^{m-1} = \left(\mathcal{A}x^{m-1}\right)_i = \sum_{k=1}^r \left(x^T u^{(k)}\right)^{m-1} u_i^{(k)} \ \forall i \in [n].$$

Together with the involved nonnegativity of each term in the summation on the right-hand side, it yields that

$$\left(x^T u^{(k)}\right)^{m-1} u_i^{(k)} = 0 \ \forall i \in [n]. \tag{3.5}$$

In addition, the condition (3.4) implies that we can pick $n$ vectors from the set $\{u^{(1)}, \ldots, u^{(r)}\}$ to span the whole space $\mathbb{R}^n$. Without loss of generality, let's say they are $u^{(1)}, \ldots, u^{(n)}$. Trivially, for any $k \in [n]$, $u^{(k)} \neq 0$. Therefore, there always exists an index $i_k \in [n]$ such that $u_{i_k}^{(k)} \neq 0$. Thus (3.5) implies that $x^T u^{(k)} = 0$ for all $k \in [n]$. This immediately leads to $x = 0$. The same contradiction arrives and hence all $H$-eigenvalues of $\mathcal{A}$ should be nonzero when $m$ is odd. On the other hand, we take any tensor $\mathcal{A} \in CP_{m,n} \cap \bar{E}_{m,n}(H)$ with its nonnegative rank-one decomposition as $\mathcal{A} = \sum_{k=1}^r (u^{(k)})^m$ with $u^{(k)} \in \mathbb{R}_+^n$ for all $k \in [r]$. Assume on the contrary that $\mathcal{A} \notin SCP_{m,n}$, which means $span\{u^{(1)}, \ldots, u^{(r)}\} \neq \mathbb{R}^n$. Thus, there exists an $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^T u^{(k)} = 0$ for all $k \in [r]$. This immediately gives us $\mathcal{A}x^m = \sum_{k=1}^r (x^T u^{(k)})^m = 0$, which is actually a contradiction to $\mathcal{A} \in \bar{E}_{m,n}(H)$ since 0 is now an $H$-eigenvalue of $\mathcal{A}$. Similarly, we can prove the $Z$-eigenvalue case. This completes the proof. ∎

An immediate observation from the nonzero property of $H$-($Z$-)eigenvalues of SCP tensors in the above theorem is that for any CP decomposition of an SCP tensor, the involved nonnegative vectors will always span the entire Euclidean space.

## 4. Easily checkable CP tensor subclasses.

Several structured tensors are introduced and proved to be CP tensors in this section, which serve as easily checkable subclasses of CP tensors.

### 4.1. Positive Cauchy tensors.

DEFINITION 4.1 (see [12]). *Let $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ with $c_i \neq 0$ for all $i \in [n]$. Suppose that $\mathcal{C} = (c_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ is defined as*

$$c_{i_1 \ldots i_m} = \frac{1}{c_{i_1} + \cdots + c_{i_m}} \ \forall i_j \in [n],\ j \in [m].$$

*Then, we say that $\mathcal{C}$ is an $m$th order $n$-dimensional* symmetric Cauchy tensor *and the vector $c$ is called the* generating vector *of $\mathcal{C}$.*

THEOREM 4.2. *Let $\mathcal{C} \in \mathbb{S}_{m,n}$ be a Cauchy tensor and $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ be its generating vector. The following statements are equivalent:*

(i) *$\mathcal{C}$ is CP.*

(ii) *$\mathcal{C}$ is strictly copositive.*

(iii) *$c > 0$.*

(iv) *The function $f_{\mathcal{C}}(x) := \mathcal{C}x^m$ is strictly monotonically increasing in $\mathbb{R}^n_+$.*

*Proof.* The implication "(ii) $\Rightarrow$ (iii)" follows readily from $0 < \mathcal{C}e_i^m = \frac{1}{mc_i}$ for any $i \in [n]$. To get "(iii) $\Rightarrow$ (i)," we can employ the proof in [10, Theorem 3.1] that for any $x \in \mathbb{R}^n$, it yields that

$$\begin{aligned}
\mathcal{C}x^m &= \sum_{i_1,\ldots,i_m \in [n]} c_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m} = \sum_{i_1,\ldots,i_m \in [n]} \frac{x_{i_1} \cdots x_{i_m}}{c_{i_1} + \cdots + c_{i_m}} \\
&= \sum_{i_1,\ldots,i_m \in [n]} \int_0^1 t^{c_{i_1} + \cdots + c_{i_m} - 1} x_{i_1} \cdots x_{i_m} dt \\
&= \int_0^1 \left( \sum_{i_1,\ldots,i_m \in [n]} t^{c_{i_1} + \cdots + c_{i_m} - 1} x_{i_1} \cdots x_{i_m} \right) dt = \int_0^1 \left( \sum_{i=1}^n t^{c_i - \frac{1}{m}} x_i \right)^m dt.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^1 \left( \sum_{i=1}^n t^{c_i - \frac{1}{m}} x_i \right)^m dt &= \lim_{k \to \infty} \sum_{j \in [k]} \left( \sum_{i=1}^n \left( \frac{j}{k} \right)^{c_i - \frac{1}{m}} x_i \right)^m / k \\
&= \lim_{k \to \infty} \sum_{j \in [k]} \left( \sum_{i=1}^n \left( \frac{j}{k} \right)^{c_i - \frac{1}{m}} x_i / k^{\frac{1}{m}} \right)^m =: \lim_{k \to \infty} \sum_{j \in [k]} (\langle u^j, x \rangle)^m
\end{aligned}$$

with

$$u^j := \left( \frac{(j/k)^{c_1 - \frac{1}{m}}}{k^{\frac{1}{m}}}, \ldots, \frac{(j/k)^{c_n - \frac{1}{m}}}{k^{\frac{1}{m}}} \right)^T \in \mathbb{R}^n_+ \ \forall j \in [k].$$

By setting $\mathcal{C}_k := \sum_{j \in [k]} (u^j)^m$, it follows that $\mathcal{C} = \lim_{k \to \infty} \mathcal{C}_k$ and $\mathcal{C}_k \in CP_{m,n}$. The closedness of $CP_{m,n}$ leads to $\mathcal{C} \in CP_{m,n}$. This implies (i). Conversely, if (i) holds, then $\mathcal{C}$ is certainly copositive, which deduces that $0 \leq \mathcal{C}e_i^m = \frac{1}{mc_i}$ for all $i \in [n]$. Thus (iii) holds. Next we prove the equivalence between (iii) and (iv). Assume that (iii) holds; for any distinct $x, y \in \mathbb{R}^n_+$, satisfying $x \geq y$, i.e., there exists an index $i \in [n]$

such that $x_i > y_i$, we have

$$f_{\mathcal{C}}(x) - f_{\mathcal{C}}(y) = \mathcal{C}x^m - \mathcal{C}y^m = \sum_{\substack{i_1,\ldots,i_m \in [n] \\ (i_1,\ldots,i_m) \neq (i,\ldots,i)}} \frac{x_{i_1}\cdots x_{i_m} - y_{i_1}\cdots y_{i_m}}{c_{i_1} + \cdots + c_{i_m}} + \frac{x_i^m - y_i^m}{mc_i} > 0.$$

Thus (iv) is obtained. Conversely, if $f_{\mathcal{C}}(x)$ is strictly monotonically increasing in $\mathbb{R}^n_+$, then for any $i \in [n]$, $0 < f_{\mathcal{C}}(e_i) - f_{\mathcal{C}}(0) = \frac{1}{mc_i}$, which implies that $c > 0$. Besides, by setting $x \in \mathbb{R}^n_+ \backslash \{0\}$ and $y = 0$, the strict monotonically increasing property of $f_{\mathcal{C}}$ also implies that $\mathcal{C}x^m > 0$. Thus (iii) and (ii) hold. ∎

PROPOSITION 4.3. *For any given Cauchy tensor $\mathcal{C} \in \mathbb{T}_{m,n}$ with its generating vector $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$, if $c > 0$, then the following statements are equivalent:*

(i) *$c_1, \ldots, c_n$ are mutually distinct.*

(ii) *$\mathcal{C}$ is strongly CP.*

*Proof.* When $m$ is even, the desired equivalence can be derived from [12, Theorem 2.3] and Theorem 3.9. Now we consider the case that $m$ is odd. To show the implication "(i)⇒ (ii)," we assume on the contrary that 0 is an $H$-eigenvalue of $\mathcal{C}$ with its associated $H$-eigenvector $x$. Then for any $i \in [n]$, we have

$$\begin{aligned} 0 = \left(\mathcal{C}x^{m-1}\right)_i &= \sum_{i_2,\ldots,i_m \in [n]} \frac{x_{i_2}\cdots x_{i_m}}{c_i + c_{i_2} + \cdots + c_{i_m}} \\ &= \sum_{i_2,\ldots,i_m \in [n]} \int_0^1 t^{c_i + c_{i_2} + \cdots + c_{i_m} - 1} x_{i_2} \cdots x_{i_m} dt \\ &= \int_0^1 t^{c_i} \left( \sum_{j \in [n]} t^{c_j - \frac{1}{m-1}} x_j \right)^{m-1} dt, \end{aligned}$$

which implies that $\sum_{j \in [n]} t^{c_j - \frac{1}{m-1}} x_j \equiv 0$ for all $t \in [0, 1]$. Thus,

$$x_1 + t^{c_2 - c_1} x_2 + \cdots + t^{c_n - c_1} x_n = 0 \ \forall t \in [0, 1].$$

By the continuity and the condition that all components of $c$ are mutually distinct, it follows readily that $x_1 = 0$. Then we have $x_2 + t^{c_3 - c_2} x_2 + \cdots + t^{c_n - c_2} x_n = 0$ for all $t \in [0, 1]$, which implies $x_2 = 0$. By repeating this process, we can gradually get $x = 0$, which contradicts to the assumption that $x$ is an $H$-eigenvalue. Thus (ii) is obtained. Conversely, to show "(ii)⇒ (i)," we still assume by contrary that $c_1, \ldots, c_n$ are not mutually distinct. Without loss of generality, we assume that $c_1 = c_2$. By setting $x \in \mathbb{R}^n$ with $x_1 = -x_2 = 1$ and other components 0, we find that for any $i \in [n]$,

$$\begin{aligned} \left(\mathcal{C}x^{m-1}\right)_i = \int_0^1 t^{c_i} \left( \sum_{j \in [2]} t^{c_j - \frac{1}{m-1}} x_j \right)^{m-1} dt \\ = \int_0^1 t^{c_i} \left( t^{c_1 - \frac{1}{m-1}} - t^{c_2 - \frac{1}{m-1}} \right)^{m-1} dt = 0, \end{aligned}$$

which indicates that 0 is an $H$-eigenvalue of $\mathcal{C}$. A contradiction to $\mathcal{C} \in E^{++}_{m,n}$ arrives. This completes the proof. ∎

**4.2. Symmetric Pascal tensors.** It is well-known that Pascal matrices have served as a convenient way to represent the famous Pascal's triangle and have found many applications in numerical analysis, filter design, image and signal processing, probability, combinatorics, numerical analysis, and electrical engineering [1, 8]. The symmetric Pascal matrix can also be extended to higher-order symmetric tensors as follows.

DEFINITION 4.4. *The tensor* $\mathcal{P} = (p_{i_1\cdots i_m}) \in \mathbb{S}_{m,n}$ *is called the* symmetric Pascal tensor *if*

$$p_{i_1\cdots i_m} = \frac{(i_1 + \cdots + i_m - m)!}{(i_1 - 1)! \cdots (i_m - 1)!} \ \forall i_1, \ldots, i_m \in [n].$$

*Furthermore, let* $c = (c_i) \in \mathbb{R}^n$ *be a nonnegative vector. The tensor* $\mathcal{P}^{(c)} = (p^{(c)}_{i_1\cdots i_m}) \in \mathbb{S}_{m,n}$ *is called a* generalized symmetric Pascal tensor *generated by* $c$ *if*

$$p^{(c)}_{i_1\cdots i_m} = \frac{\Gamma(c_{i_1} + \cdots + c_{i_m} + 1)}{\Gamma(c_{i_1} + 1) \cdots \Gamma(c_{i_m} + 1)} \ \forall i_1, \ldots, i_m \in [n],$$

*where* $\Gamma(\cdot)$ *is the gamma function.*

Obviously, by setting $c_i = i - 1$ for all $i \in [n]$, $\mathcal{P}^{(c)}$ reduces to the symmetric Pascal tensor. This is the reason why a "generalized symmetric Pascal tensor" is used for $\mathcal{P}^{(c)}$.

PROPOSITION 4.5. *Let* $c = (c_i) \in \mathbb{R}^n$ *be any given nonnegative vector and* $\mathcal{P}^{(c)} = (p^{(c)}_{i_1\cdots i_m}) \in \mathbb{S}_{m,n}$ *be the corresponding generalized symmetric Pascal tensor. Then* $\mathcal{P}^{(c)}$ *is a CP tensor.*

*Proof.* By employing the following infinite product definition of the gamma function

$$\Gamma(t) = \lim_{k\to\infty} \frac{k!k^t}{t(t+1)\cdots(t+k)} \quad \forall t \geq 0,$$

we can rewrite the entries of $\mathcal{P}^{(c)}$ as

$$p^{(c)}_{i_1\cdots i_m} = \lim_{k\to\infty} \frac{1}{(k\cdot k!)^{m-1}} \Pi_{l=1}^{k+1} \frac{(c_{i_1} + l)\cdots(c_{i_m} + l)}{(c_{i_1} + \cdots + c_{i_m} + l)} \quad \forall i_1, \ldots, i_m \in [n].$$

For any given positive integer $k$ and any integer $l \in [k+1]$, denote

$$P_l := Diag(c_1 + l, \ldots, c_n + l) \in \mathbb{S}_{2,n}, \ \mathcal{C}_l := \left(\frac{1}{c_{i_1} + \cdots + c_{i_m} + l}\right) \in \mathbb{S}_{m,n},$$
$$\mathcal{P}_l := \left(\frac{(c_{i_1} + l)\cdots(c_{i_m} + l)}{(c_{i_1} + \cdots + c_{i_m} + l)}\right) \in \mathbb{S}_{m,n}, \ \mathcal{P}(k) := \frac{1}{(k\cdot k!)^{m-1}} \Pi_{l=1}^{k+1} \frac{(c_{i_1} + l)\cdots(c_{i_m} + l)}{(c_{i_1} + \cdots + c_{i_m} + l)}.$$

It is easy to verify that

$$\mathcal{P}_l = P^m(\mathcal{C}_l), \ \mathcal{P}(k) = \frac{1}{(k\cdot k!)^{m-1}} \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{k+1}, \ \mathcal{P}^{(c)} = \lim_{k\to\infty} \mathcal{P}(k), \tag{4.1}$$

where $P^m(\cdot)$ is defined as in (2.2). In view of Theorem 4.2, [26, Theorem 2.2], [33, Proposition 1], and the closedness of the CP tensor cone, we can get the complete positivity of $\mathcal{P}^{(c)}$ as desired. ∎

**4.3. Lehmer tensors.** The Lehmer matrix, named after D. H. Lehmer, is well-known as an important type of test matrices used to evaluate the accuracy for matrix inversion programs due to fact that their exact matrix inverses are known [20, 27]. It can be naturally extended to higher-order tensors as defined in the following.

DEFINITION 4.6. *The tensor* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{S}_{m,n}$ *with* $a_{i_1\cdots i_m} = \frac{\min\{i_1,\ldots,i_m\}}{\max\{i_1,\ldots,i_m\}}$ *for all* $i_1, \ldots, i_m \in [n]$ *is called the* Lehmer tensor. *Let* $c = (c_i) \in \mathbb{R}^n$ *be a positive vector and* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{S}_{m,n}$ *with* $a_{i_1\cdots i_m} = \frac{\min\{c_{i_1},\ldots,c_{i_m}\}}{\max\{c_{i_1},\ldots,c_{i_m}\}}$ *for all* $i_1, \ldots, i_m \in [n]$. *Then* $\mathcal{A}$ *is called a* generalized Lehmer tensor *generated by* $c$.

PROPOSITION 4.7. *Let* $c = (c_i) \in \mathbb{R}^n$ *be a positive vector and* $\mathcal{A} \in \mathbb{S}_{m,n}$ *be the corresponding generalized Lehmer tensor generated by* $c$. *Then* $\mathcal{A}$ *is a CP tensor. If additionally all components of* $c$ *are mutually distinct, then* $\mathcal{A}$ *is an SCP tensor.*

*Proof.* Without loss of generality, we assume that all components in $c$ are in a nondecreasing order, i.e., $0 < c_1 \leq c_2 \leq \cdots \leq c_n$. Denote

$$\mathcal{B} = (b_{i_1\cdots i_m}) \in \mathbb{S}_{m,n} \text{ with } b_{i_1\cdots i_m} = \frac{1}{\max\{c_{i_1},\ldots,c_{i_m}\}} \ \forall i_1,\ldots,i_m \in [n],$$

and

$$\mathcal{D} = (c_{i_1\cdots i_m}) \in \mathbb{S}_{m,n} \text{ with } c_{i_1\cdots i_m} = \min\{c_{i_1},\ldots,c_{i_m}\} \ \forall i_1,\ldots,i_m \in [n].$$

Apparently, $\mathcal{A} = \mathcal{B} \circ \mathcal{D}$. Observations

$$\mathcal{B} = \frac{1}{c_n}\left(\sum_{i=1}^{n} e_i\right)^m + \left(\frac{1}{c_{n-1}} - \frac{1}{c_n}\right)\left(\sum_{i=1}^{n-1} e_i\right)^m + \cdots + \left(\frac{1}{c_1} - \frac{1}{c_2}\right)(e_1)^m$$

and

$$\mathcal{D} = c_1\left(\sum_{i=1}^{n} e_i\right)^m + (c_2 - c_1)\left(\sum_{i=2}^{n} e_i\right)^m + \cdots + (c_n - c_{n-1})(e_n)^m$$

yield the desired complete positivity of $\mathcal{C}$ since the Hadamard product preserves the complete positivity as shown in [33, Proposition 1]. Here $e_i \in \mathbb{R}^n$ is the $i$th standard basis vector. Furthermore, note that both $\{\sum_{i=1}^{k} e_i\}_{k=1}^{n}$ and $\{\sum_{i=k}^{n} e_i\}_{k=1}^{n}$ can span the entire space $\mathbb{R}^n$. The property that the Hadamard product can also preserve the strongly complete positivity as established in Proposition 3.8 leads to the remaining part of the desired assertion. ∎

COROLLARY 4.8. *The Lehmer tensor is an SCP tensor.*

*Proof.* The desired result follows directly from Proposition 4.7 by setting $c_i = i$ for each $i \in [n]$. ∎

**4.4. The fractional Hadamard powers and power mean tensors.** In the literature of matrix analysis, the fractional Hadamard powers have been introduced to characterize the so-called infinite divisibility for nonnegative symmetric positive semidefinite matrices, in which the positive semidefiniteness maintains for all of their fractional Hadamard powers with all nonnegative real exponents [5]. It is known that the Hadamard product preserves the CP of tensors, which immediately implies that the positive integer Hadamard powers of CP tensors are still CP. A natural question is, Does this property holds for any nonnegative fractional Hadamard powers of CP tensors? The answer is negative since counterexamples can be found in the matrix case: $A = [1\ 1\ 0; 1\ 2\ 1; 0\ 1\ 1] = (1\ 1\ 0)^T(1\ 1\ 0) + (0\ 1\ 1)^T(0\ 1\ 1)$ is a CP matrix, but its 1/2 Hadamard power $A^{\circ\frac{1}{2}}$ has an eigenvalue $\frac{1+\sqrt{2}-\sqrt{11-2\sqrt{2}}}{2} < 0$ and hence is not a CP matrix anymore. Inspired by the infinite divisibility of the positive Cauchy matrix, the Lehmer matrix, and the Pascal matrix [5], we will show that all nonnegative fractional Hadamard powers preserve the complete positivity of the corresponding structured CP tensors. This property comes trivially for (generalized) Lehmer tensors by definition. And for positive Cauchy tensors, we have the following.

THEOREM 4.9. *Let $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ be a positive vector and $\mathcal{C} = (\frac{1}{c_{i_1}+\cdots+c_{i_m}}) \in \mathbb{S}_{m,n}$ be its generated positive Cauchy tensor. For any nonnegative scalar $\alpha \in \mathbb{R}$, the fractional Hadamard power of $\mathcal{C}$, termed as $\mathcal{C}^{\circ\alpha} := (\frac{1}{(c_{i_1}+\cdots+c_{i_m})^\alpha}) \in \mathbb{S}_{m,n}$ is still a CP tensor.*

*Proof.* It is trivial for the case when $\alpha = 0$. For any $\alpha > 0$, using the formula of gamma function

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt \quad \forall \nu \in (0, \infty),$$

we can get

$$\begin{aligned}
\frac{1}{(c_{i_1} + \cdots + c_{i_m})^\alpha} &= \int_0^\infty e^{-t(c_{i_1}+\cdots+c_{i_m})} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \\
&= \int_0^\infty e^{-(c_{i_1}+\cdots+c_{i_m})\left(\tilde{t}\Gamma(\alpha)\alpha\right)^{1/\alpha}} d\tilde{t} \\
&= \int_0^1 e^{-(c_{i_1}+\cdots+c_{i_m})\left(\frac{1-\mu}{\mu}\Gamma(\alpha)\alpha\right)^{1/\alpha}} \frac{d\mu}{\mu^2}.
\end{aligned}$$

Thus, for any $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$,

$$\begin{aligned}
\mathcal{C}^{\circ\alpha} x^m &= \sum_{i_1,\ldots,i_m \in [n]} \int_0^1 e^{-(c_{i_1}+\cdots+c_{i_m})\left(\frac{1-\mu}{\mu}\Gamma(\alpha)\alpha\right)^{1/\alpha}} x_{i_1} \cdots x_{i_m} \frac{d\mu}{\mu^2} \\
&= \int_0^1 \sum_{i_1,\ldots,i_m \in [n]} e^{-(c_{i_1}+\cdots+c_{i_m})\left(\frac{1-\mu}{\mu}\Gamma(\alpha)\alpha\right)^{1/\alpha}} x_{i_1} \cdots x_{i_m} \frac{d\mu}{\mu^2} \\
&= \int_0^1 \left( \sum_{i\in[n]} e^{-c_i\left(\frac{1-\mu}{\mu}\Gamma(\alpha)\alpha\right)^{1/\alpha}} x_i \right)^m \frac{d\mu}{\mu^2} \\
&= \lim_{\epsilon\to 0} \left( \int_0^\epsilon \left( \sum_{i\in[n]} e^{-c_i\left(\frac{1-\mu}{\mu}\Gamma(\alpha)\alpha\right)^{1/\alpha}} x_i \right)^m \frac{d\mu}{\mu^2} \right. \\
&\qquad \left. + \int_\epsilon^1 \left( \sum_{i\in[n]} e^{-c_i\left(\frac{1-\mu}{\mu}\Gamma(\alpha)\alpha\right)^{1/\alpha}} x_i \right)^m \frac{d\mu}{\mu^2} \right) \\
&= \lim_{\epsilon\to 0} \int_\epsilon^1 \left( \sum_{i\in[n]} e^{-c_i\left(\frac{1-\mu}{\mu}\Gamma(\alpha)\alpha\right)^{1/\alpha}} x_i \right)^m \frac{d\mu}{\mu^2} \\
&= \lim_{\epsilon\to 0} \lim_{k\to\infty} \sum_{j\in[k]} \left( \sum_{i\in[n]} e^{-c_i\left(\frac{k-j(1-\epsilon)}{j(1-\epsilon)}\Gamma(\alpha)\alpha\right)^{1/\alpha}} x_i \right)^m \frac{k}{j^2(1-\epsilon)^2} \\
&=: \lim_{\epsilon\to 0} \lim_{k\to\infty} \sum_{j\in[k]} \left(\langle u^{j,\epsilon}, x\rangle\right)^m,
\end{aligned}$$

where $u^{j,\epsilon} := (\frac{k}{j^2(1-\epsilon)^2})^{\frac{1}{m}}(e^{-c_1(\frac{k-j(1-\epsilon)}{j(1-\epsilon)}\Gamma(\alpha)\alpha)^{1/\alpha}}, \ldots, e^{-c_n(\frac{k-j(1-\epsilon)}{j(1-\epsilon)}\Gamma(\alpha)\alpha)^{1/\alpha}})^T \in \mathbb{R}^n_+$. By the closedness of $CP_{m,n}$, we conclude that $\mathcal{C}^{\circ\alpha}$ is a CP tensor for any $\alpha \geq 0$. ∎

COROLLARY 4.10. *Let $c = (c_i) \in \mathbb{R}^n$ be any given nonnegative vector and $\mathcal{P}^{(c)} = (p^{(c)}_{i_1\cdots i_m}) \in \mathbb{S}_{m,n}$ be the corresponding generalized symmetric Pascal tensor. Then $(\mathcal{P}^{(c)})^{\circ\alpha}$ is a CP tensor for any $\alpha \in R_+$.*

*Proof.* This follows directly from the continuity of function $f(t) = t^\alpha$ on $[0, \infty)$ for any nonnegative $\alpha$, Theorem 4.9, and (4.1) in the proof of Proposition 4.5. ∎

Several much-studied classes of mean matrices are shown to be infinite divisible (see, e.g., [6]). Inspired by this, we will generalize the power mean matries among them to higher-order tensors and further discuss the complete positivity of these power mean tensors and their fractional Hadamard powers as well.

DEFINITION 4.11. *Let $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n_+$. For any extended real value $t \in [-\infty, +\infty]$, define $\mathcal{M}^{(t)} = (m_{i_1\ldots i_m}) \in \mathbb{S}_{m,n}$ as*

$$m_{i_1\ldots i_m} = \left(\frac{1}{m}\sum_{k=1}^{m} c_{i_k}^t\right)^{1/t} \quad \forall i_j \in [n],\ j \in [m],$$

*i.e., each element $m_{i_1\ldots i_m}$ is the $t$-power mean of $c_{i_1}, \ldots, c_{i_m}$. Then, we say that $\mathcal{M}^{(t)}$ is an $m$th order $n$-dimensional* $t$-power mean tensor *and the vector $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ is called the* generating vector *of $\mathcal{M}^{(t)}$.*

PROPOSITION 4.12. *Let $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ be positive, $\mathcal{M}^{(t)} = (m_{i_1\ldots i_m}) \in \mathbb{S}_{m,n}$ be the $t$-power mean tensor generated by $c$, and its entrywise reciprocal tensor $\mathcal{W}^{(t)} := (\frac{1}{m_{i_1\ldots i_m}}) \in \mathbb{S}_{m,n}$. We have*

(i) *for any $t \in [-\infty, 0]$, and any $\alpha \in \mathbb{R}_+$, the fractional Hadamard power $(\mathcal{M}^{(t)})^{\circ\alpha}$ is a CP tensor;*
(ii) *for any $t \in [0, +\infty]$, and any $\alpha \in \mathbb{R}_+$, the fractional Hadamard power $(\mathcal{W}^{(t)})^{\circ\alpha}$ is a CP tensor.*

*Proof.* Things are trivial for $t = 0$ since the all-one tensor is a CP tensor. (i) By Definition 4.11, for any $t \in [-\infty, 0)$ and any $\alpha \in \mathbb{R}_+$,

$$m^\alpha_{i_1\ldots i_m} = \left(\frac{1}{c^t_{i_1}/m + \cdots + c^t_{i_m}/m}\right)^{-\alpha/t} \quad \forall i_j \in [n],\ j \in [m],$$

which indicates that $(\mathcal{M}^{(t)})^{\circ\alpha}$ is actually a nonnegative fractional Hadamard power of the positive Cauchy tensor $\mathcal{A} := (\frac{1}{c^t_{i_1}/m+\cdots+c^t_{i_m}/m}) \in \mathbb{S}_{m,n}$. Thus, the desired result follows directly from Theorem 4.9. (ii) is a direct consequence of Theorem 4.9. ∎

*Remark* 4.13. Specific examples of power mean tensors are listed as follows.

(i) *"Min" tensor*: $t = -\infty$, $\mathcal{M}^{(-\infty)} = (\min\{c_{i_1}, \ldots, c_{i_m}\}) \in \mathbb{S}_{m,n}$. Without loss of generality, we may assume that all components in $c$ are in a decreasing order. Then the CP decomposition of its fractional Hadamard power is

$$\left(\mathcal{M}^{(-\infty)}\right)^{(\circ\alpha)} = c_n^\alpha \left(\sum_{i=1}^{n} e_i\right)^m + (c^\alpha_{n-1} - c^\alpha_n)\left(\sum_{i=1}^{n-1} e_i\right)^m + \cdots + (c^\alpha_1 - c^\alpha_2)(e_1)^m \quad \forall \alpha \geq 0.$$

(ii) *Harmonic mean tensor*: $t = -1$, $\mathcal{M}^{(-1)} = (\frac{m}{c_{i_1}+\cdots+c_{i_m}}) \in \mathbb{S}_{m,n}$, which is a positive Cauchy tensor generated by $\frac{c}{m}$.

(iii) *Geometric mean tensor*: $t = 0$, $\mathcal{M}^{(0)} = (\sqrt[m]{c_{i_1}\cdots c_{i_m}}) \in \mathbb{S}_{m,n}$. Apparently, the CP decompositions of $(\mathcal{M}^{(0)})^{\circ\alpha}$ and its entrywise reciprocal tensors $(\mathcal{W}^{(0)})^{\circ\alpha}$ are

$$\left(\mathcal{M}^{(0)}\right)^{\circ\alpha} = \left(c^{\circ\frac{\alpha}{m}}\right)^m, \quad \left(\mathcal{W}^{(0)}\right)^{\circ\alpha} = \left(c^{\circ\left(-\frac{\alpha}{m}\right)}\right)^m.$$

(iv) *Arithmetic mean tensor*: $t = 1$, $\mathcal{M}^{(1)} = (\frac{1}{m}(c_{i_1}+\cdots+c_{i_m})) \in \mathbb{S}_{m,n}$. Obviously, the corresponding entrywise reciprocal tensor $\mathcal{W}^{(1)}$ is exactly the positive Cauchy tensor $\mathcal{M}^{(-1)}$.

(v) *Root-mean-square tensor*: $t = 2$, $\mathcal{M}^{(2)} = (\sqrt{\frac{1}{m}(c_{i_1}^2+\cdots+c_{i_m}^2)}) \in \mathbb{S}_{m,n}$. It is easy to see that the corresponding entrywise reciprocal tensor $\mathcal{W}^{(2)}$ is exactly the fractional Hadamard power of the positive Cauchy tensor generated by $\frac{c^{\circ 2}}{m}$ with the fractional exponent $\frac{1}{2}$.

(vi) *"Max" tensor*: $t = +\infty$, $\mathcal{M}^{(+\infty)} = (\max\{c_{i_1},\ldots,c_{i_m}\}) \in \mathbb{S}_{m,n}$. Without loss of generality, we assume that the positive generating vector $c = (c_1,\ldots,c_n)$ of $\mathcal{C}$ is in an increasing order, i.e., $0 < c_1 \le c_2 \le \cdots \le c_n$. Then the CP decomposition of the fractional Hadamard power of its entrywise reciprocal tensor $\mathcal{W}^{(+\infty)}$ is

$$\begin{aligned}\left(\mathcal{W}^{(+\infty)}\right)^{\circ\alpha} &= \frac{1}{c_n^\alpha}\left(\sum_{i=1}^n e_i\right)^m + \left(\frac{1}{c_{n-1}^\alpha} - \frac{1}{c_n^\alpha}\right)\left(\sum_{i=1}^{n-1} e_i\right)^m \\ &\quad + \cdots + \left(\frac{1}{c_1^\alpha} - \frac{1}{c_2^\alpha}\right)(e_1)^m \quad \forall \alpha \ge 0.\end{aligned}$$

**5. CP-Vandermonde decomposition for positive Cauchy–Hankel tensors.** It has been known from Remark 4.13 that several types of power mean tensors including the "Min" tensor, the geometric mean tensor, and its entrywise reciprocal tensor, the entry-wise reciprocal of "Max" tensor, together with their Hadamard products and fractional Hadamard powers, can be decomposed in the CP way easily. But for other general positive Cauchy tensors, their CP decompositions are not that direct to be obtained. Recall that a tensor $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{S}_{m,n}$ is called the Hilbert tensor if $a_{i_1\ldots i_m} = \frac{1}{i_1+\cdots+i_m-m+1}$ [36]. Obviously, the Hilbert tensor is a positive Cauchy tensor with mutually distinct components of its generating vector, and hence strongly CP from Proposition 4.3. And it is also a Hankel tensor [12]. Thus, a natural question is, Do we have some special property for the class of tensors which are both positive Cauchy tensors and Hankel tensors in terms of the CP decomposition? A positive answer is given in the following theorem in which a CP decomposition in a Vandermonde manner (CP-Vandermonde decomposition) for positive Cauchy–Hankel tensors is established.

THEOREM 5.1. *Let $\mathcal{A} \in \mathbb{S}_{m,n}$ be a positive Cauchy–Hankel tensor. Then there exist real numbers $\alpha_1,\ldots,\alpha_r$ and mutually distinct nonnegative numbers $\xi_1,\ldots,\xi_r$ with an integer $r$ satisfying $\lceil\frac{m(n-1)+1}{2}\rceil \le r \le n(m-1)+1$ such that*

$$\mathcal{A} = \sum_{k=1}^r \alpha_k\left(u^{(k)}\right)^m,\ \alpha_k > 0,\ u^{(k)} := (1,\xi_k,\ldots,\xi_k^{n-1})^T \ge 0 \quad \forall k \in [r]. \tag{5.1}$$

*Proof.* Denote $N = (n-1)m+1$. By the definition of Cauchy–Hankel tensors, we can find some nonzero $a$ and $b \in \mathbb{R}$ such that $\mathcal{A} = (\frac{1}{a+b(i_1+\cdots+i_m)})$. Let $c \in \mathbb{R}^n$ with $c_k = \frac{a}{m} + kb$ for all $k \in [n]$ and denote $h := (\frac{1}{a+bm}, \frac{1}{a+b(m+1)}, \ldots, \frac{1}{a+bmn})^T \in \mathbb{R}^N$. It is easy to verify that the corresponding Cauchy tensor generated by $c$ and the Hankel tensor generated by $h$ are both exactly $\mathcal{A}$. By setting $y \in \mathbb{R}^N$ with $y_i = \frac{1}{2h_i}$ for all $i \in [N]$, it is easy to verify that the Cauchy matrix $\widetilde{A} := (\frac{1}{y_i+y_j}) \in \mathbb{S}_{2,2m(n-1)+1}$ is also a Hankel matrix whose generating vector is

$$\left(h_1,\ \frac{2h_1h_2}{h_1+h_2},\ h_2,\ \frac{2h_2h_3}{h_2+h_3},\ h_3, \ldots, \frac{2h_{N-1}h_N}{h_{N-1}+h_N},\ h_N\right)^T.$$

The vector $y = (y_i) \in \mathbb{R}^N$ is a positive vector and has all components mutually distinct due to the properties of $h$. Thus, $\widetilde{A}$ is a positive definite Hankel matrix. Invoking [37, Lemma 0.2.1], $\widetilde{A}$ admits a Vandermonde decomposition with nonnegative coefficients, that is, there exist scalars $\tau_1, \ldots, \tau_N \in \mathbb{R}$ and positive scalars $\beta_1, \ldots, \beta_N$ such that

$$\widetilde{A} = \sum_{k=1}^{N} \beta_k v^{(k)}(v^{(k)})^T, \text{ where } v^{(k)} = \left(1, \tau_k, \tau_k^2, \ldots, \tau_k^{N-1}\right)^T \ \forall k \in [N]. \tag{5.2}$$

The positive definiteness of $\widetilde{A}$ indicates that all $\beta_k$'s are positive and $\tau_1, \ldots, \tau_N$ are mutually distinct. By taking $\zeta_k = \tau_k^2$ for all $k \in [N]$, it follows readily that these $\zeta_k$'s have at least $\lceil \frac{m(n-1)+1}{2} \rceil$ distinct values and for any $j \in [N]$,

$$h_j = \sum_{k=1}^{N} \beta_k \zeta_k^{j-1}.$$

Let $r$ be the number of distinct values of $\zeta_k$'s and denote all those distinct values to be $\xi_1, \ldots, \xi_r$. Then $\xi_1, \ldots, \xi_r$ are nonnegative as $\zeta_k = \tau_k^2$ for all $k \in [N]$. Immediately, we can get those $\alpha_k$'s from $\beta_k$'s in (5.2) to decompose the Hankel tensor $\mathcal{A}$ generated by $h$ as required in (5.1). ∎

For positive Cauchy–Hankel tensors, the aforementioned CP-Vandermonde decomposition may not be unique. Such a decomposition with fewer rank-one terms will absolutely be more attractive and important for saving the storage cost. A possible way to get a numerically CP-Vandermode decomposition for a given positive Cauchy–Hankel tensor with the least terms ($r = \lceil \frac{m(n-1)+1}{2} \rceil$) of rank-one terms in (5.1) will then be proposed. Before establishing the numerical algorithm, the following proposition is stated for theoretical preparation.

PROPOSITION 5.2. *Let $N$ be any given positive integer and $h = (\frac{1}{a+bi}) \in \mathbb{R}^N$ be any given positive vector with some nonzero $a$ and $b \in \mathbb{R}$. Then the Hankel matrix generated by $\hat{h} = (h_1, 0, h_2, 0, \ldots, 0, h_N)^T$ is positive definite.*

*Proof.* Take $c_i = \frac{h_i}{2}$ for all $i \in [N]$ to generate a positive Cauchy matrix $A = (\frac{1}{c_i+c_j}) \in \mathbb{R}^{N\times N}$. Since $c$ has all its components mutually distinct, $A$ is then positive semidefinite. By direct calculation, $A$ is also a Hankel matrix generated by $\tilde{h} = (h_1, \frac{h_1h_2}{h_1+h_2}, h_2, \ldots, \frac{2h_{N-1}h_N}{h_{N-1}+h_N}, h_N)^T \in \mathbb{R}^{2N-1}$ and hence admits a Vandermonde decomposition

$$A = \sum_{k=1}^{N} \alpha_k v^{(k)}(v^{(k)})^T,\ \alpha_k > 0,\ v^{(k)} := (1, \zeta_k, \ldots, \zeta_k^{N-1})^T \geq 0$$

with all mutually distinct $\zeta_k$'s. Apparently, $h_j = \sum_{k=1}^{N} \alpha_k \zeta_k^{2(j-1)}$ for all $j \in [N]$, which immediately allows us to write the Hankel matrix generated by $\hat{h}$, say $\hat{A}$, as

$$\hat{A} = \sum_{k=1}^{N} \alpha_k \left( (v^{(k)}(v^{(k)})^T + (x^{(k)}(x^{(k)})^T \right) = A + \sum_{k=1}^{N} \alpha_k (x^{(k)}(x^{(k)})^T$$

with $x^{(k)} := (1, -\zeta_k, \ldots, (-\zeta_k)^{N-1})^T$. Since all $\alpha_k$'s are positive and $A$ is positive definite, the desired positive definiteness of $\hat{A}$ comes directly. ∎

The above proposition provides us a simple way to compute the CP-Vandermonde decomposition for a positive Cauchy–Hankel tensor corresponding to $h$ by factorizing the corresponding positive definite Hankel matrix generated by $\hat{h}$ as defined in the proposition. Thus, based on [16, Algorithm 2], we can pursue the desired CP-Vandermonde decomposition for any given positive Cauchy–Hankel tensor as follows.

**Algorithm 1.** CP-Vandermonde decomposition for positive Cauchy–Hankel tensors.

**Input:** Parameters $a, b \in \mathbb{R}$, the order $m$, and the dimension $n$ to generate a positive Cauchy–Hankel tensor $\mathcal{A} = \left(\frac{1}{a+b(i_1+\cdots+i_m)}\right) \in \mathbb{S}_{m,n}$;

**Output:** The coefficients $\alpha_k$'s and the poles $\xi_k$'s to generate a CP-Vandermonde decomposition of $\mathcal{A}$ as described in (5.1);

Step 0 Set $N = m(n-1)+1$, $\hat{h} = (h_1, 0, h_2, 0, h_3, \ldots, 0, h_N)^T \in \mathbb{R}^{2N-1}$ with $h_i = \frac{1}{a+b(m-1+i)}$. Let $H \in \mathbb{S}_{2,N}$ be the corresponding Hankel matrix generated by $\bar{h}$.

Step 1 Compute $w = H^{-1}d(\bar{h})$, where $d(\bar{h}) \in \mathbb{R}^N$ with $(d(\bar{h}))_i = \bar{h}_{i+N}$ for all $i \in [N-1]$ and $(l(\bar{h}))_N = \gamma$, $\gamma \in \mathbb{R}$ is arbitrary;

Step 2 Compute the roots $\kappa_1, \ldots, \kappa_N$ of the polynomial $p(\kappa) = \kappa^N - w_N\kappa^{N-1} - \cdots - w_2\kappa - w_1$.

Step 3 Solve the Vandermonde linear system $A\bar{\alpha} = u(\bar{h})$, where $A = (a_{ij}) \in \mathbb{S}_{2,N}$ with $a_{ij} = \xi_j^{i-1}$, and $u(\bar{h}) \in \mathbb{R}^N$ is the subvector of $\bar{h}$ formed by its first $N$ components.

Step 4 Set $\xi_k = \kappa_{i_k}^2$, and $\alpha_k = \bar{\alpha}_k$ for all $k \in [N]$.

**Return:** $\alpha_k$, $\xi_k$ for $k \in [r]$.

To get a computationally least number of rank-one items in the CP-Vandermonde decomposition from Algorithm 1, we simply choose $\gamma = 0$ to get an approximate CP-Vandermonde decomposition of only $\lceil \frac{m(n-1)+1}{2} \rceil$ terms with a very promising high accuracy. In this case, Step 1 in Algorithm 1 indicates that

$$\begin{pmatrix} h_2 & h_3 & \cdots & h_r & h_{r+1} \\ h_3 & h_4 & \cdots & h_{r+1} & h_{r+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_r & h_{r+1} & \cdots & h_{N-2} & h_{N-1} \end{pmatrix} \begin{pmatrix} w_2 \\ w_4 \\ \vdots \\ w_{N-2} \\ w_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.3}$$

when $N$ is even, and

$$\begin{pmatrix} h_1 & h_2 & \cdots & h_{r-1} & h_r \\ h_2 & h_3 & \cdots & h_r & h_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{r-1} & h_r & \cdots & h_{N-1} & h_{N-2} \end{pmatrix} \begin{pmatrix} w_1 \\ w_3 \\ \vdots \\ w_{N-2} \\ w_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.4}$$

Table 1
*CP-Vandermonde decomposition of the Hilbert tensor in $\mathbb{S}_{4,6}$.*

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_k$ | 0.0083 | 0.0304 | 0.0685 | 0.1010 | 0.1254 | 0.1396 | 0.1426 | 0.1340 | 0.1145 | 0.0857 | 0.0500 |
| $\xi_k$ | 0 | 0.9881 | 0.9383 | 0.8530 | 0.7390 | 0.6056 | 0.4635 | 0.3243 | 0.1992 | 0.0984 | 0.0301 |

Table 2
*CP-Vandermonde decomposition with inputs $(m, n, a, b) = (3, 8, 0.1, 0.25)$.*

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_k$ | 0.0749 | 0.1636 | 0.2213 | 0.2327 | 0.2007 | 0.1437 | 0.0843 | 0.0390 | 0.0132 | 0.0028 | 0.0002 |
| $\xi_k$ | 0.9926 | 0.9606 | 0.9026 | 0.8200 | 0.7170 | 0.5992 | 0.4739 | 0.3488 | 0.2321 | 0.1314 | 0.0538 |

Table 3
*CP-Vandermonde decomposition with inputs $(m, n, a, b) = (5, 5, -20.4, 5)$.*

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_k$ | 0.0025 | 0.0061 | 0.0139 | 0.0206 | 0.0258 | 0.0292 | 0.0304 | 0.0294 | 0.0260 | 0.0205 | 0.0130 |
| $\xi_k$ | 0 | 0.9880 | 0.9379 | 0.8519 | 0.7372 | 0.6031 | 0.4605 | 0.3210 | 0.1961 | 0.0959 | 0.0286 |

when $N$ is odd. In both cases, it is easy to verify that the coefficient matrices are positive definite Cauchy matrices and hence contribute the unique zero solution for the involved $w_i$'s. This further derives that if $\kappa$ is solution to the polynomial $p(\kappa)$ in Step 3, and then $-\kappa$ is also a solution. If we modify Step 4 in Algorithm 1 to be

> *Step* 4′: Reorder the sequence $\{(\kappa_1, \bar{\alpha}_1), \ldots, (\kappa_N, \bar{\alpha}_N)\}$ in the nondecreasing order with respect to the first component of each pair to get $\{(\kappa_{i_1}, \bar{\alpha}_{i_1}), \ldots, (\kappa_{i_N}, \bar{\alpha}_{i_N})\}$, and set $\xi_k = \kappa_{i_k}^2$, $\alpha_k = \bar{\alpha}_{i_k} + \bar{\alpha}_{i_{N-k+1}}$, for all $k \in [r]$ with $r = \lceil \frac{m(n-1)+1}{2} \rceil$,

then the positive Cauchy–Hankel tensor $\mathcal{A}$ can be approximately decomposed as in (5.1) with only $r = \lceil \frac{m(n-1)+1}{2} \rceil$ terms. Numerical examples are listed to show the performance of the algorithm with Step 4′.

*Example* 5.3. (1) For the Hilbert tensor (i.e., $a = 1 - m$, $b = 1$) with $m = 4$ and $n = 6$, we can get a CP-Vandermonde decomposition as in (5.1) with $\alpha_k$'s and $\xi_k$'s as in Table 1. The Frobenius norm of the residual is 6.217e-13. (2) Randomly choose real numbers $a$ and $b$ and positive integers $m$ and $n$ such that the generated Cauchy–Hankel tensor $\mathcal{A} \in \mathbb{S}_{m,n}$ has all entries positive. For low-order low-dimensional cases, for example, $N = (n-1)m + 1 \le 23$, applying Algorithm 1 with Step 4′ can always give us a CP-Vandermonde decomposition as in (5.1). Two instances are listed as follows with the accuracies 2.179e-11 and 6.074e-13, respectively.

However, for high-dimensional or high-order cases, polynomials in Step 2 may fail to admit all roots in the real field and hence the algorithm will not work anymore; probably efforts should be made on an appropriate choice of $\gamma$ in Step 1. More efficient and effective algorithms are then needed in such a case.

**6. Tractable relaxations of CP tensors.** Analogous to the matrix case, the concept of DNN tensors is introduced to serve as a tractable relaxation for CP tensors.

Definition 6.1. *An even-order symmetric tensor $\mathcal{A} = (a_{i_1 \ldots i_m})$ is said to be a* DNN tensor *if all of its entries are nonnegative and the corresponding polynomial*

$$\mathcal{A}x^m := \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m}$$

*is an SOS. An odd-order symmetric tensor* $\mathcal{A} = (a_{i_1\ldots i_m})$ *is said to be a* DNN tensor *if all its entries are nonnegative and for every* $i \in \{1, \ldots, n\}$,

$$\mathcal{A}_i x^{m-1} := \sum_{j_1,\ldots,j_{m-1}} a_{ij_1\cdots j_{m-1}} x_{j_1} \cdots x_{j_{m-1}}$$

*is an SOS.*

The following observation follows trivially from Definitions 6.1 and 1.1.

PROPOSITION 6.2. *Any CP-tensor is a DNN tensor.*

Many even-order structured tensors have been shown to be SOS in [11]. Together with the augmented Vandermonde decomposition for strong Hankel tensors that was proposed by Ding, Qi, and Wei in [16], it is easy to verify that in the setting of even-order symmetric and nonnegative tensors, the (generalized) diagonally dominant tensor, the $H$-tensor with nonnegative diagonal entries, the $MB_0$-tensor, and the strong Hankel tensor are all DNN tensors. However, the aforementioned structured tensors of odd order may not be DNN tensors anymore. A weak version of DNN tensors is then introduced due to the following spectral property of DNN tensors.

LEMMA 6.3. *A DNN tensor has all $H$-eigenvalues nonnegative.*

*Proof.* The desired nonnegativity follows directly from Definitions 6.1 and 2.1, and the equivalence between the positive semidefiniteness and the nonnegativity of all $H$-eigenvalues for real symmetric tensors as stated in [29, Theorem 5]. ∎

DEFINITION 6.4. *A nonnegative symmetric tensor is said to be a* weak DNN tensor *if all its $H$-eigenvalues are nonnegative.*

Noting that nonnegative tensors always have $H$-eigenvalues [38], the above concept is well-defined for tensors of any order (even or odd). Even though the weak DNN property of a tensor is hard to verify due to the complexity of checking nonnegativity of the minimal $H$-eigenvalue, this property coincides with the DNN property in the matrix case. Moreover, we can also show that several structured tensors are weak DNN tensors in the setting of nonnegative symmetric tensors of any order. Related concepts are recalled as follows.

DEFINITION 6.5. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{T}_{m,n}$.

(1) [40, Definition 3.14] $\mathcal{A}$ *is called a* diagonally dominant *tensor if*

$$|a_{ii\ldots i}| \geq \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |a_{ii_2\ldots i_m}| \ \forall i \in [n]. \tag{6.1}$$

$\mathcal{A}$ *is said to be* strictly diagonally dominant *if the strict inequality holds in* (6.1) *for all* $i \in [n]$.

(2) [15, 34] $\mathcal{A}$ *is called a* (strictly) generalized diagonally dominant *if there exists some positive diagonal matrix $D$ such that the tensor* $\mathcal{A}D^{1-m}\underbrace{D\cdots D}_{m-1}$ *defined as*

$$\left(\mathcal{A}D^{1-m}\underbrace{D\cdots D}_{m-1}\right)_{i_1\ldots i_m} = a_{i_1\ldots i_m} d_{i_1}^{1-m} d_{i_2}\cdots d_{i_m}, \ \forall i_j \in [n], \ j \in [m],$$

*is (strictly) diagonally dominant.*

(3) [15, 34] *$\mathcal{A}$ is called a $Z$-tensor if there exists a nonnegative tensor $\mathcal{B}$ and a real number $s$ such that $\mathcal{A} = s\mathcal{I} - \mathcal{B}$. A $Z$-tensor $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ is said to be an* $M$-tensor *if $s \geq \rho(\mathcal{B})$, where $\rho(\mathcal{B})$ is the spectral radius of $\mathcal{B}$. If $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a* strong $M$-tensor. *The* comparison tensor *of a tensor $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$, denoted by $M(\mathcal{A})$, is defined as*

$$(M(\mathcal{A}))_{i_1 \ldots i_m} := \begin{cases} |a_{i_1 \ldots i_m}| & \textit{if } \ i_1 = \cdots = i_m; \\ -|a_{i_1 \ldots i_m}| & \textit{otherwise.} \end{cases}$$

*$\mathcal{A}$ is called an $H$-tensor (strong $H$-tensor) if its comparison tensor $M(\mathcal{A})$ is an $M$-tensor (strong $M$-tensor).*

(4) [25, Definition 4] *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{T}_{m,n}$ and $\mathcal{B} = (b_{i_1 \cdots i_m}) \in \mathbb{T}_{m,n}$ with $b_{i_1 i_2 \cdots i_m} = \beta_{i_1}(\mathcal{A})$, where $\beta_i(\mathcal{A}) := \max_{\substack{j_2,\ldots,j_m \in [n] \\ (i,j_2,\ldots,j_m) \neq (i,i,\ldots,i)}} \{0, a_{ij_2 \ldots j_m}\}$. $\mathcal{A}$ is called an $MB_0$- ($MB$-)tensor if $\mathcal{A} - \mathcal{B}$ is an $M$-(a strong $M$-)tensor.*

PROPOSITION 6.6. *Let $\mathcal{A}$ be any given nonnegative symmetric tensor. If one of the following conditions holds*

(i) *$\mathcal{A}$ is a generalized diagonally dominant tensor,*
(ii) *$\mathcal{A}$ is an $H$-tensor,*
(iii) *$\mathcal{A}$ is an $MB_0$-tensor,*
(iv) *$\mathcal{A}$ is a strong Hankel tensor,*

*then $\mathcal{A}$ is a weak DNN tensor.*

*Proof.* It suffices to show the nonnegativity of all $H$-eigenvalues of the corresponding involved tensor. This desired nonnegativities for cases (i) and (ii) have been indicated in [15, 34].

To get (iii), it is known from [25, Theorem 7] that for any nonnegative symmetric $MB_0$-tensor $\mathcal{A}$, either $\mathcal{A}$ is a symmetric $M$-tensor itself or we have $\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \mathcal{E}^{J_k}$, where $\mathcal{M}$ is a symmetric $M$-tensor, $s$ is a positive integer, $h_k > 0$, $J_k \subset [n]$, and $\mathcal{E}^{J_k} \in \mathbb{S}_{m,n}$ with $(\mathcal{E}^{J_k})_{i_1 \cdots i_m} = 1$ for all $\{i_1, \ldots, i_m\} \subseteq J_k$ and otherwise 0 for $k = 1, \ldots, s$. When $m$ is even, the desired nonnegativity is trivial from the positive semidefiniteness of $\mathcal{A}$. When $m$ is odd, the assertion is obvious when $\mathcal{A}$ is an $M$-tensor itself. To show for the latter case, we first claim that for any symmetric $M$-tensor $\mathcal{M}$ and any vector $x \in \mathbb{R}^n \setminus \{0\}$, there always exists some $i \in \operatorname{supp}(x) := \{i \in [n] : x_i \neq 0\}$ such that $(\mathcal{M}x^{m-1})_i \geq 0$. Assume on the contrary that there exists some nonzero $x$ such that for any $i \in \operatorname{supp}(x)$, $(\mathcal{M}x^{m-1})_i < 0$. Let $\alpha_i = -(\mathcal{M}x^{m-1})_i / x_i^{m-1}$ for all $i \in \operatorname{supp}(x)$. Obviously, $\alpha_i > 0$ for all $i \in \operatorname{supp}(x)$. Thus, $(\bar{\mathcal{M}} + \sum_{i \in supp(x)} \alpha_i (e^{(i)})^m) \bar{x}^{m-1} = 0$, where $\bar{\mathcal{M}}$ is the principal subtensor of $\mathcal{M}$ and $\bar{x}$ the subvector of $x$ generated by the index set $\operatorname{supp}(x)$. This contradicts the fact that $\bar{\mathcal{M}} + \sum_{i \in supp(x)} \alpha_i (e^{(i)})^m$ is a strong $M$-tensor by the property of $M$-tensors. Our claim then succeeds. Now for any $H$-eigenvalue $\lambda$ of $\mathcal{A}$ with its associated $H$-eigenvector $x$, we have

$$\lambda x_i^{m-1} = \left(\mathcal{A}x^{m-1}\right)_i = \left(\mathcal{M}x^{m-1}\right)_i + \sum_{k=1}^{s} h_k \left(\mathcal{E}^{J_k} x^{m-1}\right)_i \ \forall i \in [n].$$

From our claim, we can find some $i \in \operatorname{supp}(x)$ such that $(\mathcal{M}x^{m-1})_i \geq 0$ and hence $\lambda = \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} \geq 0$.

For (iv), it is known from [16] that a strong Hankel tensor $\mathcal{A}$ always possesses an augmented Vandermonde decomposition

$$\mathcal{A} = \sum_{k=1}^{r-1} \alpha_k \left(u^{(k)}\right)^m + \alpha_r \left(e^{(n)}\right)^m,$$

where $\alpha_k > 0$, $u^{(k)} = (1, \xi_k, \ldots, \xi_k^{n-1})^T \in \mathbb{R}^n$, $\xi_k \in \mathbb{R}$, for all $k \in [r-1]$, $\alpha_r \geq 0$. When $m$ is even, the desired nonnegativity is obvious from the positive semidefiniteness of $\mathcal{A}$. Now we consider the odd-order case. For any $H$-eigenvalue $\lambda$ of $\mathcal{A}$ with its associated eigenvector $x \in \mathbb{R}^n \setminus \{0\}$, we have

$$\lambda x_i^{m-1} = \left(\mathcal{A}x^{m-1}\right)_i = \begin{cases} \sum_{k\in[r-1]} \left(x^T u^{(k)}\right)^{m-1} u_i^{(k)} + x_n^{m-1} & \text{if } i = n; \\ \sum_{k\in[r-1]} \left(x^T u^{(k)}\right)^{m-1} u_i^{(k)} & \text{otherwise.} \end{cases}$$

If $x^T u^{(k)} = 0$ for all $k \in [r-1]$, it is easy to see that for any $i \in [n]$ with $x_i \neq 0$, $\lambda = \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} \geq 0$. If there exists some $\bar{k} \in [r-1]$ such that $x^T u^{(\bar{k})} = 0$, then $\lambda x_1^{m-1} = \sum_{k\in[r-1]} (x^T u^{(k)})^{m-1} \geq (x^T u^{(\bar{k})})^{m-1} > 0$, which indicates that $\lambda > 0$. This completes the proof. ∎

The gap existing between DNN matrices and CP matrices has been extensively studied [9, 4, 18]. The remaining part of this section will be devoted to the equivalence and the gap between the tensor cones $DNN_{m,n}$ and $CP_{m,n}$. It is known from the literature of matrices that any rank-one matrix is CP if and only if it is nonnegative. This also holds for higher-order tensors as the following proposition demonstrates.

PROPOSITION 6.7. *A rank-one symmetric tensor is CP if and only if it is nonnegative.*

*Proof.* The necessity is trivial by definition. To show the sufficiency, note that for any rank-one symmetric tensor $\mathcal{A} = \lambda x^m$ to be nonnegative, we have $x \neq 0$, $\lambda \neq 0$, and $\lambda x_{i_1} \cdots x_{i_m} \geq 0$ for all $i_1, \ldots, i_m \in [n]$. If $x$ has only one nonzero element, the desired statement holds immediately. If there exists at least two nonzero elements, we claim that all nonzero elements should be of the same sign. Otherwise, if $x_i > 0$ and $x_j < 0$, then $\lambda x_i^{m-1} x_j$ and $\lambda x_i^{m-2} x_j^2$ will not be nonnegative simultaneously. Thus all elements in $x$ are either nonnegative or nonpositive. When $m$ is even, we can easily get $\lambda > 0$. Thus $\mathcal{A}$ is CP. If $m$ is odd, we can get that $\lambda^{1/m} x \geq 0$. Thus $\mathcal{A}$ is CP. ∎

The above proposition provides a special case that $CP_{m,n}$ coincides with $DNN_{m,n}$. Generally, there exists a gap between $DNN_{m,n}$ and $CP_{m,n}$. For example, any signless Laplacian tensor (always nonnegative and diagonally dominant) of a nonempty $m$-uniform hypergraph with any even $m \geq 4$ lies in the gap $\mathcal{Q} \in DNN_{m,n} \setminus CP_{m,n}$ by Proposition 3.4. Recall from [4] that for any matrix $A \in \mathbb{S}_{2,n}$, if $A$ is of rank 2 or $n \leq 4$ , $A \in DNN_{2,n}$ if and only if $A \in CP_{2,m}$. In other words, $DNN_{2,n} = CP_{2,n}$ for these two cases. How about higher-order tensors? We answer this question in a negative way as follows.

PROPOSITION 6.8. *Let $m \geq 4$ be even and $n \geq 2$. Then*

$$\left\{\alpha \left(e^{(i)} - e^{(j)}\right)^m + \alpha e^m : i, j \in [n], i \neq j, \alpha \in \mathbb{R}_{++}\right\} \subseteq DNN_{m,n} \setminus CP_{m,n}.$$

*Proof.* For simplicity, denote $GAP := \{\alpha(e^{(i)} - e^{(j)})^m + \alpha e^m : i, j \in [n], i \neq j, \alpha \in \mathbb{R}_{++}\}$. Then for any $\mathcal{A} = (a_{i_1 \ldots i_m}) \in GAP$, it follows by definition that

$\mathcal{A} \in DNN_{m,n}$. Additionally, it is easy to verify that $\mathcal{A}$ is rank-two. However, as we can see, $a_{i\ldots ij} = 0$ and $a_{i\ldots ijj} = 2$. This indicates that $\mathcal{A}$ breaks the zero-entry dominance property. Thus $\mathcal{A} \in DNN_{m,n} \setminus CP_{m,n}$. ∎

The aforementioned gap between $DNN_{m,n}$ and $CP_{m,n}$ drives us to consider tighter relaxations for $CP_{m,n}$. By employing the idea from [17] for the matrix case, an approximation hierarchy for the CP tensor cone can be proposed based on the higher-order DNN tensors. For the sake of convenience, we introduce a linear operator $G_r : \mathbb{S}_{m+r,n} \to \mathbb{S}_{m,n}$ for any nonnegative integer $r$ as follows: for any $\mathcal{Z} = (z_{i_1\cdots i_r j_1\cdots j_m}) \in \mathbb{S}_{m+r,n}$,

$$G_r(\mathcal{Z}) := (g_{j_1\cdots j_m}) \in \mathbb{S}_{m,n} \text{ with } g_{j_1\cdots j_m} = \sum_{i_1,\ldots,i_r \in [n]} z_{i_1\cdots i_r j_1\cdots j_m}, \ \forall j_1,\ldots,j_m \in [n]. \tag{6.2}$$

Apparently, $G_r(\mathcal{Z}) = \sum_{\mathcal{A}\in L_r(\mathcal{Z})} \mathcal{A}$, where

$$L_r(\mathcal{Z}) := \{\mathcal{A} = (a_{j_1\cdots j_m}) \in \mathbb{S}_{m,n} : \exists i_1,\ldots,i_r \in [n], a_{j_1\cdots j_m} = z_{i_1\cdots i_r j_1\cdots j_m} \ \forall j_1,\ldots,j_m \in [n]\}.$$

LEMMA 6.9. *For any nonnegative integer $r$, we have*

$$CP_{m,n} = \{\mathcal{A} \in \mathbb{S}_{m,n} : \exists \mathcal{Z} \in \mathbb{S}_{m+r,n}, L_r(\mathcal{Z}) \subseteq CP_{m,n}, \mathcal{A} = G_r(\mathcal{Z})\}. \tag{6.3}$$

*Proof.* For simplicity, denote

$$M := \{\mathcal{A} \in \mathbb{S}_{m,n} : \exists \mathcal{Z} \in \mathbb{S}_{m+r,n}, L_r(\mathcal{Z}) \subseteq CP_{m,n}, \mathcal{A} = G_r(\mathcal{Z})\}.$$

For any $\mathcal{A} \in M$, there exists some $\mathcal{Z} \in \mathbb{S}_{m+r,n}$ such that $L_r(\mathcal{Z}) \subseteq CP_{m,n}$ and

$$\mathcal{A} = G_r(\mathcal{Z}) = \sum_{\mathcal{C}\in L_r(\mathcal{Z})} \mathcal{C} \in CP_{m,n}$$

due to the fact that $CP_{m,n}$ is a convex cone. This indicates that $M \subseteq CP_{m,n}$. On the other hand, for any $\mathcal{A} \in CP_{m,n}$, we can always find some finite nonnegative integer $l$ and nonnegative nonzero vectors $u^{(1)},\ldots,u^{(l)} \in \mathbb{R}^n$ such that $\mathcal{A} = \sum_{k\in[l]} (u^{(k)})^m$. Set $\mathcal{Z} = \sum_{k\in[l]} \frac{(u^{(k)})^{m+r}}{(e^T u^{(k)})^r}$. Direct calculation leads to

$$\begin{aligned} L_r(\mathcal{Z}) = &\left\{ \sum_{k\in[l]} \left(u^{(k)}\right)_{i_1} \cdots \left(u^{(k)}\right)_{i_r} \left(u^{(k)}\right)^m : i_1,\ldots,i_m \in [n] \right\} \\ &\subseteq CP_{m,n}, \quad G_r(\mathcal{Z}) = \mathcal{A}. \end{aligned}$$

Thus, $\mathcal{A} \in M$ and then $CP_{m,n} \subseteq M$. This completes the proof. ∎

Inspired by a characterization in (6.3), two types of approximation hierarchies for $CP_{m,n}$ based on the higher-order tensors are then proposed as follows:

$$N^r_{m,n} := \{\mathcal{A} \in \mathbb{S}_{m,n} : \exists \mathcal{Z} \in N_{m+r,n}, \mathcal{A} = G_r(\mathcal{Z})\}, \quad r = 0,1,2,\ldots, \tag{6.4}$$

$$DNN^r_{m,n} := \{\mathcal{A} \in \mathbb{S}_{m,n} : \exists \mathcal{Z} \in S_{m+r,n}, L_r(\mathcal{Z}) \subseteq DNN_{m,n}, \mathcal{A} = G_r(\mathcal{Z})\}, \quad r = 0,1,2,\ldots. \tag{6.5}$$

PROPOSITION 6.10. *For any nonnegative integer* $r$, $N^r_{m,n}$ *and* $DNN^r_{m,n}$ *are closed convex pointed cones in* $\mathbb{S}_{m,n}$. *Moreover,*

$$CP_{m,n} \subseteq \cdots \subseteq N^{r+1}_{m,n} \subseteq N^r_{m,n} \subseteq \cdots N^1_{m,n} \subseteq N^0_{m,n} = N_{m,n}, \tag{6.6}$$

$$CP_{m,n} \subseteq \cdots \subseteq DNN^{r+1}_{m,n} \subseteq DNN^r_{m,n} \subseteq \cdots DNN^1_{m,n} \subseteq DNN^0_{m,n} = DNN_{m,n}. \tag{6.7}$$

*Proof.* The first part comes immediately from the fact that $N_{m,n}$ is a closed convex pointed cone, and $SOS_{m,n}$ is a closed, convex cone when $m \geq 2$ is an even integer. For the remaining part, by applying Lemma 6.9, together with the fact $CP_{m,n} \subseteq DNN_{m,n} \subseteq N_{m,n}$, we immediately conclude that for any nonnegative integer $r$, $CP_{m,n} \subseteq DNN^r_{m,n} \subseteq N^r_{m,n}$. For any $\mathcal{A} \in DNN^{r+1}_{m,n}$, there exists some $\mathcal{Z} = (z_{ki_1\cdots i_r j_1\cdots j_m}) \in \mathbb{S}_{m+r+1,n}$ such that $L_{r+1}(\mathcal{Z}) \subseteq DNN_{m,n}$ and $\mathcal{A} = G_{r+1}(\mathcal{Z})$. Denote $\hat{\mathcal{Z}} = (\hat{z}_{i_1\cdots i_r j_1\cdots j_m}) \in \mathbb{S}_{m+r,n}$ with $\hat{z}_{i_1\cdots i_r j_1\cdots j_m} = \sum_{k\in[n]} z_{ki_1\cdots i_r j_1\cdots j_m}$ for any $i_1,\ldots,i_r,j_1\ldots,j_m \in [n]$. By definition, each $\hat{\mathcal{A}} \in L_r(\hat{\mathcal{Z}})$ is a summation of $n$ tensors in $L_{r+1}(\mathcal{Z})$ and hence $L_r(\hat{\mathcal{Z}}) \subseteq DNN_{m,n}$ since $L_{r+1}(\mathcal{Z}) \subseteq DNN_{m,n}$ and $DNN_{m,n}$ is a convex cone. Meanwhile, by direct calculation, it is easy to get $G_r(\hat{\mathcal{Z}}) = G_{r+1}(\mathcal{Z}) = \mathcal{A}$. Henceforth, $\mathcal{A} \in DNN^r_{m,n}$, which tells us the inclusion $DNN^{r+1}_{m,n} \subseteq DNN^r_{m,n}$ for any nonnegative integer $r$. Similarly, we can prove the case of (6.6). ∎

## 7. Conclusions.

In this paper, the CP tensor has been further studied with fourfold main contributions. First, the dominance properties have been emphasized and applied to exclude a number of symmetric nonnegative tensors, such as the signless Laplacian tensors of nonempty $m$-uniform hypergraphs with $m \geq 3$, from the class of CP tensors. Second, a rich variety of subclasses of CP tensors have been investigated which contains the positive Cauchy tensors, the (generalized) symmetric Pascal tensors, the (generalized) Lehmer tensors, the power mean tensors, and their fractional Hadamard powers and Hadamard products. All these serve as new sufficient conditions and provide easily verifiable structures in the study of CP tensor verification and decomposition. Third, all positive Cauchy–Hankel tensors have been shown to admit the CP-Vandermonde decomposition, and a numerical algorithm has been proposed to achieve such a special type of CP decomposition. Last, the DNN matrices have been generalized to high-order tensors, based on which a series of tractable approximations have been proposed to approximate the CP tensor cone. All of these results can serve as a supplement to enrich tensor analysis, computation, and applications.

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## REFERENCES

[1] P. Alonso, J. Delgado, R. Gallego, and J. M. Peña, *Conditioning and accurate computations with Pascal matrices*, J. Comput. Appl. Math., 252 (2013), pp. 21–26.

[2] N. Arima, S. Kim, and M. Kojima, *Extension of completely positive cone relaxation to moment cone relaxation for polynomial optimization*, J. Optim. Theory Appl., 168 (2016), pp. 1–17.

[3] N. Arima, S. Kim, M. Kojima, and K.-C. Toh, *Lagrangian-conic Relaxations, Part* II: *Applications to Polynomial Optimization Problems*, preprint, 2014, http://www.optimization-online.org/DB_HTML/2014/01/4199.html.

[4] A. Berman and N. Shaked-Monderer, *Completely Positive Matrices*, World Scientific, River Edge, NJ, 2003.
[5] R. Bhatia, *Infinitely divisible matrices*, Amer. Math. Monthly, 113 (2006), pp. 221–235.
[6] R. Bhatia and H. Kosaki, *Mean matrices and infinite divisibility*, Linear Algebra Appl., 424 (2007), pp. 36–54.
[7] I. M. Bomze, *Copositive optimization—recent developments and applications*, European J. Oper. Res., 216 (2012), pp. 509–520.
[8] R. Brawer and M. Pirovino, *The linear algebra of the Pascal matrix*, Linear Algebra Appl., 174 (1992), pp. 13–23.
[9] S. Burer, K. M. Anstreicher, and M. Dür, *The difference between* $5 \times 5$ *doubly nonnegative and completely positive matrices*, Linear Algebra Appl., 431 (2009), pp. 1539–1552.
[10] H. Chen, G. Li, and L. Qi, *Further results on Cauchy tensors and Hankel tensors*, Appl. Math. Comput., 275 (2016), pp. 50–62.
[11] H. Chen, G. Li, and L. Qi, *SOS tensor decomposition: Theory and applications*, Commun. Math. Sci., 14 (2016), pp. 2073–2100.
[12] H. Chen and L. Qi, *Positive definiteness and semi-definiteness of even order symmetric Cauchy tensors*, J. Ind. Manag. Optim., 11 (2015), pp. 1263–1274.
[13] Z. Chen and L. Qi, *Circulant tensors with applications to spectral hypergraph theory and stochastic process*, J. Ind. Manag. Optim., 12 (2016), pp. 1227–1247.
[14] A. Cichocki, R. Zdunek, A. H. Phan, and S.-i. Amari, *Nonnegative Matrix and Tensor Factorizations: Applications to Exploratory Multi-Way Data Analysis and Blind Source Separation*, John Wiley & Sons, New York, 2009.
[15] W. Ding, L. Qi, and Y. Wei, *$M$-tensors and nonsingular $M$-tensors*, Linear Algebra Appl., 439 (2013), pp. 3264–3278.
[16] W. Ding, L. Qi, and Y. Wei, *Inheritance properties and sum-of-squares decomposition of Hankel tensors: Theory and algorithms*, BIT, 302 (2015), pp. 1–22.
[17] H. Dong, *Symmetric tensor approximation hierarchies for the completely positive cone*, SIAM J. Optim., 23 (2013), pp. 1850–1866.
[18] H. Dong and K. Anstreicher, *Separating doubly nonnegative and completely positive matrices*, Math. Program., 137 (2013), pp. 131–153.
[19] J. Fan and A. Zhou, *Completely Positive Tensor Decomposition*, preprint arXiv:1411.5149, 2014.
[20] N. J. Higham, *The Test Matrix Toolbox for MATLAB (Version* 3.0*)*, University of Manchester, Manchester, 1995.
[21] S. Kim, M. Kojima, and K.-C. Toh, *A Lagrangian-DNN relaxation: A fast method for computing tight lower bounds for a class of quadratic optimization problems*, Math. Program., 156 (2016), pp. 161–187.
[22] T. G. Kolda, *Numerical optimization for symmetric tensor decomposition*, Math. Program., 151 (2015), pp. 225–248.
[23] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim., 11 (2001), pp. 796–817.
[24] M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, in Emerging Applications of Algebraic Geometry, Springer, New York, 2009, pp. 157–270.
[25] C. Li, L. Qi, and Y. Li, *$MB$-tensors and $MB_0$-tensors*, Linear Algebra Appl., 484 (2015), pp. 141–153.
[26] Z. Luo, L. Qi, and Y. Ye, *Linear operators and positive semidefiniteness of symmetric tensor spaces*, Sci. China Math., 58 (2015), pp. 197–212.
[27] M. Newman and J. Todd, *The evaluation of matrix inversion programs*, J. SIAM, 6 (1958), pp. 466–476.
[28] J. Pena, J. C. Vera, and L. F. Zuluaga, *Completely positive reformulations for polynomial optimization*, Math. Program., 151 (2015), pp. 405–431.
[29] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 40 (2005), pp. 1302–1324.
[30] L. Qi, *Symmetric nonnegative tensors and copositive tensors*, Linear Algebra Appl., 439 (2013), pp. 228–238.
[31] L. Qi, *$H^+$-eigenvalues of Laplacian and signless Laplacian tensors*, Commun. Math. Sci., 12 (2014), pp. 1045–1064.
[32] L. Qi, *Hankel tensors: Associated Hankel matrices and Vandermonde decomposition*, Commun. Math. Sci., 13 (2015), pp. 113–125.
[33] L. Qi, C. Xu, and Y. Xu, *Nonnegative tensor factorization, completely positive tensors, and a hierarchical elimination algorithm*, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 1227–1241.
[34] M. Rajesh Kannan, N. Shaked-Monderer, and A. Berman, *Some properties of strong $H$-tensors and general $H$-tensors*, Linear Algebra Appl., 476 (2015), pp. 42–55.

[35] A. Shashua and T. Hazan, *Non-negative tensor factorization with applications to statistics and computer vision*, in Proceedings of the 22nd International Conference on Machine Learning, ACM, 2005, pp. 792–799.

[36] Y. Song and L. Qi, *Infinite and finite dimensional Hilbert tensors*, Linear Algebra Appl., 451 (2014), pp. 1–14.

[37] E. E. Tyrtyshnikov, *How bad are Hankel matrices?*, Numer. Math., 67 (1994), pp. 261–269.

[38] Y. Yang and Q. Yang, *Further results for Perron–Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2517–2530.

[39] A. Yoshise and Y. Matsukawa, *On optimization over the doubly nonnegative cone*, in Proceedings of the 2010 IEEE International Symposium on Computer-Aided Control System Design (CACSD), IEEE, 2010, pp. 13–18.

[40] L. Zhang, L. Qi, and G. Zhou, *$M$-tensors and some applications*, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 437–452.