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SOME RESULTS ON l^k-EIGENVALUES OF TENSOR AND RELATED SPECTRAL RADIUS

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Abstract. In this paper, we study the l^k-eigenvalues/vectors of a real symmetric square tensor. Specially, we investigate some properties on the related l^k-spectral radius of a real nonnegative symmetric square tensor.

1. **Introduction.**

A *tensor*, denoted by $\mathcal{A}$, is a multidimensional array, and its order is the number of dimensions. Let m and n be positive integers. We call $\mathcal{A} = (a_{i_1\ldots i_m})$, where $a_{i_1\ldots i_m} \in \Re$ for $i_l = 1,\ldots,n$ and $l = 1,\ldots,m$, a real m-th order n-dimensional square tensor. The eigenvalues and eigenvectors of such square tensor were introduced by Qi [6], and were introduced independently by Lim [4]. Since tensors and eigenvalues/vectors of tensors have many applications in various fields such as medical resonance imaging [1, 8], higher-order Markov chains [5] and best-rank one approximation in date analysis [7], many nice properties such as the Perron-Frobenius theorem for eigenvalues/vectors of nonnegative square tensor have been established, see, e.g., [2, 9].

In this paper, we study the l^k-eigenvalues/vectors of a real symmetric square tensor and some related properties. This paper is organized as follows. After introducing the l^k-eigenvalues and eigenvectors of the high order tensor in Section 2, we in Section 3 investigate some properties of l^k-spectral radius of a real nonnegative symmetric square tensor. Final remarks are given in Section 4.

Some words about the notation. $\Re^n$ denotes the real Euclidean space of column vectors of length n. For a vector $\mathbf{x} = (x_1,\ldots,x_n)^\top \in \Re^n$, we denote $|\mathbf{x}| = (|x_1|,\ldots,|x_n|)^\top$ and $\mathbf{x}^{[\alpha]} = (x_1^\alpha,\ldots,x_n^\alpha)^\top$ (i.e., taking αth power coordinatewise), where α is a given positive integer. Similarly, for the tensor $\mathcal{A} = (a_{i_1\ldots i_m})$, we define $|\mathcal{A}| = (|a_{i_1\ldots i_m}|)$. Inequalities between real tensors or vectors will be understood componentwise, e.g, for two real tensors $\mathcal{A} = (a_{i_1\ldots i_m})$ and $\mathcal{B} = (b_{i_1\ldots i_m})$, the inequality $\mathcal{A} \geq \mathcal{B}$ means $a_{i_1\ldots i_m} \geq b_{i_1\ldots i_m}$ for every $i_l = 1,\ldots,n$ and $l = 1,\ldots,m$. The

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set of all nonnegative vectors in $\Re^n$ is denoted by $\Re^n_+$, i.e., $\Re^n_+ = \{\mathbf{x} \in \Re^n \mid \mathbf{x} \geq \theta\}$, where θ is the zero vector in an appropriate finite dimensional space. Similarly, we denote $\Re^n_{++} = \{\mathbf{x} \in \Re^n \mid \mathbf{x} > \theta\}$.

2. l^k-eigenvalues and eigenvectors of $\mathcal{A}$.

Let us equip $\Re^n$ with the l^k-norm, $\|\cdot\|_k$, defined by

$$\|\mathbf{x}\|_k := (|x_1|^k + \ldots + |x_n|^k)^{1/k},$$

where $k = 2, \ldots, m$. Recall that for $k \geq 2$, the l^k-norm is a continuously differentiable function on $\Re^n \backslash \{\theta\}$. For $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \Re^n$, we denote $\varphi_\alpha(\mathbf{x}) := (\mathrm{sign}(x_1)|x_1|^\alpha, \ldots, \mathrm{sign}(x_n)|x_n|^\alpha)^\top$, where α is a given integer and

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

It is clear that the gradient of the l^k-norm is given by

$$\nabla \|\mathbf{x}\|_k = \frac{\varphi_{k-1}(\mathbf{x})}{\|\mathbf{x}\|_k^{k-1}}. \tag{1}$$

Let $\mathcal{A} = (a_{i_1 \ldots i_m})$ be a real m-th order n-dimensional square tensor, where $m \geq 2$. We say that $\mathcal{A}$ is symmetric, if $a_{i_1 \ldots i_m}$ is invariant under any permutation of indices among $i_1, \ldots, i_m$, i.e., $a_{\pi(i_1 \ldots i_m)} = a_{i_1 \ldots i_m}$ for any $\pi \in S_m$, where S_m is the permutation group of m indices. By using $\mathcal{A}$, we define the operator A on $\Re^n$ into itself:

$$A(\mathbf{x}) = \mathcal{A}\mathbf{x}^{m-1},$$

where $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in $\Re^n$ such that

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{1 \leq i_2, \ldots, i_m \leq n} a_{i i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \text{ for } i = 1, 2, \ldots, n.$$

It is obviously seen that if $\mathcal{A}$ is nonnegative, then the associate nonlinear operator $A : \Re^n_+ \to \Re^n_+$ and

$$A(\mathbf{x}) \leq A(\mathbf{y}), \text{ for any } \mathbf{x}, \mathbf{y} \in \Re^n_+ \text{ with } \mathbf{x} \leq \mathbf{y}.$$

If there are a number λ and a nonzero vector $\mathbf{x} \in \Re^n$ are solution of the following homogeneous polynomial equations:

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

then λ is called the eigenvalue of $\mathcal{A}$ and $\mathbf{x}$ the eigenvector of $\mathcal{A}$ associated with the eigenvalue λ. This definition was introduced by Qi [6] when m is even and $\mathcal{A}$ is symmetric. Independently, Lim [4] gave such a definition but restricted $\mathbf{x}$ to be a real vector and λ to be a real number. The following definition was proposed by Lim [4].

Definition 2.1. Let $\mathcal{A}$ be a real m-th order n-dimensional square tensor. For the given integer $k \in \{2, \ldots, m\}$, if $(\lambda, \mathbf{x}) \in \Re \times (\Re^n \backslash \{\theta\})$ is a solution of the following systems:

$$\begin{cases} \mathcal{A}\mathbf{x}^{m-1} = \lambda \varphi_{k-1}(\mathbf{x}), \\ \|\mathbf{x}\|_k = 1, \end{cases} \tag{2}$$

then we call $\mathbf{x}$ and λ an l^k-eigenvector and l^k-eigenvalue of $\mathcal{A}$, respectively.

In general, the unit-norm constraint in (2) is not superfluous, since the first expression in (2) is not a homogeneous system and $\mathbf{x}$ can not be scaled by positive scalar γ, except $k = m$.

For the existence of the l^k-eigenvalues/vectors of $\mathcal{A}$, we have the following theorem.

Theorem 2.2. *Let $\mathcal{A}$ be a real m-th order n-dimensional symmetric square tensor. Then for every $k \in \{2, \ldots, m\}$, its l^k-eigenvalues and eigenvectors always exist.*

Proof. For every given integer $k \in \{2, \ldots, m\}$, we consider the optimization problem as follows

$$\begin{array}{ll} \min & \mathcal{A}\mathbf{x}^m := \sum\limits_{1 \le i_1, \ldots, i_m \le n} a_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m} \\ \text{s.t.} & \|\mathbf{x}\|_k = 1. \end{array} \tag{3}$$

It is clear that the feasible set of (3) is nonempty compact, which means that the optimal solution always exist. Moreover, since the objective function is continuously differentiable and the constraints of (3) satisfy the linear independence constraint qualification, it is easy to see that every local optimal solution $\bar{\mathbf{x}}$ of (3) satisfies the following optimality conditions:

$$\begin{cases} \mathcal{A}\bar{\mathbf{x}}^{m-1} = \bar{\lambda}\varphi_{k-1}(\bar{\mathbf{x}}), \\ \|\bar{\mathbf{x}}\|_k = 1, \end{cases} \tag{4}$$

where $m\bar{\lambda}$ is the optimal Lagrangian multiplier. By (4), we know that $\bar{\mathbf{x}}$ is an l^k-eigenvector of $\mathcal{A}$, associated with the l^k-eigenvalue $\bar{\lambda}$. We obtain the desired result and complete the proof. ☐

Theorem 2.3. *Let $\mathcal{A}$ be a real m-th order n-dimensional square tensor. If $\mathbf{x}$ is an l^k-eigenvector of $\mathcal{A}$, associated with the l^k-eigenvalue λ, then*

$$\lambda = \mathcal{A}\mathbf{x}^m.$$

Proof. Since $\text{sign}(a)a = |a|$ for any $a \in \Re$, by the first expression in (2), we have $\mathcal{A}\mathbf{x}^m = \lambda\|\mathbf{x}\|_k^k$. Consequently, by the second equation in (2), we obtain the desired result. ☐

We now recall that a m-order n-dimensional square tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \cdots, n\}$ such that

$$a_{i_1 \cdots i_m} = 0, \text{ for } i_1 \in I \text{ and } i_2, \cdots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

For the nonnegative irreducible square tensor, we have the following lemmas, which were proved in [2] and [3] respectively, and will be used later.

Lemma 2.4. *Let $\mathcal{A}$ be a real m-th order n-dimensional nonnegative square tensor. If $\mathcal{A}$ is irreducible, then*

$$\sum_{1 \le i_2, \ldots, i_m \le n} a_{i i_2 \ldots i_m} > 0, \text{ for every } i = 1, \ldots, n.$$

Lemma 2.5. *Let $\mathcal{A}$ be a real m-th order n-dimensional nonnegative square tensor. If $\mathcal{A}$ is irreducible, then for any $\mathbf{x} \in \Re^n_+ \backslash \{\theta\}$, it holds that $\mathcal{A}\mathbf{x}^{m-1} \ne \theta$.*

Theorem 2.6. *Let $\mathcal{A}$ be a real m-th order n-dimensional nonnegative symmetric square tensor. Assume that $\mathcal{A}$ is irreducible, then for every $k \in \{2, \ldots, m\}$, there exist $\bar{\lambda} > 0$ and $\bar{\mathbf{x}} \in \Re^n_{++}$ with $\|\bar{\mathbf{x}}\|_k = 1$, such that $\bar{\mathbf{x}}$ is an l^k-eigenvector of $\mathcal{A}$, associated with the l^k-eigenvalue $\bar{\lambda}$.*

Proof. Denote $D_n = \{z = (z_1, \ldots, z_n) \in \Re^n_+ : \sum_{i=1}^n z_i = 1\}$. Provided by Lemma 2.5, the map F on D_n into itself:

$$F(\mathbf{x}) = \left(\frac{(\mathcal{A}\mathbf{x}^{m-1})_1^{\frac{1}{k-1}}}{\sum_{l=1}^n (\mathcal{A}\mathbf{x}^{m-1})_l^{\frac{1}{k-1}}}, \ldots, \frac{(\mathcal{A}\mathbf{x}^{m-1})_n^{\frac{1}{k-1}}}{\sum_{l=1}^n (\mathcal{A}\mathbf{x}^{m-1})_l^{\frac{1}{k-1}}} \right)$$

is well-defined.

According to the Brouwer fixed point Theorem, there exists $\hat{\mathbf{x}} \in D_n$ such that

$$\mathcal{A}\hat{\mathbf{x}}^{m-1} = \hat{\mu}\ \hat{\mathbf{x}}^{[k-1]} = \hat{\mu}\ \varphi_{k-1}(\hat{\mathbf{x}}), \tag{5}$$

where

$$\hat{\mu} = \left(\sum_{l=1}^n (\mathcal{A}\hat{\mathbf{x}}^{m-1})_l^{\frac{1}{k-1}} \right)^{k-1}.$$

Moreover, by taking $\bar{\lambda} = \hat{\mu}/\|\hat{\mathbf{x}}\|_k^{m-k}$ and $\bar{\mathbf{x}} = \hat{\mathbf{x}}/\|\hat{\mathbf{x}}\|_k$, we know that $\|\bar{\mathbf{x}}\|_k = 1$ and $(\bar{\lambda}, \bar{\mathbf{x}})$ is a solution of (2).

It is clear that $\bar{\lambda} > 0$ by Lemma 2.5. Moreover, since $\mathcal{A}$ is irreducible, we can prove that $\bar{\mathbf{x}} \in \Re^n_{++}$ by a similar way used in the proof of Theorem 4 in [3]. We obtained the desired result and complete the proof. □

Remark 1. In [2], Chang et al. introduced a definition of eigenvalue/vector of tensor, which generalized those proposed in [4, 6]. Moreover, the Perron-Frobenius theorem for nonnegative tensors was proposed by Chang et al [2], in which the authors proved that the nonnegative eigenvector is unique up to a multiplication constant. However, the uniqueness result for $\bar{\lambda}$ in Theorem 2.6 can not be proved yet. In next section, we will study the spectral radius of $\mathcal{A}$ related with the l^k-eigenvalues.

3. l^k-Spectral radius of $\mathcal{A}$.

In this section we study the spectral radius of $\mathcal{A}$ with respect to l^k-eigenvalue. We first introduce the following definition.

Definition 3.1. Let $\mathcal{A}$ be a real m-th order n-dimensional square tensor. We call $\rho_k(\mathcal{A})$ the l^k-spectral radius of $\mathcal{A}$ if it equals the largest absolute l^k-eigenvalues of $\mathcal{A}$, i.e., $\rho_k(\mathcal{A}) = \max\{\ |\lambda| : \lambda \text{ is an } l^k\text{−eigenvalue of } \mathcal{A}\}$.

Notice that $\rho_k(\mathcal{A})$ itself may not be an l^k-eigenvalue of that tensor. Let $u \in \Re^n_{++}$, we denote the order interval

$$B_n(\mathbf{u}) = \{\mathbf{x} \in \Re^n : -\mathbf{u} \le \mathbf{x} \le \mathbf{u}\},$$

which is a convex body in $\Re^n$, i.e., $B_n(\mathbf{u})$ is closed and convex and for any $\mathbf{x} \in \Re^n$, there exists a positive number t such that $\mathbf{x} \in tB_n(\mathbf{u})$, and

$$\mathbf{x} \in B_n(\mathbf{u}),\ |\alpha| \le 1 \Rightarrow \alpha\mathbf{x} \in B_n(\mathbf{u}).$$

Moreover, we recall the norm on $\Re^n$ defined by

$$\|\mathbf{x}\|_{\mathbf{u}} = \inf\{t \ge 0 : \mathbf{x} \in tB_n(\mathbf{u})\}, \tag{6}$$

which will be used in the analysis of l^k-spectral radius. It is easy to see that $\|\mathbf{u}\|_{\mathbf{u}} = 1$ and $\|\cdot\|_{\mathbf{u}}$ is monotonic with respect to $\Re^n_+$, i.e., $\|\mathbf{x}\|_{\mathbf{u}} \le \|\mathbf{y}\|_{\mathbf{u}}$ whenever $\theta \le \mathbf{x} \le \mathbf{y}$. Based upon the norm $\|\cdot\|_{\mathbf{u}}$ on $\Re^n$, we introduce the induced operator norm on A as follows

$$\|A\|_{\mathbf{u}} = \sup_{\|\mathbf{x}\|_{\mathbf{u}}=1} \|A(\mathbf{x})\|_{\mathbf{u}}.$$

We have the following proposition.

Proposition 1. *Let* $\mathbf{u} \in \Re^n_{++}$. *Then for any* $\mathbf{x} \in \Re^n$, *it holds that* $\left\|\mathbf{x}^{[\alpha]}\right\|_{\mathbf{u}^{[\alpha]}} = \|\mathbf{x}\|^\alpha_{\mathbf{u}}$, *where* α *is a given positive integer.*

Proof. It is clear that $\mathbf{u}^{[\alpha]} \in \Re^n_{++}$. The conclusion comes from the definition. ☐

Proposition 2. *Let* $\mathbf{u} \in \Re^n_{++}$. *Suppose that* $\mathcal{A}$ *is a real* m*-th order* n*-dimensional nonnegative symmetric square tensor. Then it holds that*

$$\sup_{\|\mathbf{x}\|_{\mathbf{u}}=1} \|A(\mathbf{x})\|_{\mathbf{u}^{[k-1]}} = \|A(\mathbf{u})\|_{\mathbf{u}^{[k-1]}} \tag{7}$$

for every integer $k \in \{2, \ldots, m\}$.

Proof. It is clear that $\mathbf{u}^{[k-1]} \in \Re^n_{++}$ for every $k \in \{2, \ldots, m\}$. We write $\|A(\mathbf{u})\|_{\mathbf{u}^{[k-1]}} := \mu$, then it holds that

$$-\mu\mathbf{u}^{[k-1]} \le \mathcal{A}\mathbf{u}^{m-1} \le \mu\mathbf{u}^{[k-1]}. \tag{8}$$

Since $\|\mathbf{u}\|_{\mathbf{u}} = 1$, the conclusion that $\sup_{\|\mathbf{x}\|_{\mathbf{u}}=1} \|A(\mathbf{x})\|_{\mathbf{u}^{[k-1]}} \ge \mu$ is trivial.

We now prove the conclusion that $\sup_{\|\mathbf{x}\|_{\mathbf{u}}=1} \|A(\mathbf{x})\|_{\mathbf{u}^{[k-1]}} \le \mu$. For any $\mathbf{x} \in \Re^n$ with $\|\mathbf{x}\|_{\mathbf{u}} = 1$, we have that $|\mathbf{x}| \le \mathbf{u}$. Consequently, since $\mathcal{A}$ is nonnegative, we have

$$|\mathcal{A}\mathbf{x}^{m-1}| \le \mathcal{A}|\mathbf{x}|^{m-1} \le \mathcal{A}\mathbf{u}^{m-1}.$$

Moreover, by (8), it holds that

$$\mu\mathbf{u}^{[k-1]} - \mathcal{A}\mathbf{x}^{m-1} = \mu\mathbf{u}^{[k-1]} - \mathcal{A}\mathbf{u}^{m-1} + \mathcal{A}\mathbf{u}^{m-1} - \mathcal{A}\mathbf{x}^{m-1} \ge \theta$$

and

$$\mu\mathbf{u}^{[k-1]} + \mathcal{A}\mathbf{x}^{m-1} = \mu\mathbf{u}^{[k-1]} - \mathcal{A}\mathbf{u}^{m-1} + \mathcal{A}\mathbf{u}^{m-1} + \mathcal{A}\mathbf{x}^{m-1} \ge \theta.$$

By (6), we know that $\|\mathcal{A}\mathbf{x}^{m-1}\|_{\mathbf{u}^{[k-1]}} \le \mu$ for any $\mathbf{x} \in \Re^n$ with $\|\mathbf{x}\|_{\mathbf{u}} = 1$. Hence, we have

$$\sup_{\|\mathbf{x}\|_{\mathbf{u}}=1} \|A(\mathbf{x})\|_{\mathbf{u}^{[p-1]}} \le \mu.$$

We obtain the desired result and complete the proof. ☐

Theorem 3.2. *Suppose that* $\mathcal{A}$ *is a real* m*-th order* n*-dimensional nonnegative symmetric square tensor. If there exist* $\alpha > 0$ *and* $\bar{\mathbf{u}} \in \Re^n_+\backslash\{\theta\}$, *such that* $\mathcal{A}\bar{\mathbf{u}}^{m-1} \ge \alpha\bar{\mathbf{u}}^{[k-1]}$, *then*

$$\rho_k(\mathcal{A}) \ge \|\bar{\mathbf{u}}\|_k^{k-m}\alpha. \tag{9}$$

Specially, in addition, if $\|\bar{\mathbf{u}}\|_k = 1$, *then* $\rho_k(\mathcal{A}) \ge \alpha$.

Proof. We consider the following optimization problem

$$\begin{array}{ll} \max & \mathcal{A}\mathbf{x}^m \\ \text{s.t.} & \|\mathbf{x}\|_k = 1. \end{array} \tag{10}$$

By the compactness of feasible set of (10), the maximum and minimum objective values, denoted $\lambda_{\max}$ and $\lambda_{\min}$ respectively, must exist. For any local optimal solution $\bar{\mathbf{x}}$ of (10), associated with the optimal value $\bar{\lambda}$, there exists $\gamma \in \Re$ such that

$$\left\{ \begin{array}{l} \mathcal{A}\bar{\mathbf{x}}^{m-1} - \gamma\nabla\|\bar{\mathbf{x}}\|_k = 0, \\ \|\bar{\mathbf{x}}\|_k = 1. \end{array} \right.$$

Consequently, since $\|\bar{\mathbf{x}}\|_k = 1$, by (1), it holds that $\mathcal{A}\bar{\mathbf{x}}^{m-1} = \gamma\varphi_{k-1}(\bar{\mathbf{x}})$, which implies that $\gamma = \bar{\lambda}$. Hence, $\bar{\mathbf{x}}$ is an l^k-eigenvector of $\mathcal{A}$, associated with the l^k-eigenvalue $\bar{\lambda}$. By this, we know that $\rho_k(\mathcal{A}) \geq \max\{|\lambda_{\min}|, |\lambda_{\max}|\}$. On the other hand, by the given condition, we know that

$$\alpha\|\bar{\mathbf{u}}\|_k^{k-m} \leq \mathcal{A}\hat{\mathbf{u}}^m,$$

where $\hat{\mathbf{u}} = \bar{\mathbf{u}}/\|\bar{\mathbf{u}}\|_k$. Consequently, from the fact that $\|\hat{\mathbf{u}}\|_k = 1$, we have that $\mathcal{A}\hat{\mathbf{u}}^m \leq \max\{|\lambda_{\min}|, |\lambda_{\max}|\}$. Hence, it holds that (9). Specially, in addition, if $\|\bar{\mathbf{u}}\|_k = 1$, the desired result follows directly from (9). We complete the proof. ☐

Corollary 1. *Let $\mathcal{A}, \mathcal{B}$ be two m-th order n-dimensional symmetric square tensors satisfying $|\mathcal{B}| \leq \mathcal{A}$. Then $\rho_k(\mathcal{B}) \leq \rho_k(\mathcal{A})$ for every $k \in \{2, \ldots, m\}$.*

Proof. Let $k \in \{2, \ldots, m\}$. For any l^k-eigenvalue β and corresponding l^k-eigenvector $\bar{\mathbf{u}}$ of $\mathcal{B}$, it holds that

$$\begin{cases} \mathcal{B}\bar{\mathbf{u}}^{m-1} = \beta\varphi_{k-1}(\bar{\mathbf{u}}), \\ \|\bar{\mathbf{u}}\|_k = 1, \end{cases}$$

which implies that $|\beta||\bar{\mathbf{u}}|^{[k-1]} = |\beta||\varphi_{k-1}(\bar{\mathbf{u}})| \leq |\mathcal{B}||\bar{\mathbf{u}}|^{m-1} \leq \mathcal{A}|\bar{\mathbf{u}}|^{m-1}$. It is clear that $|\bar{\mathbf{u}}| \in \Re^n_+\backslash\{\theta\}$. By Theorem 3.2, we know that $\rho_k(\mathcal{A}) \geq \|\bar{\mathbf{u}}\|_k^{k-m}|\beta|$. Moreover, $\rho_k(\mathcal{A}) \geq |\beta|$ since $\|\bar{\mathbf{u}}\|_k = 1$. Hence, it holds that $\rho_k(\mathcal{A}) \geq \rho_k(\mathcal{B})$. We obtain the desired result and complete the proof. ☐

Theorem 3.3. *Suppose that $\mathcal{A}$ is a real m-th order n-dimensional nonnegative symmetric square tensor. For the given integer $k \in \{2, \ldots, m\}$, if there exist $\alpha > 0$ and $\bar{\mathbf{u}} \in \Re^n_{++}$, such that $\mathcal{A}\bar{\mathbf{u}}^{m-1} \leq \alpha\bar{\mathbf{u}}^{[k-1]}$, then*

$$\rho_k(\mathcal{A}) \leq \left(\max_{1\leq i\leq n}\{1/\bar{u}_i\}\right)^{m-k}\alpha. \tag{11}$$

Specially, in the case where $k = m$, it holds that $\rho_k(\mathcal{A}) \leq \alpha$.

Proof. For any l^k-eigenvector $\bar{\mathbf{x}} \in \Re^n\backslash\{\theta\}$ of $\mathcal{A}$, associated with the l^k-eigenvalue $\bar{\lambda}$, we claim that

$$\|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}^k \leq 1/\min_{1\leq i\leq n}\{\bar{u}_i^k\}. \tag{12}$$

In fact, by (6), it holds that $|\bar{x}_i| \leq \|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}\bar{u}_i$ for $i = 1, \ldots, n$, and at least one equality hold. Without loss of generality, we assume $|\bar{x}_1| = \|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}\bar{u}_1$. Then we have that

$$\|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}^k = |\bar{x}_1|^k/\bar{u}_1^k \leq 1/\min\{\bar{u}_i^k\},$$

where the inequality comes from the fact that $|\bar{x}_1|^k \leq \sum_{i=1}^n |\bar{x}_i|^k = 1$.

Since $\mathcal{A}\bar{\mathbf{x}}^{m-1} = \bar{\lambda}\,\varphi_{k-1}(\bar{\mathbf{x}})$, by the fact that $\|\varphi_{k-1}(\bar{\mathbf{x}})\|_{\bar{\mathbf{u}}^{[k-1]}} = \|\bar{\mathbf{x}}^{[k-1]}\|_{\bar{\mathbf{u}}^{[k-1]}}$ and Proposition 1, we know that $\|\mathcal{A}\bar{\mathbf{x}}^{m-1}\|_{\bar{\mathbf{u}}^{[k-1]}} = |\bar{\lambda}|\|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}^{k-1}$. Consequently, it holds that $|\bar{\lambda}|\|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}^{k-m} = \|\mathcal{A}\hat{\mathbf{x}}^{m-1}\|_{\bar{\mathbf{u}}^{[k-1]}}$, where $\hat{\mathbf{x}} = \bar{\mathbf{x}}/\|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}$. Hence,

$$\begin{array}{rcl} |\bar{\lambda}|\|\bar{\mathbf{x}}\|_{\bar{\mathbf{u}}}^{k-m} & = & \|\mathcal{A}\hat{\mathbf{x}}^{m-1}\|_{\bar{\mathbf{u}}^{[k-1]}} \\ & \leq & \sup\limits_{\|\mathbf{x}\|_{\bar{\mathbf{u}}}=1} \|A(\mathbf{x})\|_{\bar{\mathbf{u}}^{[k-1]}} \\ & = & \|A(\bar{\mathbf{u}})\|_{\bar{\mathbf{u}}^{[k-1]}} \\ & \leq & \alpha\|\bar{\mathbf{u}}^{[k-1]}\|_{\bar{\mathbf{u}}^{[k-1]}} \\ & = & \alpha, \end{array}$$

where the last second equality is due to Proposition 2, and the last inequality comes from the given condition and the fact that $\|\cdot\|_{\mathbf{u}}$ is monotonous. Moreover, by (12), we obtain

$$|\bar{\lambda}| \le \left(\max_{1\le i\le n}\{1/\bar{u}_i\}\right)^{m-k}\alpha,$$

which implies that

$$\rho_k(\mathcal{A}) \le \left(\max_{1\le i\le n}\{1/\bar{u}_i\}\right)^{m-k}\alpha.$$

If $k=m$, the conclusion comes from (11) immediately. We obtain the desired result and complete the proof. $\square$

For every $i=1,\ldots,n$, let us denote $b_i = \sum\limits_{1\le i_2,\ldots,i_m\le n} a_{ii_2\ldots i_m}$.

Corollary 2. *Let $\mathcal{A}$ be a real m-th order n-dimensional nonnegative symmetric square tensor. Then*

$$\rho_k(\mathcal{A}) \le \max_{1\le i\le n}\{b_i\}$$

for every $k\in\{2,\ldots,m\}$. In particular, if $b_i=\alpha$ for $i=1,\ldots,n$, then $\rho_k(\mathcal{A})=\alpha$ for every $k\in\{2,\ldots,m\}$.

Proof. From Lemma 2.4, we know that $b_i>0$ for $i=1,\ldots,n$. By taking $\bar{\mathbf{u}} = (1,\ldots,1)^\top \in \Re^n_{++}$, we know that for every $k=2,\ldots,m$,

$$\mathcal{A}\bar{\mathbf{u}}^{m-1} = (b_1,\ldots,b_n)^\top \le \max_{1\le i\le n}\{b_i\}\bar{\mathbf{u}}^{[k-1]},$$

i.e., the condition (9) in Theorem 3.3 holds. By Theorem 3.3, we obtain that $\rho_k(\mathcal{A}) \le \max\limits_{1\le i\le n}\{b_i\}$ for $k=2,\ldots,m$. In particular, if $b_i=\alpha$ for $i=1,\ldots,n$, then it is clear that for $k=2,\ldots,m$, $\mathcal{A}\bar{\mathbf{u}}^{m-1} = \alpha\bar{\mathbf{u}}^{[k-1]}$, which means that $\bar{\mathbf{u}}$ is an l^k-eigenvector of $\mathcal{A}$, associated with the eigenvalue α. Hence, $\rho_k(\mathcal{A})\ge\alpha$ for $k=2,\ldots,m$. Therefore, we obtain that $\rho_k(\mathcal{A})=\alpha$ and complete the proof. $\square$

For any $k\in\{2,\ldots,m\}$, let us denote $\mathcal{I}_k = (\delta_{i_1\ldots i_m})$ with

$$\delta_{i_1\ldots i_m} = \begin{cases} 1, & \text{if } i_1=\ldots=i_m, \\ 1/C^k_{m-1}, & \text{if } \{i_1,\ldots,i_m\} = \{\overbrace{i,\ldots,i}^{k},\overbrace{j,\ldots,j}^{m-k}\} \text{ with } i\ne j, \\ 0, & \text{otherwise,} \end{cases}$$

where $C^k_n = \frac{n!}{k!(n-k)!}$. It is easy to see that for every $k\in\{2,\ldots,m\}$, $\mathcal{I}_k$ is symmetric, and $\mathcal{A}+\mathcal{I}_k$ is irreducible if and only if $\mathcal{A}$ if irreducible.

Theorem 3.4. *Let $\mathcal{A}$ be a real m-th order n-dimensional nonnegative symmetric square tensor and $\mathcal{B} = a(\mathcal{A}+b\mathcal{I}_{m/2})$, where $a>0$, $|b|\le\rho_{m/2}(\mathcal{A})$ and m is even. Then*

$$\rho_{m/2}(\mathcal{B}) \ge a\left(\rho_{m/2}(\mathcal{A})+b\right).$$

Moreover, in addition, if $\mathcal{B}$ is nonnegative and $|b|\le\min\left\{\rho_{m/2}(\mathcal{A}), a^{-1}\rho_{m/2}(\mathcal{B})\right\}$, then $\rho_{m/2}(\mathcal{B}) = a\left(\rho_{m/2}(\mathcal{A})+b\right)$.

Proof. Let $\bar{\mathbf{x}}$ be an $l^{m/2}$-eigenvector of $\mathcal{A}$, associated with the eigenvalue $\bar{\lambda}$ satisfying $|\bar{\lambda}| = \rho_{m/2}(\mathcal{A})$. Since $\mathcal{A}\bar{\mathbf{x}}^{m-1} = \bar{\lambda}\varphi_{m/2-1}(\bar{\mathbf{x}})$, it holds that $\rho_{m/2}(\mathcal{A})|\bar{\mathbf{x}}|^{[m/2-1]} \le$

$\mathcal{A}|\bar{\mathbf{x}}|^{m-1}$ from the given condition that $\mathcal{A}$ is nonnegative. Consequently, since $\|\bar{\mathbf{x}}\|_{m/2} = 1$, we know that

$$\begin{aligned} \mathcal{B}|\bar{\mathbf{x}}|^{m-1} &= a(\mathcal{A} + b\mathcal{I}_{m/2})|\bar{\mathbf{x}}|^{m-1} \\ &\geq a\left(\rho_{m/2}(\mathcal{A}) + b\|\bar{\mathbf{x}}\|_{m/2}^{m/2}\right)|\bar{\mathbf{x}}|^{[m/2-1]} \\ &= a\left(\rho_{m/2}(\mathcal{A}) + b\right)|\bar{\mathbf{x}}|^{[m/2-1]}, \end{aligned}$$

i.e., the condition (9) with $\bar{\mathbf{u}} = |\bar{\mathbf{x}}|$, $k = m/2$ and $\alpha = a\left(\rho_{m/2}(\mathcal{A}) + b\right)$ holds. Since $\mathcal{B}$ is symmetric, by Theorem 3.2, we know that

$$\rho_{m/2}(\mathcal{B}) \geq \|\bar{\mathbf{x}}\|_{m/2}^{-m/2} a(\rho_{m/2}(\mathcal{A}) + b) = a(\rho_{m/2}(\mathcal{A}) + b).$$

Moreover, if $\mathcal{B}$ is nonnegative and $|b| \leq \min\left\{\rho_{m/2}(\mathcal{A}), a^{-1}\rho_{m/2}(\mathcal{B})\right\}$, then we can prove that

$$\rho_{m/2}(\mathcal{A}) \geq a^{-1}(\rho_{m/2}(\mathcal{B}) - ab)$$

by the similar way used above. Therefore, $\rho_{m/2}(\mathcal{B}) = a(\rho_{m/2}(\mathcal{A}) + b)$. The proof is completed. □

4. Final remarks.

In the case where $k = m$, the l^k-eigenvalue and l^k-eigenvector are reduced the H-eigenvalue and H-eigenvector , respectively, which were introduced in [6] and [4]. In [2, 9], the Perron-Frobenius theorem and some further properties of H-eigenvalues/vectors of tensor were studied. If $k = 2$, then the l^k-eigenvalues/vectors becomes the Z-eigenvalues/vectors given in [6]. In general, for the case that $2 \leq k \leq m-1$, the uniqueness of the l^k-eigenvalue $\bar{\lambda}$ which associated with the positive l^k-eigenvector does not hold. So, as future work, we need to find some appropriate conditions which ensure the uniqueness of the l^k-eigenvalue $\bar{\lambda}$ in Theorem 2.6.

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