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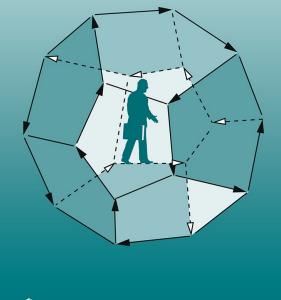## Springer

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Honghai Li[1] · Jia-Yu Shao[2] · Liqun Qi[3]

Abstract In this paper, we study some extremal problems of three kinds of spectral radii of k-uniform hypergraphs (the adjacency spectral radius, the signless Laplacian spectral radius and the incidence Q-spectral radius). We call a connected and acyclic k-uniform hypergraph a supertree. We introduce the operation of "moving edges" for hypergraphs, together with the two special cases of this operation: the edge-releasing operation and the total grafting operation. By studying the perturbation of these kinds of spectral radii of hypergraphs under these operations, we prove that for all these three kinds of spectral radii, the hyperstar $\mathcal{S}_{n,k}$ attains uniquely the maximum spectral radius among all k-uniform supertrees on n vertices. We also determine the unique k-uniform supertree on n vertices with the second largest spectral radius (for these three kinds of spectral radii). We also prove that for all these three kinds of spectral radii, the loose path $\mathcal{P}_{n,k}$ attains uniquely the minimum spectral radius among all k-th power hypertrees of n vertices. Some bounds on the incidence Q-spectral radius are given. The relation between the incidence Q-spectral radius and the spectral radius of the matrix product of the incidence matrix and its transpose is discussed.

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1 Introduction

In 2005, Qi (2005) and Lim (2005) independently introduced the concept of tensor eigenvalues and the spectra of tensors. In 2008, Lim (2008) proposed the study of the spectra of hypergraphs via the spectra of tensors.

In 2012, Cooper and Dutle (2012) defined the eigenvalues (and the spectrum) of a uniform hypergraph as the eigenvalues (and the spectrum) of the adjacency tensor of that hypergraph, and obtained a number of interesting results on the spectra of hypergraphs. They also generalized some basic spectral results from graphs to hypergraphs. The (adjacency) spectrum of a uniform hypergraph were further studied in Pearson and Zhang (2014), Xie and Chang (2013b).

In 2013, Qi (2014) proposed a simple and natural definition for the Laplacian tensor $\mathcal{L}$ and the signless Laplacian tensor $\mathcal{Q}$ as $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ respectively, where $\mathcal{A}$ is the adjacency tensor of the hypergraph (defined as in Cooper and Dutle (2012)), and $\mathcal{D}$ is the degree diagonal tensor of the hypergraph. The properties of these Laplacian and signless Laplacian tensors were further studied in Hu and Qi (2014), Hu et al. (2013, 2015), Qi et al. (2014), Shao et al. (2015). (In Xie and Chang (2013a, c), they proposed a different definition for the signless Laplacian tensor of an even uniform hypergraph.)

Since then, the study of various kinds of spectra of hypergraphs has attracted extensive attention and research interest.

In this paper, we study some extremal spectral problems for three kinds of spectral radii of k-uniform hypergraphs: the adjacency spectral radius, the signless Laplacian spectral radius and the incidence Q-spectral radius, and for two special classes of uniform hypergraphs: the class of k-uniform supertrees on n vertices, and the class of kth power hypertrees on n vertices.

The incidence Q-tensor of a k-uniform hypergraph G, denoted by $\mathcal{Q}^*(G)$ (or simply $\mathcal{Q}^*$), is defined as the tensor product (in the sense of Shao (2013), Bu et al. (2014)) $\mathcal{Q}^*(G) = R\mathbb{I}R^T$, where R is the incidence matrix of G, and $\mathbb{I}$ is the identity tensor. The spectral radius of the incidence Q-tensor is called the incidence Q-spectral radius, of that k-uniform hypergraph. The incidence Q-tensor $\mathcal{Q}^*$ coincides with the "signless Laplacian tensor" proposed by Xie and Chang (2013a, c), for even uniform hypergraphs (which is different from the signless Laplacian tensor $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ studied in this paper).

It is easy to see that both $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ and $\mathcal{Q}^*(G)$ are generalizations of the signless Laplacian matrix from ordinary graphs to uniform hypergraphs.

For the purpose of studying the extremal problems of that three kinds of spectral radii, we introduce the operation of "moving edges" for hypergraphs, together with the two special cases of this operation: the edge-releasing operation and the total grafting operation. We study the perturbation of these three kinds of spectral radii of hypergraphs under these operations, and show that all these three kinds of spectral

radii of supertrees strictly increase under the edge-releasing operation and the inverse of the total grafting operation.

Using these perturbation results, we prove that for all these three kinds of spectral radii, the hyperstar $\mathcal{S}_{n,k}$ attains uniquely the maximum spectral radius among all k-uniform supertrees on n vertices. We also determine the unique k-uniform supertree on n vertices with the second largest spectral radius (for these three kinds of spectral radii). Meanwhile, the corresponding minimization problems for these three kinds of spectral radii of hypertrees are investigated, and we show that for all these three kinds of spectral radii, the loose path $\mathcal{P}_{n,k}$ attains uniquely the minimum spectral radius among all k-th power hypertrees of n vertices.

This paper is organized as follows. In Sect. 2, notation and some definitions about tensors and hypergraphs are given. In Sect. 3, supertrees are defined and some properties of supertrees are discussed. In Sect. 4, the incidence Q-tensor $\mathcal{Q}^*(G)$ of a uniform hypergraph G is defined, and we show in Sect. 4 that $\mathcal{Q}^*(G)$ is irreducible if and only if the hypergraph G is connected. In Sect. 5, we introduce the above-mentioned three operations on hypergraphs, and investigate the perturbation of the three kinds of spectral radii of the supertrees under these operations. As applications, we determine the unique k-uniform supertrees on n vertices with the largest and second largest spectral radii, and determine the unique k-uniform hypertree on n vertices with the smallest spectral radius (for all these three kinds of spectral radii). In the last section, some bounds on the incidence Q-spectral radius are presented, and the relation between the incidence Q-spectral radius and the spectral radius of the matrix product RR^T is discussed.

2 Preliminaries

A kth-order n-dimensional real tensor $\mathcal{T}$ consists of n^k entries in real numbers:

$$\mathcal{T} = (\mathcal{T}_{i_1 i_2 \ldots i_k}), \ \mathcal{T}_{i_1 i_2 \ldots i_k} \in \mathbb{R}, \ 1 \leq i_1, i_2, \ldots, i_k \leq n.$$

$\mathcal{T}$ is called *symmetric* if the value of $\mathcal{T}_{i_1 i_2 \ldots i_k}$ is invariant under any permutation of its indices $i_1, i_2, \ldots, i_k$. A real symmetric tensor $\mathcal{T}$ of order k dimension n uniquely defines a kth degree homogeneous polynomial function f with real coefficient by $f(x) = \mathcal{T}x^k$, which is a real scalar defined as

$$\mathcal{T}x^k = \sum_{i_1, \ldots, i_k = 1}^{n} \mathcal{T}_{i_1 \ldots i_k} x_{i_1} \ldots x_{i_k}.$$

$\mathcal{T}$ is called positive semi-definite if $f(x) = \mathcal{T}x^k \geq 0$ for all $x \in \mathbb{R}^n$. Clearly, for the nontrivial case, k must be even.

Recall that $\mathcal{T}x^{k-1}$ is a vector in $\mathbb{R}^n$ with its ith component as

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2, \ldots, i_k = 1}^{n} \mathcal{T}_{i i_2 \ldots i_k} x_{i_2} \ldots x_{i_k}. \tag{1}$$

Definition 1 (Qi 2005) Let $\mathcal{T}$ be a kth-order n-dimensional tensor and $\mathbb{C}$ be the set of all complex numbers. Then λ is an eigenvalue of $\mathcal{T}$ and $0 \neq x \in \mathbb{C}^n$ is an eigenvector corresponding to λ if (λ, x) satisfies

$$\mathcal{T}x^{k-1} = \lambda x^{[k-1]},$$

where $x^{[k-1]} \in \mathbb{C}^n$ with $(x^{[k-1]})_i = (x_i)^{k-1}$.

Several kinds of eigenvalues of tensors were defined in Qi (2005) and we focus on the one above in this paper.

A hypergraph G is a pair (V, E), where $E \subseteq \mathcal{P}(V)$ and $\mathcal{P}(V)$ stands for the power set of V. The elements of $V = V(G)$, labeled as $[n] = \{1, \ldots, n\}$, are referred to as vertices and the elements of $E = E(G)$ are called edges. A hypergraph G is said to be k-uniform for an integer $k \geq 2$ if, for all $e \in E(G)$, $|e| = k$. For a subset $S \subset V$, we denote by E_S the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. It is the set of edges containing the vertex i, i.e., $E_i = \{e \in E \mid i \in e\}$. The cardinality $|E_i|$ of the set E_i is defined as the degree of the vertex i, which is denoted by d_i. A hypergraph is *regular of degree* r if $d_1 = \cdots = d_n = r$.

A hypergraph G is called a *linear hypergraph* (Bretto 2013), if each pair of the edges of G has at most one common vertex. We assume that G is simple throughout the paper, i.e. $e_i \neq e_j$ if $i \neq j$. In a hypergraph, two vertices are said to be *adjacent* if there is an edge that contains both of these vertices. Two edges are said to be *adjacent* if their intersection is not empty. A vertex v is said to be *incident* to an edge e if $v \in e$. In a hypergraph G, a *path of length* q is defined to be an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_q, e_q, v_{q+1}$ such that

(1) $v_1, \ldots, v_{q+1}$ are all distinct vertices of G,
(2) $e_1 \ldots, e_q$ are all distinct edges of G,
(3) $v_r, v_{r+1} \in e_r$ for $r = 1, \ldots, q$.

If $q > 1$ and $v_1 = v_{q+1}$, then this path is called a *cycle of length* q. A hypergraph G is *connected* if there exists a path starting at v and terminating at u for all $v, u \in V$, and is called *acyclic* if it contains no cycle. These definitions can be found in Berge (1976) and Bretto (2013).

The following definitions on power hypergraphs and hypertrees can be found in Hu et al. (2013).

Definition 2 (Hu et al. 2013) Let $G = (V, E)$ be an ordinary graph. For every $k \geq 3$, the kth power of G, $G^k := (V^k, E^k)$ is defined as the k-uniform hypergraph with the edge set $E^k := \{e \cup \{i_{e,1}, \ldots, i_{e,k-2}, \} | e \in E\}$ and the vertex set $V^k := V \cup (\cup_{e \in E}\{i_{e,1}, \ldots, i_{e,k-2}\})$.

Definition 3 (Hu et al. 2013) The kth power of an ordinary tree is called a hypertree.

3 Supertrees

In ordinary graph theory, a tree is defined to be a connected graph without cycles. Analogously, we introduce the concept of supertree as follows.

Definition 4 A *supertree* is a hypergraph which is both connected and acyclic.

From this definition and the definition of hypertrees we can see that all hypertrees are supertrees.

A characterization of acyclic hypergraph has been given in Berge's (1976) textbook and particularly for the connected case is the following result.

Proposition 5 [Berge (1976), Proposition 4, p. 392] *If G is a connected hypergraph with n vertices and m edges, then it is acyclic if and only if $\sum_{i \in [m]}(|e_i| - 1) = n - 1$.*

In particular, if G is a connected k-uniform hypergraph with n vertices and m edges, then it is acyclic if and only if $m = \frac{n-1}{k-1}$.

Proposition 6 *A supertree G is a linear hypergraph. If in addition, G is a k-uniform supertree on n vertices, then it has $\frac{n-1}{k-1}$ edges.*

Proof Suppose on the contrary that G is not a linear hypergraph, then there exist two distinct edges e_i and e_j having at least two common vertices, say $\{v_1, v_2\} \subseteq e_i \cap e_j$. Then v_1, e_i, v_2, e_j, v_1 would be a cycle of length 2, contradicting that G is acyclic. So G is a linear hypergraph.

By Proposition 5, we know that a k-uniform supertree on n vertices has $\frac{n-1}{k-1}$ edges. □

Proposition 7 *Let n, k be positive integers with $n \geq k$. Then there exists a k-uniform supertree with n vertices if and only if $n - 1$ is a multiple of $k - 1$.*

Proof The necessity follows from Proposition 6. Now if $n - 1$ is a multiple of $k - 1$. Let $n' = \frac{n-1}{k-1} + 1$, and take G to be the k-th power of an ordinary tree T of order n'. Then it is easy to verify that G is a k-uniform supertree with n vertices. □

Definition 8 (Hu et al. 2013) Let $G = (V, E)$ be a k-uniform hypergraph with n vertices and m edges.

- If there is a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \cdots \cup V_m$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_m| = k - 1$, and $E = \{V_0 \cup V_i \mid i \in [m]\}$, then G is called a hyperstar, denoted by $\mathcal{S}_{n,k}$.
- If G is a path $(v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m)$ such that the vertices $v_1, \ldots, v_{m-1}$ are of degree two, and all the other vertices of G are of degree one, then G is called a loose path, denoted by $\mathcal{P}_{n,k}$.

Note that both hyperstar and loose path are supertrees. Also, in the literature, hyperstars are more commonly called sunflowers. Here we choose to call it hyperstar since it is a hypertree, and is actually the k-th power of an ordinary star.

4 The incidence $\boldsymbol{Q}$-tensors of uniform hypergraphs

In Shao (2013), Shao introduced a definition for tensor product and then Bu et al. (2014) generalized it as follows:

Definition 9 Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times \cdots \times n_{k+1}}$ be order $m \geq 2$ and $k \geq 1$ tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ with entries

$$c_{i\alpha_1 \ldots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m \in [n_2]} a_{ii_2 \ldots i_m} b_{i_2 \alpha_1} \ldots b_{i_m \alpha_{m-1}},$$

where $i \in [n_1], \alpha_1, \ldots, \alpha_{m-1} \in [n_3] \times \cdots \times [n_{k+1}]$.

Note that by Definition 9, now $\mathcal{T} x^{k-1}$ defined in (1) can be simply written as $\mathcal{T} x$.

Let $G = (V, E)$ be a k-uniform hypergraph with vertex set $V = \{v_1, \ldots, v_n\}$, and edge set $E = \{e_1 \ldots, e_m\}$. In Bretto (2013), the *incidence matrix* of G is defined to be a matrix R whose rows and columns are indexed by the vertices and edges of G, respectively. The (i, j)-entry of R is

$$r_{ij} = \begin{cases} 1, & \text{if } v_i \in e_j; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbb{I}$ denote the identity tensor of appropriate dimension, e.g., $\mathbb{I}_{i_1 \ldots i_k} = 1$ if $i_1 = \cdots = i_k \in [m]$, and zero otherwise when the dimension is m. Consider the tensor $R \mathbb{I} R^T$. By Definition 9, it is a tensor of order k and dimension n, whose $(i_1, i_2, \ldots, i_k)$-entry is

$$(R \mathbb{I} R^T)_{i_1, i_2, \ldots, i_k} = \sum_{j=1}^m r_{i_1 j} (\mathbb{I} R^T)_{ji_2 \ldots i_k}$$

$$= \sum_{j=1}^m r_{i_1 j} \left(\sum_{l_2, \ldots, l_k = 1}^m \mathbb{I}_{jl_2 \ldots l_k} r_{i_2 l_2} \ldots r_{i_k l_k} \right)$$

$$= \sum_{j=1}^m r_{i_1 j} r_{i_2 j} \ldots r_{i_k j} \tag{2}$$

which is the number of edges e of G such that $i_t \in e$ for all $t = 1, \ldots, k$.

Note that then $R \mathbb{I} R^T$ is a symmetric tensor of order k and dimension n. Consider the homogeneous polynomial $f(x) := x^T (R \mathbb{I} R^T x)$, and let $y = R^T x$, write

$$x(e) = \sum_{i \in e} x_i.$$

Then by $y = R^T x$ we have $y_j = \sum_{i \in e_j} x_i = x(e_j)$, and so

$$x^T (R \mathbb{I} R^T x) = x^T (R \mathbb{I} y) = x^T (R y^{[k-1]}) = y^T y^{[k-1]} = \sum_{j \in [m]} y_j^k$$

$$= \sum_{j \in [m]} x(e_j)^k = \sum_{e \in E} x(e)^k. \tag{3}$$

Thus it can be seen that when k is even, $f(x) \geq 0$ for any $x \in \mathbb{R}^n$ and so $R\mathbb{I}R^T$ is positive semi-definite.

Definition 10 The incidence Q-tensor of the uniform hypergraph G, denoted by $\mathcal{Q}^* = \mathcal{Q}^*(G)$, is defined as $\mathcal{Q}^* = R\mathbb{I}R^T$.

From this definition of the incidence Q-tensor and the above formula (2) for the entries of $\mathcal{Q}^*$ we also have that

$$
\begin{aligned}
(\mathcal{Q}^* x)_i &= \sum_{i_2,\dots,i_k=1}^{n} \mathcal{Q}^*_{i i_2,\dots,i_k} x_{i_2} \dots x_{i_k} \\
&= \sum_{i_2,\dots,i_k=1}^{n} \left(\sum_{j=1}^{m} r_{ij} r_{i_2 j} \dots r_{i_k j} \right) x_{i_2} \dots x_{i_k} \\
&= \sum_{i_2,\dots,i_k=1}^{n} \left(\sum_{e \in E_i, i_2 \in e, \dots, i_k \in e} x_{i_2} \dots x_{i_k} \right) \\
&= \sum_{e \in E_i} \left(\sum_{i_2 \in e} x_{i_2} \right) \dots \left(\sum_{i_k \in e} x_{i_k} \right) \\
&= \sum_{e \in E_i} x(e)^{k-1} \tag{4}
\end{aligned}
$$

Xie and Chang (2013c) defined the signless Laplacian tensor T_Q of an even uniform hypergraph as the symmetric tensor associated with the polynomial

$$
T_Q x^k := \sum_{e_p \in E} Q(e_p) x^k, \quad \forall x \in \mathbb{R}^n, \tag{5}
$$

where $Q(e_p) x^k = (x_{i_1} + x_{i_2} + \dots + x_{i_k})^k$, for $e_p = \{i_1, i_2, \dots, i_k\} \subseteq V$. Comparing Eqs. (5) and (3), clearly $f(x) = T_Q x^k$ and so $T_Q = \mathcal{Q}^*$. That is, our incidence Q-tensor coincides with the signless Laplacian tensor T_Q introduced by Xie and Chang, who defined it for even uniform hypergraph. Note that we do not put restriction on the parity of k, namely, the incidence Q-tensor $\mathcal{Q}^*$ applies to both even and odd uniform hypergraphs.

We prefer not to call $\mathcal{Q}^*$ the signless Laplacian tensor of G. In spectral graph theory, the Laplacian matrix and the signless Laplacian matrix appear together. The Laplacian matrix is defined as $L = D - A$, and the signless Laplacian matrix is defined as $Q = D + A$, where D is the degree diagonal matrix and A is the adjacency matrix of the graph. Such a definition was generalized to hypergraphs in Qi (2014) and further studied in Hu and Qi (2014), Hu et al. (2013, 2015), Qi et al. (2014), Shao et al. (2015). On the other hand, the tensor $\mathcal{Q}^* = R\mathbb{I}R^T$ is closely related to the incidence matrix R of the hypergraph. Thus, it is also adequate to be called the incidence Q-tensor of that hypergraph.

A kth-order n-dimensional tensor $\mathcal{T} = (\mathcal{T}_{i_1 i_2 \ldots i_k})$ is called *reducible*, if there exists a nonempty proper index subset $I \subset [n]$ such that

$$\mathcal{T}_{i_1 i_2 \ldots i_k} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_k \notin I.$$

$\mathcal{T}$ is called *weakly reducible* (Friedland et al. 2013), if there exists a nonempty proper index subset $I \subset [n]$ such that

$$\mathcal{T}_{i_1 i_2 \ldots i_k} = 0, \quad \forall i_1 \in I, \quad \text{and at least one of the } i_2, \ldots, i_k \notin I.$$

If $\mathcal{T}$ is not reducible, then $\mathcal{T}$ is called *irreducible*. If $\mathcal{T}$ is not weakly reducible, then $\mathcal{T}$ is called *weakly irreducible*.

It is easy to see from the definition that irreducibility implies weak irreducibility.

It is proved in Friedland et al. (2013) and Yang and Yang (2011) that a uniform hypergraph G is connected if and only if its adjacency tensor $\mathcal{A}$ is weakly irreducible (and also if and only if its signless Laplacian tensor $\mathcal{Q}$ is weakly irreducible). Now for the incidence Q-tensor $\mathcal{Q}^*$, we have the following result.

Lemma 11 *Let G be a k-uniform hypergraph on n vertices. Then the incidence Q-tensor $\mathcal{Q}^*$ is irreducible if and only if G is connected.*

Proof Sufficiency. Suppose that G is connected. For any nonempty proper subset $I \subset [n]$, choose arbitrarily two vertices i, j such that $i \in I$ and $j \in V \backslash I$. Because G is connected, there exists a path $i_1 (= i), e_1, i_2, e_2, i_3, \ldots, e_q, i_{q+1} (= j)$ such that $i_l, i_{l+1} \in e_l$ for $l = 1, \ldots, q$. Since $i_1 = i \in I$ and $i_{q+1} = j \in V \backslash I$, clearly there exists some $r (1 \le r \le q)$ such that $i_r \in I$ and $i_{r+1} \in V \backslash I$. This implies that $\mathcal{Q}^*_{i_r i_{r+1} \ldots i_{r+1}} \ne 0$ as there exists at least one edge (e.g. e_{i_r}) contains both i_r and i_{r+1}. Thus $\mathcal{Q}^*$ is irreducible.

Necessity. Suppose that G is disconnected. Assume that G_1 is a connected component of G, with vertex set $V_1 = V(G_1)$. Then for any $i_1 \in V_1$ and $i_2, \ldots, i_k \in V \backslash V_1$, there exists no edge containing all vertices $i_1, i_2, \ldots, i_k$. So $\mathcal{Q}^*_{i_1 i_2 \ldots i_k} = 0$, and thus $\mathcal{Q}^*$ is reducible. $\square$

Consequently, if G is connected, then $\mathcal{Q}^*$ is irreducible. Since irreducible nonnegative tensor with a nonzero diagonal is primitive, this incidence Q-tensor $\mathcal{Q}^*$ is also a primitive tensor.

Let $\mathcal{T}$ be a kth-order n-dimensional nonnegative tensor. The spectral radius of $\mathcal{T}$ is defined as $\rho(\mathcal{T}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{T}\}$. Part of Perron-Frobenius theorem for nonnegative tensors is stated in the following for reference, and for more details one can refer to a survey (Chang et al. 2013).

Theorem 12 *If $\mathcal{T}$ is a nonnegative tensor, then $\rho(\mathcal{T})$ is an eigenvalue with a nonnegative eigenvector x corresponding to it.*

If furthermore $\mathcal{T}$ is symmetric and weakly irreducible, then x is positive. Moreover, the nonnegative eigenvector is unique up to a constant multiple.

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \ge 0\}$.

Lemma 13 (Hu and Qi 2015) *Let $\mathcal{T}$ be a symmetric nonnegative tensor of order k and dimension n. Then*

$$\rho(\mathcal{T}) = \max\left\{ x^T(\mathcal{T}x) \mid x \in \mathbb{R}^n_+, \sum_{i=1}^{n} x_i^k = 1 \right\}. \qquad (6)$$

Furthermore, $x \in \mathbb{R}^n_+$ with $\sum_{i=1}^{n} x_i^k = 1$ is an eigenvector of $\mathcal{T}$ corresponding to $\rho(\mathcal{T})$ if and only if it is an optimal solution of the maximization problem (6).

From Lemma 13, it follows immediately that $\rho(\mathcal{T})$ can also be expressed as follows

$$\rho(\mathcal{T}) = \max\left\{ \frac{x^T(\mathcal{T}x)}{x^T(\mathbb{I}x)} \mid x \in \mathbb{R}^n_+, x \neq 0 \right\},$$

where $\mathcal{T}$ and $\mathbb{I}$ have the same order and dimension. Note that $x^T(\mathbb{I}x) = \sum_{i \in [n]} x_i^k = \| x \|_k^k$. By Theorem 12, for a symmetric weakly irreducible nonnegative tensor $\mathcal{T}$, it has a unique positive eigenvector x with $\| x \|_k = 1$ corresponding to $\rho(\mathcal{T})$ and then we call x the *principal eigenvector* of $\mathcal{T}$.

5 The extremal spectral radii of k-uniform supertrees and kth power hypertrees on n vertices

In this section, we prove our main results (see Theorems 19, 21, 26 and 30 below). For this purpose, we first introduce the operation of "moving edges" for hypergraphs, together with the two special cases of this operation: the edge-releasing operation and the total grafting operation. We study the perturbation of the three kinds of spectral radii of hypergraphs under these operations: the adjacency spectral radius, the signless Laplacian spectral radius and the incidence Q-spectral radius. We show that all these three kinds of spectral radii of supertrees strictly increase under the edge-releasing operation and the inverse of the total grafting operation.

Using these perturbation results, we prove that for all these three kinds of spectral radii, the hyperstar $\mathcal{S}_{n,k}$ attains uniquely the maximum spectral radius among all k-uniform supertrees on n vertices, and we give the exact values of these three kinds of spectral radii of $\mathcal{S}_{n,k}$. We also determine (in Theorem 21) that for all these three kinds of spectral radii, $S^k(1, n'-3)$ attains uniquely the second largest spectral radius among all k-uniform supertrees on n vertices (where $n' = \frac{n-1}{k-1} + 1$). We also prove that for all these three kinds of spectral radii, the loose path $\mathcal{P}_{n,k}$ attains uniquely the minimum spectral radius among all k-th power hypertrees on n vertices.

By Proposition 7, we know that there exists a k-uniform supertree with n vertices if and only if $n-1$ is a multiple of $k-1$. So in this section, we always assume that $n-1$ is a multiple of $k-1$.

Recall the Laplacian tensor and signless Laplacian tensor proposed by Qi (2014). Let $G = (V, E)$ be a k-uniform hypergraph. The *adjacency tensor* of G was defined in Cooper and Dutle (2012) as the k-th order n-dimensional tensor $\mathcal{A}$ whose $(i_1 \ldots i_k)$-entry is:

$$a_{i_1\dots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1,\dots,i_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}$ be a k-th order n-dimensional diagonal tensor with its diagonal element $d_{i\dots i}$ being d_i, the degree of vertex i in G, for all $i \in [n]$. Then $\mathcal{L} = \mathcal{D} - \mathcal{A}$ is the *Laplacian tensor* of the hypergraph G, and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ is the *signless Laplacian tensor* of the hypergraph G.

For a vector x of dimension n and a subset $U \subseteq [n]$, we write

$$x^U = \prod_{i \in U} x_i$$

By Cooper and Dutle (2012), we have

$$x^T(\mathcal{A}(G)x) = \sum_{e \in E(G)} kx^e$$

and

$$(\mathcal{A}(G)x)_i = \sum_{e \in E_i(G)} x^{e \setminus \{i\}}$$

Also it is easy to calculate for the signless Laplacian tensor $\mathcal{Q}(G)$ that:

$$x^T(\mathcal{Q}(G)x) = \sum_{\{j_1,\dots,j_k\} \in E(G)} (x_{j_1}^k + \cdots + x_{j_k}^k + kx_{j_1}\dots x_{j_k}) = \sum_{e \in E(G)} (x^{[k]}(e) + kx^e)$$

where $x^{[k]}(e) = x_{j_1}^k + \cdots + x_{j_k}^k$ for $e = \{j_1,\dots,j_k\}$, and

$$(\mathcal{Q}(G)x)_i = d_i(G)x_i^{k-1} + \sum_{e \in E_i(G)} x^{e \setminus \{i\}}$$

Now we introduce the operation of *moving edges* on hypergraphs.

Definition 14 Let $r \geq 1$, $G = (V, E)$ be a hypergraph with $u \in V$ and $e_1,\dots,e_r \in E$, such that $u \notin e_i$ for $i = 1,\dots,r$. Suppose that $v_i \in e_i$ and write $e_i' = (e_i \setminus \{v_i\}) \cup \{u\}$ $(i = 1,\dots,r)$. Let $G' = (V, E')$ be the hypergraph with $E' = (E \setminus \{e_1,\dots,e_r\}) \cup \{e_1',\dots,e_r'\}$. Then we say that G' is obtained from G by moving edges $(e_1,\dots,e_r)$ from $(v_1,\dots,v_r)$ to u.

Remark (1) The vertices $v_1,\dots,v_r$ need not be distinct. That is, the repetition of some vertices in $v_1,\dots,v_r$ is allowed.
(2) Generally speaking, the new hypergraph G' may contain multiple edges. But if G is acyclic and there is an edge $e \in E$ containing all the vertices $u, v_1,\dots,v_r$, then G' contains no multiple edges.

Theorem 15 *Let $r \geq 1$, G be a connected hypergraph, G' be the hypergraph obtained from G by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to u, and G' contains no multiple edges. Then we have:*

(1) *If x is the principal eigenvector of $\mathcal{A}(G)$ corresponding to $\rho(\mathcal{A}(G))$, and suppose that $x_u \geq \max_{1 \leq i \leq r}\{x_{v_i}\}$, then $\rho(\mathcal{A}(G')) > \rho(\mathcal{A}(G))$.*
(2) *If x is the principal eigenvector of $\mathcal{Q}(G)$ corresponding to $\rho(\mathcal{Q}(G))$, and suppose that $x_u \geq \max_{1 \leq i \leq r}\{x_{v_i}\}$, then $\rho(\mathcal{Q}(G')) > \rho(\mathcal{Q}(G))$.*
(3) *If x is the principal eigenvector of $\mathcal{Q}^*(G)$ corresponding to $\rho(\mathcal{Q}^*(G))$, and suppose that $x_u \geq \max_{1 \leq i \leq r}\{x_{v_i}\}$, then $\rho(\mathcal{Q}^*(G')) > \rho(\mathcal{Q}^*(G))$*

Proof Let $e_i' = (e_i \backslash \{v_i\}) \cup \{u\}$ $(i = 1, \ldots, r)$ as in Definition 14.

(1) By the hypothesis we obviously have $x^{e_i'}/x^{e_i} = x_u/x_{v_i} \geq 1$. Thus by using Lemma 13 and the above expression for $x^T(\mathcal{A}(G)x)$ we have

$$\rho(\mathcal{A}(G')) - \rho(\mathcal{A}(G)) \geq x^T(\mathcal{A}(G')x) - \rho(\mathcal{A}(G))$$
$$= x^T(\mathcal{A}(G')x) - x^T(\mathcal{A}(G)x)$$
$$= \sum_{e \in E(G')} kx^e - \sum_{e \in E(G)} kx^e$$
$$= k \sum_{i=1}^{r} (x^{e_i'} - x^{e_i}) \geq 0.$$

If the equality holds, then $\rho(\mathcal{A}(G')) = x^T(\mathcal{A}(G')x)$ and so x is the eigenvector of $\mathcal{A}(G')$ corresponding to $\rho(\mathcal{A}(G')) = \rho(\mathcal{A}(G))$ by Lemma 13. In this case, using the above expression for $(\mathcal{A}(G)x)_u$ and $(\mathcal{A}(G')x)_u$, we have

$$0 = (\rho(\mathcal{A}(G')) - \rho(\mathcal{A}(G)))x_u^{k-1}$$
$$= (\mathcal{A}(G')x)_u - (\mathcal{A}(G)x)_u$$
$$= \sum_{e \in E_u(G')} x^{e \backslash \{u\}} - \sum_{e \in E_u(G)} x^{e \backslash \{u\}}$$
$$= \sum_{i=1}^{r} x^{e_i' \backslash \{u\}} > 0$$

a contradiction.

(2) By the hypothesis we have $x^{[k]}(e_i') - x^{[k]}(e_i) = x_u^k - x_{v_i}^k \geq 0$, and $x^{e_i'}/x^{e_i} = x_u/x_{v_i} \geq 1$. Thus by using Lemma 13 and the above expression for $x^T(\mathcal{Q}(G)x)$ we have

$$\rho(\mathcal{Q}(G')) - \rho(\mathcal{Q}(G)) \geq x^T(\mathcal{Q}(G')x) - \rho(\mathcal{Q}(G))$$
$$= x^T(\mathcal{Q}(G')x) - x^T(\mathcal{Q}(G)x)$$

$$= \sum_{e \in E(G')} (x^{[k]}(e) + kx^e) - \sum_{e \in E(G)} (x^{[k]}(e) + kx^e)$$

$$= \sum_{i=1}^{r} (x^{[k]}(e_i') - x^{[k]}(e_i)) + k \sum_{i=1}^{r} (x^{e_i'} - x^{e_i}) \geq 0,$$

Also by using the above expression for $(\mathcal{Q}(G)x)_u$, the strict inequality can be obtained similarly from the following relation:

$$(\mathcal{Q}(G')x)_u - (\mathcal{Q}(G)x)_u = (d_u(G') - d_u(G))x_u^{k-1} + \sum_{i=1}^{r} x^{e_i' \setminus \{u\}} > 0$$

where $d_u(G') - d_u(G) = r$.

(3) By the hypothesis we have:

$$x(e_i') - x(e_i) = x_u - x_{v_i} \geq 0 \quad (i = 1, \ldots, r).$$

Thus we have $x(e_i')^k \geq x(e_i)^k$, since x is a positive vector. Now by Lemma 13 and the Eq. (3) we have

$$\rho(\mathcal{Q}^*(G')) - \rho(\mathcal{Q}^*(G)) \geq x^T(\mathcal{Q}^*(G')x) - \rho(\mathcal{Q}^*(G))$$

$$= x^T(\mathcal{Q}^*(G')x) - x^T(\mathcal{Q}^*(G)x)$$

$$= \sum_{e \in E(G')} x(e)^k - \sum_{e \in E(G)} x(e)^k$$

$$= \sum_{i=1}^{r} (x(e_i')^k - x(e_i)^k) \geq 0,$$

If the equality holds, then $\rho(\mathcal{Q}^*(G')) = x^T(\mathcal{Q}^*(G')x)$ and so x is the eigenvector of $\mathcal{Q}^*(G')$ corresponding to $\rho(\mathcal{Q}^*(G')) = \rho(\mathcal{Q}^*(G))$ by Lemma 13. In this case, applying eigenvalue equations and Eq. (4) to the vertex u in G' and G, we find

$$0 = (\rho(\mathcal{Q}^*(G')) - \rho(\mathcal{Q}^*(G)))x_u^{k-1}$$

$$= (\mathcal{Q}^*(G')x)_u - (\mathcal{Q}^*(G)x)_u$$

$$= \sum_{e \in E_u(G')} x(e)^{k-1} - \sum_{e \in E_u(G)} x(e)^{k-1}$$

$$= \sum_{i=1}^{r} x(e_i')^{k-1} > 0,$$

a contradiction.   □

Recall that a linear hypergraph is a hypergraph each pair of whose edges has at most one common vertex. We have proved in Proposition 6 that all supertrees are linear hypergraphs.

In a k-uniform linear hypergraph G, an edge e is called a *pendent edge* if e contains exactly $k - 1$ vertices of degree one. If e is not a pendent edge, then it is also called a *non-pendent edge*.

The following *edge-releasing* operation on linear hypergraphs is a special case of the above defined edge moving operation.

Definition 16 Let G be a k-uniform linear hypergraph, e be a non-pendent edge of G and $u \in e$. Let $\{e_1, e_2, \ldots, e_r\}$ be all the edges of G adjacent to e but not containing u, and suppose that $e_i \cap e = \{v_i\}$ for $i = 1, \ldots, r$. Let G' be the hypergraph obtained from G by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to u. Then G' is said to be obtained from G by an edge-releasing operation on e at u.

In other words, edge-releasing a non-pendent edge e of G at u means moving all the edges adjacent to e but not containing u from their commom vertices with e to u.

Since e is a non-pendent edge of G, e contains at least one non-pendent vertex different from u. So by the definition of linear hypergraph, there exists at least one edge "adjacent to e but not containing u". This means that edge-releasing operation is a special case of the edge-moving operation in Definition 14.

From the above definition we can see that, if G' and G'' are the hypergraphs obtained from a k-uniform linear hypergraph G by an edge-releasing operation on some edge e at u and at v, respectively. Then G' and G'' are isomorphic. Also, if G is acyclic, then G' contains no multiple edges.

Proposition 17 *Let G' be a hypergraph obtained from a k-uniform supertree G by edge-releasing a non-pendent edge e of G. Then G' is also a supertree.*

Proof Since G is connected, it is easy to see that G' is also connected. Also by the definition of the edge-releasing operation we can see that, G and G' have the same number of edges. Thus we have $|E(G')| = |E(G)| = \frac{n-1}{k-1}$. So by Proposition 5 we conclude that G' is also a supertree. □

Theorem 18 *Let G' be a supertree obtained from a k-uniform supertree G by edge-releasing a non-pendent edge e of G at u. Then we have $\rho(\mathcal{A}(G')) > \rho(\mathcal{A}(G))$, $\rho(\mathcal{Q}(G')) > \rho(\mathcal{Q}(G))$, and $\rho(\mathcal{Q}^*(G')) > \rho(\mathcal{Q}^*(G))$.*

Proof To prove $\rho(\mathcal{A}(G')) > \rho(\mathcal{A}(G))$, take x to be the principal eigenvector of $\mathcal{A}(G)$ corresponding to $\rho(\mathcal{A}(G))$, and take $v \in e$ such that $x_v = \max_{i \in e}\{x_i\}$. Let G'' be the supertree obtained from G by edge-releasing the edge e of G at v, then G' and G'' are isomorphic. But by Definition 16, G'' is obtained from G by moving some edges from some vertices of e to v. So by Theorem 15 we have $\rho(\mathcal{A}(G')) = \rho(\mathcal{A}(G'')) > \rho(\mathcal{A}(G))$.

Similarly we can prove $\rho(\mathcal{Q}(G')) > \rho(\mathcal{Q}(G))$ and $\rho(\mathcal{Q}^*(G')) > \rho(\mathcal{Q}^*(G))$. □

Theorem 19 *Let $\mathfrak{T}$ be a k-uniform supertree on n vertices with m edges (here $m = \frac{n-1}{k-1}$). Then*

$$\rho(\mathcal{A}(\mathfrak{T})) \le \rho(\mathcal{A}(\mathcal{S}_{n,k}))$$

and

$$\rho(\mathcal{Q}(\mathfrak{T})) \le \rho(\mathcal{Q}(\mathcal{S}_{n,k}))$$

and

$$\rho(\mathcal{Q}^*(\mathfrak{T})) \leq \rho(\mathcal{Q}^*(\mathcal{S}_{n,k}))$$

with either one of the equalities holding if and only if $\mathfrak{T}$ is the hyperstar $\mathcal{S}_{n,k}$.

Proof We use induction on the number of non-pendent vertices (vertices with degrees at least two) $N_2(\mathfrak{T})$. If $N_2(\mathfrak{T}) = 1$, then $\mathfrak{T}$ is the hyperstar $\mathcal{S}_{n,k}$. Now we assume that $N_2(\mathfrak{T}) \geq 2$, namely $\mathfrak{T}$ is not a hyperstar. Suppose x and y be two non-pendent vertices. Then there must be some non-pendent edge e in the path from x to y. Let $\mathfrak{T}'$ be the supertree obtained from $\mathfrak{T}$ by edge-releasing the non-pendent edge e of $\mathfrak{T}$. Then by Theorem 18 we have $\rho(\mathcal{A}(\mathfrak{T})) < \rho(\mathcal{A}(\mathfrak{T}'))$. On the other hand, we have $N_2(\mathfrak{T}') < N_2(\mathfrak{T})$. So by the inductive hypothesis we have $\rho(\mathcal{A}(\mathfrak{T}')) \leq \rho(\mathcal{A}(\mathcal{S}_{n,k}))$. Combining the above two relations we obtain $\rho(\mathcal{A}(\mathfrak{T})) < \rho(\mathcal{A}(\mathcal{S}_{n,k}))$.

Using the same arguments we can prove the second and the third inequalities. □

Next we determine the supertree with the second largest spectral radius (also for the three kinds of spectral radii).

Let $S(a, b)$ be the ordinary tree with $a + b + 2$ vertices obtained from an edge e by attaching a pendent edges to one end vertex of e, and attaching b pendent edges to the other end vertex of e. Let $S^k(a, b)$ be the kth power of $S(a, b)$. We have the following lemma for the comparison of the spectral radii of $S^k(a, b)$ and $S^k(c, d)$ when $a + b = c + d$.

Lemma 20 *Let a, b, c, d be nonnegative integers with $a + b = c + d$. Suppose that $a \leq b$, $c \leq d$ and $a < c$, then we have:*

$$\rho(\mathcal{A}(S^k(a, b))) > \rho(\mathcal{A}(S^k(c, d))),$$

and

$$\rho(\mathcal{Q}(S^k(a, b))) > \rho(\mathcal{Q}(S^k(c, d))),$$

and

$$\rho(\mathcal{Q}^*(S^k(a, b))) > \rho(\mathcal{Q}^*(S^k(c, d))).$$

Proof Let x, y be the (only) two non-pendent vertices of $S^k(c, d)$ with the degrees $d(x) = c + 1$ and $d(y) = d + 1$. Let G' be obtained from $S^k(c, d)$ by moving $c - a$ pendent edges from x to y, and G'' be obtained from $S^k(c, d)$ by moving $d - a$ pendent edges from y to x. Then both G' and G'' are isomorphic to $S^k(a, b)$.

On the other hand, it can be verified that at least one of G' and G'' will satisfy the condition (1) of Theorem 15. So by Theorem 15 we have

$$max(\rho(\mathcal{A}(G')), \rho(\mathcal{A}(G''))) > \rho(\mathcal{A}(S^k(c, d))).$$

Thus we have

$$\rho(\mathcal{A}(S^k(a, b))) = max(\rho(\mathcal{A}(G')), \rho(\mathcal{A}(G''))) > \rho(\mathcal{A}(S^k(c, d))).$$

The other two inequalities can be proved in exactly the same way. $\qquad\square$

The following theorem shows that for all these three kinds of spectral radii, $S^k(1, n' - 3)$ attains uniquely the second largest spectral radius among all k-uniform supertrees on n vertices (where $n' = \frac{n-1}{k-1} + 1$).

Theorem 21 *Let $\mathfrak{T}$ be a k-uniform supertree on n vertices (with $m = n' - 1$ edges where $n' = \frac{n-1}{k-1} + 1$). Suppose that $\mathfrak{T} \neq \mathcal{S}_{n,k}$, then we have*

$$\rho(\mathcal{A}(\mathfrak{T})) \leq \rho(\mathcal{A}(S^k(1, n' - 3))),$$

and

$$\rho(\mathcal{Q}(\mathfrak{T})) \leq \rho(\mathcal{Q}(S^k(1, n' - 3))),$$

and

$$\rho(\mathcal{Q}^*(\mathfrak{T})) \leq \rho(\mathcal{Q}^*(S^k(1, n' - 3))),$$

with either one of the equalities holding if and only if $\mathfrak{T} \cong S^k(1, n' - 3)$.

Proof We use induction on the number of non-pendent vertices $N_2(\mathfrak{T})$. Since $\mathfrak{T} \neq \mathcal{S}_{n,k}$, we have $N_2(\mathfrak{T}) \geq 2$. Now we assume that $\mathfrak{T} \neq S^k(1, n' - 3)$.

If $N_2(\mathfrak{T}) = 2$, then the two non-pendent vertices (say, x and y) of $\mathfrak{T}$ must be adjacent (otherwise, all the internal vertices of the path between x and y would be non-pendent vertices other than x and y, contradicting $N_2(\mathfrak{T}) = 2$), and so it can be easily verified that $\mathfrak{T} = S^k(c, d)$ for some positive integers $2 \leq c \leq d$ with $c + d = n' - 2$ ($2 \leq c$ since $\mathfrak{T} \neq S^k(1, n' - 3)$). So by Lemma 20 we get the desired results.

If $N_2(\mathfrak{T}) \geq 3$, let x, y be two non-pendent vertices of $\mathfrak{T}$. Let $x, e_1, x_1, \ldots, e_r, y$ be a path from x to y. Let $\mathfrak{T}_1$ be obtained from $\mathfrak{T}$ by moving all the edges incident with x (except e_1) to y, and $\mathfrak{T}_2$ be obtained from $\mathfrak{T}$ by moving all the edges incident with y (except e_r) to x. Then both $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are still supertrees (since they are still connected, and have the same number of edges as $\mathfrak{T}$), and we have

$$2 \leq N_2(\mathfrak{T}_i) = N_2(\mathfrak{T}) - 1 < N_2(\mathfrak{T}) \qquad (i = 1, 2).$$

So by induction and Theorem 15 (since at least one of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ will satisfy the condition (1) of Theorem 15) we have

$$\rho(\mathcal{A}(\mathfrak{T})) < max(\rho(\mathcal{A}(\mathfrak{T}_1)), \rho(\mathcal{A}(\mathfrak{T}_2))) \leq \rho(\mathcal{A}(S^k(1, n' - 3)))$$

Using the same arguments we can prove the second and the third inequalities. $\qquad\square$

Next we consider the minimal problems for these three kinds of spectral radii. By introducing the operation of *total grafting* and studying the perturbation of the spectral radii under this operation, we are able to determine that the loose path $\mathcal{P}_{n,k}$ attains uniquely the minimum spectral radius among all k-th power hypertrees on n vertices.

A path $P = (v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ in a k-uniform hypergraph H is called a *pendent path* (starting from v_0), if all the vertices $v_1, \ldots, v_{p-1}$ are of degree two,

the vertex v_p is of degree one, and all the $k - 2$ vertices in the set $e_i \setminus \{v_{i-1}, v_i\}$ are of degree one in H $(i = 1, \ldots, p)$.

Definition 22 Let G be a connected k-uniform linear hypergraph and v be a vertex of G. Let $G(v; p, q)$ be a k-uniform linear hypergraph obtained from G by adding two pendent paths $P = (v, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ and $Q = (v, e'_1, u_1, \ldots, u_{q-1}, e'_q, u_q)$ at v, where $V(P) \cap V(Q) = \{v\}$. Then we say that $G(v; p + q, 0)$ is obtained from $G(v; p, q)$ by a *total grafting* operation at v.

Proposition 23 *Let $G(v; p + q, 0)$ be the k-uniform linear hypergraph obtained from G by adding a pendent path $P = (v, e_1, v_1, \ldots, v_{p-1}, e_p, v_p, e_{p+1}, \ldots, e_{p+q}, v_{p+q})$ at v. Let G_1 be the hypergraph obtained from $G(v; p + q, 0)$ by moving the edge e_{p+1} from v_p to v, and let G_2 be the hypergraph obtained from $G(v; p + q, 0)$ by moving all edges incident to v (except e_1) from v to v_p. Then both G_1 and G_2 are isomorphic to $G(v; p, q)$.*

Proof The proof of this result is obvious. □

Theorem 24 *Let $G(v; p, q)$ and $G(v; p + q, 0)$ be defined as above (where G is connected). If both p and q are not zero, then*

$$\rho(\mathcal{A}(G(v; p, q))) > \rho(\mathcal{A}(G(v; p + q, 0))).$$

and

$$\rho(\mathcal{Q}(G(v; p, q))) > \rho(\mathcal{Q}(G(v; p + q, 0))).$$

and

$$\rho(\mathcal{Q}^*(G(v; p, q))) > \rho(\mathcal{Q}^*(G(v; p + q, 0))).$$

Proof Let G_1 be the hypergraph obtained from $G(v; p + q, 0)$ by moving the edge e_{p+1} from v_p to v, and let G_2 be the hypergraph obtained from $G(v; p + q, 0)$ by moving all edges incident to v (except e_1) from v to v_p. Then both G_1 and G_2 are isomorphic to $G(v; p, q)$ by Proposition 23. Since G is connected, $G(v; p + q, 0)$ is also connected and so we can assume that x is the principal eigenvector of $\mathcal{A}(G(v; p + q, 0))$ corresponding to $\rho(\mathcal{A}(G(v; p + q, 0)))$. Consider the components x_v, x_{v_p} of x corresponding to v and v_p. Obviously, either $x_v \geq x_{v_p}$ or $x_v \leq x_{v_p}$. Thus by Theorem 15, we have

$$\rho(\mathcal{A}(G(v; p, q))) = max\{\rho(\mathcal{A}(G_1)), \rho(\mathcal{A}(G_2))\} > \rho(\mathcal{A}(G(v; p + q, 0))),$$

Similarly we can show that

$$\rho(\mathcal{Q}(G(v; p, q))) = max\{\rho(\mathcal{Q}(G_1)), \rho(\mathcal{Q}(G_2))\} > \rho(\mathcal{Q}(G(v; p + q, 0))).$$

and

$$\rho(\mathcal{Q}^*(G(v; p, q))) = max\{\rho(\mathcal{Q}^*(G_1)), \rho(\mathcal{Q}^*(G_2))\} > \rho(\mathcal{Q}^*(G(v; p + q, 0))).$$

□

The following lemma is about the total grafting operation on ordinary trees.

Lemma 25 *Let T be an ordinary tree of order n which is not a path. Then the path P_n can be obtained from T by several times of total grafting operations.*

Proof Let $N_3(T)$ be the number of vertices in T with degree at least 3. Then $T \neq P_n \iff N_3(T) \geq 1$. We then use induction on $N_3(T)$.

Let v be a vertex of T, let u be a vertex with degree at least 3 which is furthest to v (since $N_3(T) \geq 1$). Then there are at least $(d(u) - 1)$ many pendant paths starting from u. By using $(d(u) - 2)$ many total grafting operations at u on these pendant paths, we finally obtain a tree T' of order n with $N_3(T') = N_3(T) - 1$ (since the vertex u has degree 2 in the new tree T'). By using induction on the tree T', we arrive our desired result. □

Theorem 26 *Let T^k be the kth power of an ordinary tree T, defined as in* Hu et al. *(2013). Suppose that T^k has n vertices. Then we have*

$$\rho(\mathcal{A}(\mathcal{P}_{n,k})) \leq \rho(\mathcal{A}(T^k)) \leq \rho(\mathcal{A}(\mathcal{S}_{n,k}))$$

and

$$\rho(\mathcal{Q}(\mathcal{P}_{n,k})) \leq \rho(\mathcal{Q}(T^k)) \leq \rho(\mathcal{Q}(\mathcal{S}_{n,k}))$$

and

$$\rho(\mathcal{Q}^*(\mathcal{P}_{n,k})) \leq \rho(\mathcal{Q}^*(T^k)) \leq \rho(\mathcal{Q}^*(\mathcal{S}_{n,k}))$$

where either one of the left equalities holds if and only if $T^k \cong \mathcal{P}_{n,k}$, and either one of the right equalities holds if and only if $T^k \cong \mathcal{S}_{n,k}$.

Proof If $T^k \neq \mathcal{P}_{n,k}$, then T is a tree of order $n' = \frac{n-1}{k-1} + 1$ which is not a path. By Lemma 25, $P_{n'}$ can be obtained from T by several times of total grafting operations. Accordingly, $\mathcal{P}_{n,k}$ can be obtained from T^k by several times of total grafting operations. So by Theorem 24, we have $\rho(\mathcal{A}(\mathcal{P}_{n,k})) < \rho(\mathcal{A}(T^k))$, and $\rho(\mathcal{Q}(\mathcal{P}_{n,k})) < \rho(\mathcal{Q}(T^k))$, and $\rho(\mathcal{Q}^*(\mathcal{P}_{n,k})) < \rho(\mathcal{Q}^*(T^k))$.

Since T^k is a supertree, the right inequalities follow immediately as a special case of Theorem 19. □

It is proved in Theorem 4.1 of Hu et al. (2015) that

$$\rho(\mathcal{Q}(\mathcal{S}_{n,k})) = 1 + \alpha^*,$$

where $\alpha^* \in (m - 1, m]$ is the largest real root of $x^k - (m - 1)x^{k-1} - m = 0$, and $m = \frac{n-1}{k-1}$ is the number of edges of $\mathcal{S}_{n,k}$.

Now we compute the value of $\rho(\mathcal{Q}^*(\mathcal{S}_{n,k}))$.

Recall that an automorphism of a k-uniform hypergraph G is a permutation σ of $V(G)$ such that $\{i_1, i_2, \ldots, i_k\} \in E(G)$ if and only if $\{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\} \in E(G)$, for any $i_j \in V(G)$, $j = 1, \ldots, k$. The group of all automorphisms of G is denoted by $Aut(G)$.

In Shao (2013), Shao introduced the concept of permutational similarity for tensors as follows: for two order k and dimension n tensors $\mathcal{A}$ and $\mathcal{B}$, if there exists

a permutation matrix $P = P_\sigma$ (corresponding to a permutation $\sigma \in S_n$) such that $\mathcal{B} = P\mathcal{A}P^T$, then $\mathcal{A}$ and $\mathcal{B}$ are called permutational similar. Note that if $\mathcal{B} = P\mathcal{A}P^T$, then $b_{i_1,\ldots,i_k} = a_{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_k)}$. Shao (2013) showed that similar tensors have the same characteristic polynomials and thus have the same spectra.

Proposition 27 *A permutation $\sigma \in S_n$ is an automorphism of a k-uniform hypergraph G on n vertices if and only if $P_\sigma \mathcal{Q}^* = \mathcal{Q}^* P_\sigma$.*

Proof Let $P = P_\sigma$ be the permutation matrix corresponding to σ, and $\mathcal{Q}' = P\mathcal{Q}^* P^T$. Then we have

$$\mathcal{Q}'_{i_1,\ldots,i_k} = \mathcal{Q}^*_{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_k)}.$$

So by the definition of automorphism and the associative law of the tensor product we have

$$\begin{aligned}
\sigma \in Aut(G) &\Longleftrightarrow \mathcal{Q}^*_{i_1,\ldots,i_k} = \mathcal{Q}^*_{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_k)} = \mathcal{Q}'_{i_1,\ldots,i_k} \quad (\forall i_1,\ldots,i_k \in [n]) \\
&\Longleftrightarrow \mathcal{Q}^* = \mathcal{Q}' = P\mathcal{Q}^* P^T \\
&\Longleftrightarrow P\mathcal{Q}^* = \mathcal{Q}^* P
\end{aligned}$$

$\square$

If x is an eigenvector of $\mathcal{Q}^*$ corresponding to the eigenvalue λ, then for each automorphism σ of G we have

$$\mathcal{Q}^* P_\sigma x = P_\sigma \mathcal{Q}^* x = \lambda P_\sigma x^{[k-1]} = \lambda (P_\sigma x)^{[k-1]}.$$

Thus $P_\sigma x$ is also an eigenvector of $\mathcal{Q}^*$ corresponding to the eigenvalue λ. This simple observation leads to what follows.

Lemma 28 *Let G be a connected k-uniform hypergraph, $\mathcal{Q}^* = \mathcal{Q}^*(G)$ be its (irreducible) incidence Q-tensor. If x is the principal eigenvector of $\mathcal{Q}^*$ corresponding to $\lambda = \rho(\mathcal{Q}^*)$, then we have:*

(1) *$P_\sigma x = x$ for each automorphism σ of G.*
(2) *For any orbit Ω of $Aut(G)$ and each pair of vertices $i, j \in \Omega$, the corresponding components x_i, x_j of x are equal.*

Proof (1) By hypothesis we have $\mathcal{Q}^* x = \lambda x^{[k-1]}$. For each automorphism σ of G we have

$$\mathcal{Q}^* P_\sigma x = P_\sigma \mathcal{Q}^* x = \lambda P_\sigma x^{[k-1]} = \lambda (P_\sigma x)^{[k-1]}.$$

Thus $P_\sigma x$ is also an eigenvector of $\mathcal{Q}^*$ corresponding to the eigenvalue λ. Since $\mathcal{Q}^*$ is nonnegative irreducible, by Theorem 12 the nonnegative eigenvector of $\mathcal{Q}^*$ corresponding to $\lambda = \rho(\mathcal{Q}^*)$ is unique up to a constant multiple. So $P_\sigma x = cx$ for some $c \in \mathbb{R}$. Thus $c^2 x^T x = (x^T P_\sigma^T) P_\sigma x = x^T x$, and so $c = 1$ since both $P_\sigma x$ and x are nonnegative.

The result (2) follows directly from result (1). $\square$

Now we can obtain the value of the incidence Q-spectral radius of the hyperstar as in the following theorem.

Theorem 29 *Let $\mathcal{S}_{n,k}$ be a k-uniform hyperstar on n vertices. Then*

$$\rho(\mathcal{Q}^*(\mathcal{S}_{n,k})) = (m^{1/(k-1)} + k - 1)^{k-1},$$

where $m = \frac{n-1}{k-1}$ is the number of edges of $\mathcal{S}_{n,k}$.

Proof Let $V_0 \cup V_1 \cup \cdots \cup V_m$ be the disjoint partition of $V(\mathcal{S}_{n,k})$ such that $|V_0| = 1$, $|V_1| = \cdots |V_m| = k - 1$ and $E = \{V_0 \cup V_i \,|i = 1, \ldots, m\}$. Note that V_0 and $V_1 \cup \cdots \cup V_m$ are two orbits of automorphism group $Aut(\mathcal{S}_{n,k})$. Let x be the principal eigenvector of $\mathcal{Q}^*(\mathcal{S}_{n,k})$. Since $\mathcal{S}_{n,k}$ is connected, by Lemma 28 we have that the components of x corresponding to vertices in V_0 and $V \setminus V_0$ are constant respectively, and let a and b be these common values respectively. By the eigenvalue equation $\mathcal{Q}^*(\mathcal{S}_{n,k})x = \rho x^{[k-1]}$ and the Eq. (4), where ρ denotes $\rho(\mathcal{Q}^*(\mathcal{S}_{n,k}))$ for convenience, we have

$$\rho a^{k-1} = m(a + (k - 1)b)^{k-1},$$
$$\rho b^{k-1} = (a + (k - 1)b)^{k-1}.$$

Dividing the first equation by the second equation, we obtain $(\frac{a}{b})^{k-1} = m$. Thus $\frac{a}{b} = m^{1/(k-1)}$. So by the second equation we have

$$\rho = \left(\frac{a}{b} + k - 1\right)^{k-1} = \left(m^{1/(k-1)} + k - 1\right)^{k-1}.$$

□

Next we show that $\rho(\mathcal{A}(\mathcal{S}_{n,k})) = m^{1/k}$. Similarly as in the proof of Theorem 29, let x be the principal eigenvector of $\mathcal{A}(\mathcal{S}_{n,k})$. Let u be the center of $\mathcal{S}_{n,k}$ (the unique non-pendent vertex). Let $a = x_u$ and b be the common value of all the other components of x. Then by the eigenvalue equation $\mathcal{A}(\mathcal{S}_{n,k})x = \rho x^{[k-1]}$, we have

$$\rho a^{k-1} = mb^{k-1},$$
$$\rho b^{k-1} = ab^{k-2}.$$

where $\rho = \rho(\mathcal{A}(\mathcal{S}_{n,k}))$. From this we solve that $\rho = a/b = m^{1/k}$.

Theorem 30 *Let $\mathfrak{T}$ be a k-uniform supertree on n vertices with $m = \frac{n-1}{k-1}$ edges. Then*

$$\rho(\mathcal{A}(\mathfrak{T})) \le m^{1/k},$$

and

$$\rho(\mathcal{Q}(\mathfrak{T})) \le 1 + \alpha^*,$$

where $\alpha^* \in (m-1, m]$ is the largest real root of $x^k - (m-1)x^{k-1} - m = 0$, and

$$\rho(\mathcal{Q}^*(\mathfrak{T})) \leq (m^{1/(k-1)} + k - 1)^{k-1}$$

where either one of the equalities holds if and only if $\mathfrak{T}$ is the hyperstar $\mathcal{S}_{n,k}$.

Proof The results follow directly from Theorems 19 and 29, and the fact $\rho(\mathcal{A}(\mathcal{S}_{n,k})) = m^{1/k}$, and the proof of Theorem 4.1 in Hu et al. (2015). □

6 Some other properties and bounds on incidence Q-spectral radius

In this section, we first give a characterization of regular hypergraphs in terms of their incidence Q-tensors, and then using it to give some upper and lower bounds of the incidence Q-spectral radii of uniform hypergraphs.

Proposition 31 *A k-uniform hypergraph G is regular (of degree r) if and only if its incidence Q-tensor has an all-1 eigenvector (with corresponding eigenvalue $k^{k-1}r$).*

Proof Let $\mathcal{Q}^* = (\mathcal{Q}^*_{i_1 i_2 \ldots i_k})$ be the incidence Q-tensor of G, and $x = \mathbf{1} = (1, \ldots, 1)^T$ be the all-1 vector. From the Eq. (4), we have

$$(\mathcal{Q}^*\mathbf{1})_i = \sum_{e \in E_i} x(e)^{k-1} = k^{k-1}d_i, \quad (\forall\, i = 1, \ldots, n).$$

Thus

    G is regular of degree $r \iff d_1 = \cdots = d_n = r \iff \mathcal{Q}^*\mathbf{1} = (k^{k-1}r)\mathbf{1} = (k^{k-1}r)\mathbf{1}^{[k-1]} \iff \mathbf{1}$ is an eigenvector of $\mathcal{Q}^*$ with corresponding eigenvalue $k^{k-1}r$. □

A natural way to bound the spectral radius of a symmetric nonnegative tensor is to use the expression of spectral radius presented in the form of maximization problem (6).

Theorem 32 *Let G be a k-uniform hypergraph with maximum degree Δ and average degree d. Then*

$$d \leq \frac{1}{k^{k-1}}\rho(\mathcal{Q}^*) \leq \Delta. \tag{7}$$

with equality holding in either of these inequalities if and only if G is regular.

Proof Since $\mathcal{Q}^*$ is a symmetric nonnegative tensor, we have

$$\rho(\mathcal{Q}^*) = \max\left\{x^T(\mathcal{Q}^*x) \mid x \in \mathbb{R}^n_+, \parallel x \parallel_k = 1\right\}.$$

Let $x = (1/\sqrt[k]{n})\mathbf{1} = (1/\sqrt[k]{n})(1, \ldots, 1)^T$. Then by Eq. (3) and the fact that $d = \frac{\sum_{i \in [n]} d_i}{n} = \frac{km}{n}$ we have

$$\rho(\mathcal{Q}^*) \geq x^T(\mathcal{Q}^*x) = \sum_{\{j_1, \ldots, j_k\} \in E} (x_{j_1} + \cdots + x_{j_k})^k = m\frac{k^k}{n} = k^{k-1}d,$$

If equality holds, then we have $\rho(\mathcal{Q}^*) = x^T(\mathcal{Q}^*x)$ and so x is a eigenvector of $\mathcal{Q}^*$ by Lemma 13. Thus all-1 vector $\mathbf{1}$ is an eigenvector of $\mathcal{Q}^*$ and so G is regular by Proposition 31.

Now for the right inequality. Let x be a nonnegative eigenvector of $\mathcal{Q}^*$ corresponding to $\rho(\mathcal{Q}^*)$ with $\|x\|_k = 1$. Then we have

$$
\begin{aligned}
\rho(\mathcal{Q}^*) = x^T(\mathcal{Q}^*x) &= \sum_{\{j_1,\ldots,j_k\}\in E} (x_{j_1} + \cdots + x_{j_k})^k \\
&\le k^{k-1} \sum_{\{j_1,\ldots,j_k\}\in E} (x_{j_1}^k + \cdots + x_{j_k}^k) \\
&= k^{k-1} \sum_{i\in[n]} d_i x_i^k \\
&\le k^{k-1}\Delta \sum_{i\in[n]} x_i^k \\
&= k^{k-1}\Delta,
\end{aligned}
$$

where the first inequality follows from Jensen's inequality $(\frac{x_{j_1}+\cdots+x_{j_k}}{k})^k \le \frac{x_{j_1}^k+\cdots+x_{j_k}^k}{k}$. If $\rho(\mathcal{Q}^*) = k^{k-1}\Delta$, then all inequalities above must be all equalities. Thus $d_1 = \cdots = d_n = \Delta$.

Conversely, if G is regular, then $d = \Delta$. By the inequalities (7), both sides become equalities. $\square$

Because $\mathcal{Q}^* = R\mathbb{I}R^T$, so it naturally attracts us to find the relation between the spectral radii $\rho(\mathcal{Q}^*)$ and $\rho(RR^T)$. For this purpose, we need the following inequalities.

Lemma 33 (Hardy et al. 1988) *If $0 < r < s$ and $a_1,\ldots,a_k \ge 0$, then we have*

$$
(a_1^s + a_2^s + \cdots + a_k^s)^{1/s} < (a_1^r + a_2^r + \cdots + a_k^r)^{1/r} \tag{8}
$$

unless all $a_1,\ldots,a_k$ but one are zero, and

$$
\left(\frac{a_1^s + a_2^s + \cdots + a_k^s}{k}\right)^{1/s} > \left(\frac{a_1^r + a_2^r + \cdots + a_k^r}{k}\right)^{1/r} \quad (a_1,\ldots,a_k > 0) \tag{9}
$$

Theorem 34 *Let G be a k-uniform ($k \ge 3$) connected hypergraph on n vertices, and $\mathcal{Q}^* = R\mathbb{I}R^T$ be its incidence Q-tensor, where R is the incidence matrix of G. Then*

$$
\rho(RR^T) < \rho(\mathcal{Q}^*) < k^{k-2}\rho(RR^T).
$$

Proof Let x be a nonnegative eigenvector of RR^T with unit length corresponding to its spectral radius $\rho(RR^T)$, and let $y = x^{[2/k]}$. Then $\sum_{i\in[n]} y_i^k = \sum_{i\in[n]} x_i^2 = 1$, and by inequality (8) we have

$$\rho(RR^T) = x^T(RR^T)x = \sum_{\{j_1,\dots,j_k\}\in E}(x_{j_1}+\cdots+x_{j_k})^2$$

$$= \sum_{\{j_1,\dots,j_k\}\in E}((y_{j_1}^{k/2}+\cdots+y_{j_k}^{k/2})^{2/k})^k$$

$$\leq \sum_{\{j_1,\dots,j_k\}\in E}(y_{j_1}+\cdots+y_{j_k})^k$$

$$= y^T(Q^*y)$$

$$\leq \rho(Q^*),$$

if equality holds in the last inequality, then y is a positive vector since the connectedness of G implies that Q^* is nonnegative irreducible. Then the first inequality must be strict by inequality (8). So we always have $\rho(RR^T) < \rho(Q^*)$.

For the second inequality, let y be the principle eigenvector of Q^* corresponding to its spectral radius $\rho(Q^*)$, and let $x = y^{[k/2]}$. Then $\sum_{i\in[n]}x_i^2 = \sum_{i\in[n]}y_i^k = 1$, and by inequality (9) we have

$$(y_{j_1}+\cdots+y_{j_k})^k = (x_{j_1}^{2/k}+\cdots+x_{j_k}^{2/k})^k = \left(\left(\frac{x_{j_1}^{2/k}+\cdots+x_{j_k}^{2/k}}{k}\right)^{k/2}\right)^2 \cdot k^k$$

$$< \left(\frac{x_{j_1}+\cdots+x_{j_k}}{k}\right)^2 \cdot k^k = (x_{j_1}+\cdots+x_{j_k})^2 \cdot k^{k-2}$$

From this inequality we have

$$\rho(Q^*) = y^T(Q^*y) = \sum_{\{j_1,\dots,j_k\}\in E}(y_{j_1}+\cdots+y_{j_k})^k$$

$$< \sum_{\{j_1,\dots,j_k\}\in E}(x_{j_1}+\cdots+x_{j_k})^2\cdot k^{k-2} = x^T(RR^T)x\cdot k^{k-2} \leq \rho(RR^T)\cdot k^{k-2}$$

$\square$

For the purpose of comparing these bounds in Theorem 34, take G_1 and G_2 be the k-uniform s-path and s-cycle on n vertices respectively, where $1 \leq s \leq \frac{k}{2}$ (Qi et al. 2014). Let R_1 and R_2 denote the incidence matrices of G_1 and G_2 respectively, and let m_i denotes the number of edges of G_i for $i = 1, 2$. Then we have

$$R_1^T R_1 = kI + sA(P_{m_1}), \quad R_2^T R_2 = kI + sA(C_{m_2})),$$

where $A(P_{m_1})$ and $A(C_{m_2})$ are the adjacency matrices of ordinary path and cycle on m_1 and m_2 vertices, respectively. Note that $\rho(R_1^T R_1) = k + s\rho(A(P_{m_1})) = k +$

$2s \cos \frac{\pi}{m_1+1}$ and $\rho(R_2^T R_2) = k + s\rho(A(C_{m_2})) = k + 2s$. Thus we have

$$k + 2s \cos \frac{\pi}{m_1 + 1} < \rho(\mathcal{Q}^*(G_1)) < k^{k-2}\left(k + 2s \cos \frac{\pi}{m_1 + 1} \right),$$
$$k + 2s < \rho(\mathcal{Q}^*(G_2)) < k^{k-2}(k + 2s).$$

Generally, these lower bounds are not better than the lower bound $k^{k-1}d$ in Theorem 32. However, these upper bounds are better than the upper bound $2k^{k-1}$ in Theorem 32, because $2s \leq k$ and so $1 + \frac{2s}{k}$ and $1 + \frac{2s}{k} \cos \frac{\pi}{m_1+1}$ are not more than 2.

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