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MB-tensors and MB_0-tensors

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The class of MB- (MB_0-)tensors, which is a generation of B- (B_0-)tensors and quasi-double B- (B_0-)tensors, is proposed. And we prove that an even order symmetric MB- (MB_0-)tensor is positive (semi-)definite. By giving some conditions to determine MB- (MB_0-)tensors, some checkable sufficient conditions for the positive (semi-)definiteness of tensors are given.

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1. Introduction

A real mth order n-dimension tensor $\mathcal{A} = (a_{i_1\cdots i_m})$, denoted by $\mathcal{A} \in R^{[m,n]}$, consists of n^m real entries:

$$a_{i_1\cdots i_m} \in R,$$

where $i_j \in N = \{1, 2, \ldots, n\}$ for $j = 1, \ldots, m$ [2,3,5,11,17]. It is obvious that a matrix is an order 2 tensor. Moreover, a tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is called symmetric [12,17] if

$$a_{i_1\cdots i_m} = a_{\pi(i_1\cdots i_m)}, \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. And an mth order n-dimension tensor is called the unit tensor denoted by $\mathcal{I} = (\delta_{i_1\cdots i_m}) \in R^{[m,n]}$ [2,26], where

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

For a tensor $\mathcal{A} \in R^{[m,n]}$, if there are a nonzero vector $x = (x_1, \ldots, x_n)^T \in R^n$ and a number $\lambda \in R$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}$$

and $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$, then λ is called an H-eigenvalue of $\mathcal{A}$ and x is called a corresponding H-eigenvector of $\mathcal{A}$ [17]. As shown in [17], Qi used H-eigenvalues of real symmetric tensors to determine positive (semi-)definite tensors, that is, an even order real symmetric tensor is positive (semi-)definite if and only if all its H-eigenvalues are positive (non-negative). Here a tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is called positive (semi-)definite [22,23] if for any nonzero vector x in $\mathbb{R}^n$,

$$\mathcal{A}x^m > (\geq) 0,$$

where $\mathcal{A}x^m = \sum\limits_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2\cdots i_m} x_{i_1} \cdots x_{i_m}$.

Positive definiteness and semi-definiteness of real symmetric tensors have applications in automatical control, polynomial problems, magnetic resonance imaging and spectral hypergraph theory [1,3–10,18–21,23–25,27].

It is not effective by using H-eigenvalues in some cases to determine that a real symmetric tensor $\mathcal{A}$ is positive (semi-)definite because it is not easy to compute the smallest

H-eigenvalue of that tensor when its order and dimension are large. Hence one tries to give some checkable sufficient conditions [11–13,22,23,27,28]. In [23], Song and Qi introduced the class of B- (B_0-)tensors, which is a natural extension of B-matrices [15,16], to provide a checkable sufficient condition for positive (semi-)definite tensors.

Definition 1. (See [23].) Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$. $\mathcal{A}$ is called a B-tensor (B_0-tensor, resp.) if for all $i \in N$

$$\sum_{i_2,\ldots,i_m \in N} a_{ii_2 \cdots i_m} > (\geq)\ 0$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,\ldots,i_m \in N} a_{ii_2 \cdots i_m} \right) > (\geq)\ a_{ij_2 \cdots j_m}, \text{ for } j_2, \ldots, j_m \in N, \delta_{ij_2 \cdots j_m} = 0.$$

In [13], Li and Li provided the following sufficient and necessary conditions for B-tensors and B_0-tensors, respectively.

Proposition 1. *(See [13, Proposition 2].) Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$.*

(I) $\mathcal{A}$ is a B-tensor if and only if for each $i \in N$,

$$a_{ii \cdots i} - \beta_i(\mathcal{A}) > \Delta_i(\mathcal{A}),$$

where $\beta_i(\mathcal{A}) = \max\limits_{\substack{j_2,\ldots,j_m \in N, \\ \delta_{ij_2 \ldots j_m} = 0}} \{0, a_{ij_2 \cdots j_m}\}$ and $\Delta_i(\mathcal{A}) = \sum\limits_{\substack{i_2 \ldots i_m \in N, \\ \delta_{ii_2 \ldots i_m} = 0}} (\beta_i(\mathcal{A}) - a_{ii_2 \cdots i_m})$.

(II) $\mathcal{A}$ is a B_0-tensor if and only if for each $i \in N$,

$$a_{ii \cdots i} - \beta_i(\mathcal{A}) \geq \Delta_i(\mathcal{A}).$$

To give other checkable sufficient conditions for positive (semi-)definite tensors, Li and Li [13] proposed two new classes of tensors: quasi-double B-tensors and quasi-double B_0-tensors, and proved that a B-tensor is a quasi-double B-tensor, that is, the class of quasi-double B-tensors is a generalization of B-tensors; see Proposition 4 in [13].

Definition 2. (See [13].) Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$ with $a_{i \cdots i} > \beta_i(\mathcal{A})$ for all $i \in N$. $\mathcal{A}$ is called a quasi-double B- (B_0-)tensor if for all $i, j \in N$, $i \neq j$,

$$(a_{i \cdots i} - \beta_i(\mathcal{A})) \left(a_{j \cdots j} - \beta_j(\mathcal{A}) - \Delta_j^i(\mathcal{A}) \right) > (\geq) \left(\beta_j(\mathcal{A}) - a_{ji \cdots i} \right) \Delta_i(\mathcal{A}), \tag{1}$$

where

$$\Delta_j^i(\mathcal{A}) = \Delta_j(\mathcal{A}) - (\beta_j(\mathcal{A}) - a_{ji \cdots i}) = \sum_{\substack{\delta_{jj_2 \ldots j_m} = 0, \\ \delta_{ij_2 \ldots j_m} = 0}} (\beta_j(\mathcal{A}) - a_{jj_2 \cdots j_m}).$$

As shown in [13,22,23,27], an even order symmetric B-tensor is positive definite, an even order symmetric quasi-double B-tensor is positive definite, and an even order symmetric B_0-tensor is positive semi-definite. For quasi-double B_0-tensors, Li and Li only give the following conjecture.

Conjecture 1. *(See [13, Conjecture 1].) An even order symmetric quasi-double B_0-tensor is positive semi-definite.*

In this paper, we introduce two new classes of tensors: MB-tensors and MB_0-tensors, prove that the class of MB-tensors (MB_0-tensors) is a generalization of B-tensors and quasi-double B-tensor (B_0-tensors and quasi-double B_0-tensor, respectively), and that an even order symmetric MB- (MB_0-)tensor is positive (semi-)definite, which provides a positive answer to Conjecture 1 in [13]. In addition, by giving some conditions to determine MB- (MB_0-)tensors, some checkable sufficient conditions for the positive (semi-)definiteness of tensors are given.

2. MB-tensor and MB_0-tensor

We first define MB-tensors and MB_0-tensors involved with (strong) M-tensors.

Definition 3. (See [3,5,28].) Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$. $\mathcal{A}$ is called

(I) a Z-tensor if all the off-diagonal entries of $\mathcal{A}$ are non-positive, that is, $a_{i_1\ldots i_m} \leq 0$, for $i_j \in N$, $j = 1, 2, \ldots, m$, and $\delta_{i_1 i_2\cdots i_n} = 0$;

(II) an (a strong) M-tensor if $\mathcal{A}$ is a Z-tensor with the form $\mathcal{A} = c\mathcal{I} - \mathcal{B}$ such that $\mathcal{B}$ is a non-negative tensor and $c \geq (>)\rho(\mathcal{B})$, where $\rho(\mathcal{B})$ is the spectral radius of $\mathcal{B}$.

For a real tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, we can write it as

$$\mathcal{A} = \mathcal{B}^+ + \ \mathcal{C}, \tag{2}$$

where $\mathcal{B}^+ = (b_{i_1\cdots i_m}) \in R^{[m,n]},\ \mathcal{C} = (c_{i_1\cdots i_m}) \in R^{[m,n]}$,

$$b_{ii_2\cdots i_m} = a_{ii_2\cdots i_m} - \beta_i(\mathcal{A}) \text{ for } i \in N,$$

and

$$c_{ii_2\cdots i_m} = \beta_i(\mathcal{A}) \text{ for } i \in N.$$

Since $b_{ii_2\cdots i_m} = a_{ii_2\cdots i_m} - \beta_i(\mathcal{A}) \leq 0$ for $i, i_2, \ldots, i_n \in N$ and $\delta_{ii_2\cdots i_n} = 0$, we have that B^+ is a Z-tensor.

Definition 4. Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, and $\mathcal{A} = \mathcal{B}^+ + \ \mathcal{C}$.

(I) $\mathcal{A}$ is called an MB-tensor if $\mathcal{B}^+$ is a strong M-tensor.

(II) $\mathcal{A}$ is called an MB_0-tensor if $\mathcal{B}^+$ is an M-tensor.

Obviously, an MB-tensor is an MB_0-tensor, and the class of MB-tensors is a generalization of MB-matrices [14]. We next give the relationships of MB- (MB_0-)tensors and the positive (semi-)definite tensors. First, recall some results on M-tensors.

Lemma 1. *(See [28, Theorem 3.9].) Let $\mathcal{A} \in R^{[m,n]}$ be a Z-tensor. Then*
(I) $\mathcal{A}$ is a strong M-tensor if and only if $\tau(\mathcal{A}) = \min_{\lambda \in \sigma(\mathcal{A})} Re\lambda > 0$.
(II) $\mathcal{A}$ is an M-tensor if and only if $\tau(\mathcal{A}) \geq 0$.

As shown in [28], $\tau(\mathcal{A})$ is the smallest H-eigenvalue of an M-tensor $\mathcal{A}$. Hence, according to Lemma 1, Zhang et al. give the following result.

Lemma 2. *(See [28, Corollary 3.10].) Let $\mathcal{A} \in R^{[m,n]}$ be a Z-tensor. Then $\mathcal{A}$ is an (a strong) M-tensor if and only if all its H-eigenvalues are (positive) non-negative.*

Lemma 3. *(See [28, Theorem 4.1].) Let $\mathcal{A} \in R^{[m,n]}$ be a symmetric Z-tensor and m be even. Then*
(I) $\mathcal{A}$ is positive definite if and only if A a strong M-tensor.
(II) $\mathcal{A}$ is positive semi-definite if and only if A an M-tensor.

Remark 1. In Theorem 4.1 of [28], Zhang et al. only give the part (I) of Lemma 3. For the part (II), we can prove it directly by using the fact that if all its H-eigenvalues of a symmetric tensor $\mathcal{A}$ is non-negative, then $\mathcal{A}$ is positive semi-definite; for details, see Theorem 5 in [17].

Lemma 4. *Let $\mathcal{A}_1 = s\mathcal{I} - \mathcal{B}_1$ and $\mathcal{A}_2 = s\mathcal{I} - \mathcal{B}_2$, where $\mathcal{B}_1$, $\mathcal{B}_2 \in R^{[m,n]}$ are non-negative. And let $\mathcal{A}_1 \leq \mathcal{A}_2$. If $\mathcal{A}_1$ is an (a strong) M-tensor, then $\mathcal{A}_2$ is an (a strong) M-tensor.*

Proof. Since $\mathcal{A}_1 \leq \mathcal{A}_2$, then $\mathcal{B}_1 \geq \mathcal{B}_2$. Hence by Lemma 3.2 in [26], we have $\rho(\mathcal{B}_1) \geq \rho(\mathcal{B}_2)$. If $\mathcal{A}_1$ is an M-tensor, then $s \geq \rho(\mathcal{B}_1)$, consequently, $s \geq \rho(\mathcal{B}_2)$. Hence, $\mathcal{A}_2$ is an M-tensor. Similarly, we can prove that if $\mathcal{A}_1$ is a strong M-tensor, then $\mathcal{A}_2$ is a strong M-tensor. $\square$

Now by Lemma 3 and Lemma 4, we give the relationships of MB- (MB_0-)tensors and positive (semi-)definite tensors. Before that we give the definitions of P- (P_0-)tensors [22,23] and partially all one tensors [22]. A real tensor $\mathcal{A} \in R^{[m,n]}$ is called a P- (P_0-)tensor if for any nonzero x in R^n,

$$\max_{i \in N} x_i(\mathcal{A}x^{m-1})_i > (\geq) 0.$$

Suppose that $\mathcal{A} \in R^{[m,n]}$ is a symmetric tensor, and has a principal sub-tensor $\mathcal{A}_r^J$ with $J \in N$ and $|J| = r (1 \leq r \leq n)$ such that all the entries of $\mathcal{A}_r^J$ are one, and all the other entries of $\mathcal{A}$ are zero, then $\mathcal{A}$ is called a partially all one tensor, and denoted by ε^J. If

$J = N$, then we denote ε^J simply by ε and call it an all one tensor. And an even order partially all one tensor is positive semi-definite; for details, see [22].

Theorem 5. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ be a symmetric MB-tensor. Then either $\mathcal{A}$ is a strong M-tensor itself, or we have*

$$\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k}, \tag{3}$$

where $\mathcal{M}$ is a strong M-tensor, s is a positive integer, $h_k > 0$ and $\hat{J}_k \subseteq N$, for $k = 1, 2, \cdots, s$.

Proof. Let $\hat{J}(\mathcal{A}) = \{i \in N :$ there is at least one positive off-diagonal entry in the ith row of $\mathcal{A}\}$. Obviously, $\hat{J}(\mathcal{A}) \subseteq N$. If $\hat{J}(\mathcal{A}) = \emptyset$, then $\mathcal{A} = \mathcal{B}^+$, and hence $\mathcal{A}$ is a strong M-tensor by the fact that $\mathcal{A}$ is an MB-tensor. The conclusion follows in the case.

Now we suppose that $\hat{J}(\mathcal{A}) \neq \emptyset$, let $\mathcal{A}_1 = \mathcal{A} = (a^{(1)}_{i_1\cdots i_m})$, and let $d^{(1)}_i$ be the value of the largest off-diagonal entry in the ith row of $\mathcal{A}_1$, that is,

$$d^{(1)}_i = \max_{\substack{i_2\ldots i_m \in N,\\ \delta_{ii_2\ldots i_m}=0}} a^{(1)}_{ii_2\cdots i_m}.$$

Furthermore, let $\hat{J}_1 = \hat{J}(\mathcal{A}_1)$, $h_1 = \min\limits_{i\in\hat{J}_1} d^{(1)}_i$ and

$$J_1 = \{i \in \hat{J}_1 : d^{(1)}_i = h_1\}.$$

Then $J_1 \subseteq \hat{J}_1$ and $h_1 > 0$.

Consider $\mathcal{A}_2 = \mathcal{A}_1 - h_1\varepsilon^{\hat{J}_1} = (a^{(2)}_{i_1\cdots i_m})$. Obviously, $\mathcal{A}_2$ is also symmetric by the definition of $\varepsilon^{\hat{J}_1}$. Note that

$$a^{(2)}_{i_1\cdots i_m} = \begin{cases} a^{(1)}_{i_1\cdots i_m} - h_1, & i_1, i_2, \ldots, i_m \in \hat{J}_1 \\ a^{(1)}_{i_1\cdots i_m}, & \text{otherwise}, \end{cases} \tag{4}$$

for $i \in J_1$,

$$\beta_i(\mathcal{A}_2) = \beta_i(\mathcal{A}_1) - h_1 = 0, \tag{5}$$

and that for $i \in \hat{J}_1 \backslash J_1$,

$$\beta_i(\mathcal{A}_2) = \beta_i(\mathcal{A}_1) - h_1 > 0. \tag{6}$$

Furthermore, let

$$\mathcal{A}_1 = \mathcal{A} = \mathcal{B}^+_1 + \mathcal{C}_1,\ \mathcal{A}_2 = \mathcal{B}^+_2 + \mathcal{C}_2 \tag{7}$$

where $\mathcal{B}_1^+ = (b^{(1)}_{i_1\cdots i_m}) \in R^{[m,n]}$, $\mathcal{B}_2^+ = (b^{(2)}_{i_1\cdots i_m}) \in R^{[m,n]}$ and

$$b^{(1)}_{ii_2\cdots i_m} = a^{(1)}_{ii_2\cdots i_m} - \beta_i(\mathcal{A}_1),\ b^{(2)}_{ii_2\cdots i_m} = a^{(2)}_{ii_2\cdots i_m} - \beta_i(\mathcal{A}_2) \text{ for } i \in N.$$

Combining (4), (5), (6), (7) with the fact that for each $j \notin \hat{J}_1$, $\beta_i(\mathcal{A}_2) = \beta_i(\mathcal{A}_1)$, we have

$$\mathcal{B}_2^+ = \mathcal{B}_1^+ + \ h_1 \varepsilon^{\hat{J}_1}.$$

Since $\mathcal{A}_1 = \mathcal{A}$ is an *MB*-tensor, $\mathcal{B}_1^+$ is a strong M-tensor. Note that $\mathcal{B}_2^+ \geq \mathcal{B}_1^+$, then by Lemma 4, we have that $\mathcal{B}_2^+$ is also a strong M-tensor, and hence $\mathcal{A}_2$ is a symmetric *MB*-tensor.

Now replace $\mathcal{A}_1$ by $\mathcal{A}_2$, and repeat this process. Let $\hat{J}(\mathcal{A}_2) = \{i \in N :$ there is at least one positive off-diagonal entry in the ith row of $\mathcal{A}_2\}$. Then $\hat{J}(\mathcal{A}_2) = \hat{J}_1 \backslash J_1$. Repeat this process until $\hat{J}(\mathcal{A}_{s+1}) = \emptyset$. Let $\mathcal{M} = \mathcal{A}_{s+1}$. Then (3) holds. □

Theorem 6. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ be a symmetric MB-tensor. If m is even, then $\mathcal{A}$ is positive definite, consequently, $\mathcal{A}$ is a P-tensor.*

Proof. If m is even, then $\mathcal{A}$ an even order symmetric *MB*-tensor. By Theorem 5, we have that if $\mathcal{A}$ itself is a symmetric strong M-tensor, then it is positive definite by Lemma 3. Otherwise, (3) holds with $s > 0$. For $x \in R^n$, by (3) and the fact that $\mathcal{M}$ is positive definite, we have

$$\mathcal{A}x^m = \mathcal{M}x^m + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k} x^m = \mathcal{M}x^m + \sum_{k=1}^{s} h_k ||x_{\hat{J}_k}||_m^m \geq \mathcal{M}x^m > 0.$$

This implies that $\mathcal{A}$ is positive definite. Note that a symmetric tensor is a P-tensor if and only if it is positive definite [23], therefore $\mathcal{A}$ is a P-tensor. The proof is complete. □

Similar to Theorems 5 and 6, by the part (II) of Lemma 3, Lemma 4, and the fact that a symmetric tensor is a P_0-tensor if and only it is positive semi-definite [23], we easily have that an even order symmetric MB_0-tensor is positive semi-definite and a P_0-tensor.

Theorem 7. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ be a symmetric MB_0-tensor. Then either $\mathcal{A}$ is an M-tensor itself, or we have*

$$\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k},$$

where $\mathcal{M}$ is an M-tensor, s is a positive integer, $h_k > 0$ and $\hat{J}_k \subseteq N$, for $k = 1, 2, \cdots, s$.

Theorem 8. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ be a symmetric MB_0-tensor. If m is even, then $\mathcal{A}$ is positive semi-definite, consequently, $\mathcal{A}$ is a P_0-tensor.*

Since an even order real symmetric tensor is positive (semi-)definite if and only if all its H-eigenvalues are positive (non-negative) [17], by Theorems 6 and 8 we have the following results.

Corollary 1. *All the H-eigenvalues of an even order symmetric MB-tensor are positive.*

Corollary 2. *All the H-eigenvalues of an even order symmetric MB_0-tensor are non-negative.*

3. Relationships between B- (B_0-)tensors, quasi-double B- (B_0-)tensors and MB-(MB_0-)tensors

In [13], Li and Li gave the relationship between B-tensors and quasi-double B-tensors as follows.

Proposition 2. *(See [13, Proposition 4].) Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, $n \geq 2$. If $\mathcal{A}$ is a B-tensor, then $\mathcal{A}$ is a quasi-double B-tensor.*

Now, we prove that a B_0-tensor is a quasi-double B_0-tensor.

Proposition 3. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, $n \geq 2$. If $\mathcal{A}$ is a B_0-tensor, then $\mathcal{A}$ is a quasi-double B_0-tensor.*

Proof. If $\mathcal{A}$ is a B_0-tensor, then by Proposition 1 for any $i \in N$,

$$a_{i\cdots i} - \beta_i(\mathcal{A}) \geq \Delta_i(\mathcal{A}),$$

that is,

$$a_{i\cdots i} - \beta_i(\mathcal{A}) - \Delta_i^k(\mathcal{A}) \geq \beta_i(\mathcal{A}) - a_{ik\cdots k}, \text{ for } k \neq i.$$

Obviously, for $i, j \in N$, $j \neq i$,

$$a_{i\cdots i} - \beta_i(\mathcal{A}) \geq \Delta_i(\mathcal{A}) \geq 0,$$

and

$$a_{j\cdots j} - \beta_j(\mathcal{A}) - \Delta_j^i(\mathcal{A}) \geq \beta_j(\mathcal{A}) - a_{ji\cdots i} \geq 0.$$

It is easy to see that Inequality (1) holds, i.e., $\mathcal{A}$ is a quasi-double B_0-tensor by Definition 2. The proof is complete. □

Next, we establish the relationships between quasi-double B- (B_0-)tensors and MB- (MB_0-)tensors. Before that a lemma is given.

Lemma 9. *(See [12, Theorem 2.1].) Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m,n]}$, $n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{\substack{i,j \in N, \\ j \neq i}} \mathcal{K}_{i,j}(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ *is the spectrum of* $\mathcal{A}$, *that is, the set of all eigenvalues of* $\mathcal{A}$,

$$\mathcal{K}_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : \left(|z - a_{i \cdots i}| - r_i^j(\mathcal{A}) \right) |z - a_{j \cdots j}| \leq |a_{ij \cdots j}| r_j(\mathcal{A}) \right\}$$

and

$$r_i^j(\mathcal{A}) = r_i(\mathcal{A}) - |a_{ij \cdots j}| = \sum_{\substack{\delta_{i,i_2,\ldots,i_m}=0, \\ \delta_{j,i_2,\ldots,i_m}=0}} |a_{ii_2 \cdots i_m}|.$$

Theorem 10. *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$, $n \geq 2$. *If* $\mathcal{A}$ *is a quasi-double B- (B_0-)tensor, then* $\mathcal{A}$ *is an MB- (MB_0-)tensor.*

Proof. We first prove that a quasi-double B_0-tensor is an MB_0-tensor. Let

$$\mathcal{A} = \mathcal{B}^+ + \ \mathcal{C}$$

where $\mathcal{B}^+ = (b_{i_1 \cdots i_m}) \in R^{[m,n]}$ and

$$b_{ii_2 \cdots i_m} = a_{ii_2 \cdots i_m} - \beta_i(\mathcal{A}) \text{ for } i \in N.$$

Since $\mathcal{A}$ is a quasi-double B_0-tensor, we have by Definition 2 that for all $i, j \in N$, $i \neq j$,

$$(a_{i \cdots i} - \beta_i(\mathcal{A})) \left(a_{j \cdots j} - \beta_j(\mathcal{A}) - \Delta_j^i(\mathcal{A}) \right) \geq (\beta_j(\mathcal{A}) - a_{ji \cdots i}) \Delta_i(\mathcal{A}). \tag{8}$$

Note that $b_{ii_2 \cdots i_m} = a_{ii_2 \cdots i_m} - \beta_i(\mathcal{A}) \leq 0$ for $\delta_{i,i_2,\ldots,i_m} = 0$, that is, $\mathcal{B}^+$ is a Z-tensor. Hence, Inequality (8) is equivalent to

$$b_{i \cdots i}(b_{j \cdots j} - r_j^i(\mathcal{B}^+)) \geq |b_{ji \cdots i}| r_i(\mathcal{B}^+) \text{ for all } i, j \in N, \ i \neq j.$$

We now prove that $\tau(\mathcal{B}^+) = \min\limits_{\lambda \in \sigma(\mathcal{B}^+)} Re\lambda \geq 0$. Suppose that $\tau(\mathcal{B}^+) < 0$, then there is $\lambda_0 \in \sigma(\mathcal{B}^+)$ such that $Re\lambda_0 = \tau(\mathcal{B}^+) < 0$. Since A a quasi-double B_0-tensor, we have by Definition 2 $b_{i \cdots i} = a_{i \cdots i} - \beta_i(\mathcal{A}) > 0$, consequently, $b_{j \cdots j} - r_j^i(\mathcal{B}^+) \geq 0$ for $j \neq i$. This implies that for all $i, j \in N$, $i \neq j$,

$$\begin{aligned}
|\lambda_0 - b_{i \cdots i}|(|\lambda_0 - b_{j \cdots j}| - r_j^i(\mathcal{B}^+)) &\geq |Re\lambda_0 - b_{i \cdots i}|(|Re\lambda_0 - b_{j \cdots j}| - r_j^i(\mathcal{B}^+)) \\
&> |b_{i \cdots i}|(|b_{j \cdots j}| - r_j^i(\mathcal{B}^+)) \\
&= b_{i \cdots i}(b_{j \cdots j} - r_j^i(\mathcal{B}^+)) \\
&\geq |b_{ji \cdots i}| r_i(\mathcal{B}^+),
\end{aligned}$$

equivalently, $\lambda_0 \notin \mathcal{K}_{j,i}(\mathcal{B}^+)$ for all $i, j \in N$, $i \neq j$. Hence, $\lambda_0 \notin \mathcal{K}(\mathcal{B}^+)$, which contradicts Lemma 9. Therefore, $\tau(\mathcal{B}^+) \geq 0$. Furthermore, note that $\mathcal{B}^+$ is a Z-tensor, by Lemma 1 and Definition 4 we have that $\mathcal{B}^+$ is an M-tensor, and that $\mathcal{A}$ is an MB_0-tensor.

Similarly, we can obtain that a quasi-double B-tensor is an MB-tensor. The proof is complete. □

By Proposition 2, Proposition 3 and Theorem 10, we easily get that

$$\{B_0\text{-tensors}\} \subseteq \{\text{quasi-double } B_0\text{-tensors}\} \subseteq \{MB_0\text{-tensors}\},$$

and that

$$\{B\text{-tensors}\} \subseteq \{\text{quasi-double } B\text{-tensors}\} \subseteq \{MB\text{-tensors}\}.$$

Furthermore, as shown in [22,23,27], an odd order B- (B_0-)tensor may not be a P- (P_0-)tensor, and an even order nonsymmetric B- (B_0-)tensor may not be a P- (P_0-)tensor. Hence, we conclude that an odd order MB-tensor may not be a P-tensor, an odd order MB_0-tensor may not be a P_0-tensor, an even order nonsymmetric MB-tensor may not be a P-tensor, and an even order nonsymmetric MB_0-tensor may not be a P_0-tensor.

Since an even order symmetric MB_0-tensor is positive semi-definite, and an even order symmetric MB-tensor is positive definite, we have immediately the following result.

Corollary 3. *(I) An even order symmetric B_0-tensor is positive semi-definite;*

(II) An even order symmetric quasi-double B_0-tensor is positive semi-definite;

(III) An even order symmetric B-tensor is positive definite;

(IV) An even order symmetric quasi-double B-tensor is positive definite.

Remark 2. The part (II) of Corollary 3 is exactly Conjecture 1 in [13].

4. Sufficient conditions for MB- (MB_0-)tensors

In this section, we give some checkable sufficient conditions for MB- (MB_0-)tensors, also for the positive (semi-)definiteness of tensors. Before that a lemma in [12] is given as follows.

Lemma 11. *(See [12, Theorem 2.2].) Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m,n]}$, $n \geq 2$. And let S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) = \left(\bigcup_{i \in S, j \in \bar{S}} \mathcal{K}_{i,j}(\mathcal{A}) \right) \bigcup \left(\bigcup_{i \in \bar{S}, j \in S} \mathcal{K}_{i,j}(\mathcal{A}) \right),$$

where $K_{i,j}(\mathcal{A})$ is defined as in Lemma 9.

Theorem 12. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, $n \geq 2$, with*

$$\mathcal{A} = \mathcal{B}^+ + \mathcal{C}$$

where $\mathcal{B}^+ = (b_{i_1\cdots i_m}) \in R^{[m,n]}$ and

$$b_{ii_2\cdots i_m} = a_{ii_2\cdots i_m} - \beta_i(\mathcal{A}) \text{ for } i \in N.$$

If $b_{i\cdots i} > 0$ for $i \in N$, and there is a nonempty proper subset S of N such that for each $i \in S$ and each $j \in \bar{S}$,

$$\left(b_{i\cdots i} - r_i^j(\mathcal{B}^+)\right) b_{j\cdots j} \geq r_j(\mathcal{B}^+)|b_{ij\cdots j}|$$

and

$$\left(b_{j\cdots j} - r_j^i(\mathcal{B}^+)\right) b_{i\cdots i} \geq r_i(\mathcal{B}^+)|b_{ji\cdots i}|,$$

then $\mathcal{A}$ is an MB_0-tensor, and positive semi-positive.

Proof. By Definition 2, we only prove that $\mathcal{B}^+$ is an M-tensor. Note that $\mathcal{B}^+$ is a Z-tensor. Hence, we only prove $\tau(\mathcal{B}^+) \geq 0$.

Suppose that $\tau(\mathcal{B}^+) < 0$. Then there is $\lambda_0 \in \sigma(\mathcal{B}^+)$ such that $Re\lambda_0 = \tau(\mathcal{B}^+) < 0$. Similar to the proof of Theorem 10, we can get that for each $i \in S$ and each $j \in \bar{S}$,

$$|\lambda_0 - b_{j\cdots j}|(|\lambda_0 - b_{i\cdots i}| - r_i^j(\mathcal{B}^+)) > |b_{ij\cdots j}|r_j(\mathcal{B}^+),$$

and

$$|\lambda_0 - b_{i\cdots i}|(|\lambda_0 - b_{j\cdots j}| - r_j^i(\mathcal{B}^+)) > |b_{ji\cdots i}|r_i(\mathcal{B}^+),$$

that is, $\lambda_0 \notin \mathcal{K}^S(\mathcal{A})$. This contradicts Lemma 11. Hence, $\tau(\mathcal{B}^+) \geq 0$, consequently, $\mathcal{A}$ is an MB_0-tensor, and positive semi-positive. □

Similar to the proof of Theorem 12, by Lemma 11 we easily obtain a sufficient condition for MB-tensors.

Theorem 13. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, $n \geq 2$, with*

$$\mathcal{A} = \mathcal{B}^+ + \mathcal{C}$$

where $\mathcal{B}^+$ is defined as in Theorem 12. If $b_{i\cdots i} > 0$ for $i \in N$, and there is a nonempty proper subset S of N such that for each $i \in S$ and each $j \in \bar{S}$,

$$\left(b_{i\cdots i} - r_i^j(\mathcal{B}^+)\right) b_{j\cdots j} > r_j(\mathcal{B}^+)|b_{ij\cdots j}|$$

and

$$\left(b_{j\cdots j} - r_j^i(\mathcal{B}^+)\right) b_{i\cdots i} > r_i(\mathcal{B}^+)|b_{ji\cdots i}|,$$

then $\mathcal{A}$ is an MB-tensor, and positive semi-positive.

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