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Programmable criteria for strong $\mathcal{H}$-tensors

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Abstract Strong $\mathcal{H}$-tensors play an important role in identifying positive semidefiniteness of even-order real symmetric tensors. We provide several simple practical criteria for identifying strong $\mathcal{H}$-tensors. These criteria only depend on the elements of the tensors; therefore, they are easy to be verified. Meanwhile, a sufficient and necessary condition of strong $\mathcal{H}$-tensors is obtained. We also propose an algorithm for identifying the strong $\mathcal{H}$-tensors based on these criterions. Some numerical results show the feasibility and effectiveness of the algorithm.

Keywords Strong $\mathcal{H}$-tensors · Positive semidefiniteness · Irreducible

1 Introduction

We start with some preliminaries. First, denote $[n] := \{1, 2, \cdots, n\}$. A complex (real) order m dimension n tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ consists of n^m complex (real) entries:

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{C}\ (\mathbb{R}),$$

where $i_j \in [n]$ for $j \in [m]$ [5, 10, 12, 16, 25]. It is obvious that a matrix is an order 2 tensor. Moreover, a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is called symmetric [9, 18] if

$$a_{i_1 \cdots i_m} = a_{\pi(i_1 \cdots i_m)},\ \forall \pi \in \Pi_m,$$

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where Π_m is the permutation group of m indices. And a real tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is called nonnegative if each entry is nonnegative. An order m dimension n tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m})$ is called the unit tensor [22], where

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & if\ i_1 = i_2 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

Given an order m dimension n complex tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, if there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called the eigenvalue of $\mathcal{A}$ and x the eigenvector of $\mathcal{A}$ associated with λ [6, 8, 11, 13, 17–20, 23], where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith component are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in [n]} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively.

In addition, the spectral radius of a tensor $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda\ is\ an\ eigenvalue\ of\ \mathcal{A}\}.$$

Analogous with that of M-matrices, comparison matrices and H-matrices, the definitions of $\mathcal{M}$-tensors, comparison tensors and strong $\mathcal{H}$-tensors are given by:

Definition 1 [26] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a real tensor of order m dimension n. $\mathcal{A}$ is called an $\mathcal{M}$-tensor if there exists a nonnegative tensor $\mathcal{B}$ and a positive real number $\eta \geq \rho(\mathcal{B})$ such that $\mathcal{A}=\eta\mathcal{I}$-$\mathcal{B}$. If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}$-tensor.

Definition 2 [7] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a complex tensor of order m dimension n. We call another tensor $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \cdots i_m})$ as the comparison tensor of $\mathcal{A}$ if

$$m_{i_1 i_2 \cdots i_m} = \begin{cases} +|a_{i_1 i_2 \cdots i_m}|, & if\ (i_2, i_3, \cdots, i_m) = (i_1, i_1, \cdots, i_1), \\ -|a_{i_1 i_2 \cdots i_m}|, & if\ (i_2, i_3, \cdots, i_m) \neq (i_1, i_1, \cdots, i_1). \end{cases}$$

Definition 3 [14] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a complex tensor of order m dimension n. $\mathcal{A}$ is called a strong $\mathcal{H}$-tensor if there is an entrywise positive vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ such that for all $i \in [n]$,

$$|a_{i\cdots i}| x_i^{m-1} > \sum_{\substack{i_2, i_3, \ldots, i_m \in [n], \\ \delta_{i i_2 \ldots i_m} = 0}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m}. \quad (1)$$

Moreover, Ding, Qi, and Wei [7] also provided the following definition of strong $\mathcal{H}$-tensor, which is equivalent to the Definition 3.

Definition 4 [7] We call a tensor an $\mathcal{H}$-tensor, if its comparison tensor is an $\mathcal{M}$-tensor, we call it as a strong $\mathcal{H}$-tensor, if its comparison tensor is a strong $\mathcal{M}$-tensor.

For an mth degree homogeneous polynomial of n variables $f(x)$ denoted as

$$f(x) = \sum_{i_1,\ldots,i_m \in [n]} a_{i_1 i_2 \cdots i_m} x_{i_1} \cdots x_{i_m}, \tag{2}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. When m is even, $f(x)$ is called positive definite if

$$f(x) > 0, \; for \; any \; x \in \mathbb{R}^n, x \neq 0.$$

The homogeneous polynomial $f(x)$ in (2) is equivalent to the tensor product of an order m dimensional n symmetric tensor $\mathcal{A}$ and x^m defined by

$$f(x) = \mathcal{A}x^m = \sum_{i_1,\ldots,i_m \in [n]} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}, \tag{3}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. The positive definiteness of multivariate polynomial $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control, such as the multivariate network realizability theory [2], a test for Lyapunov stability in multivariate filters [3], a test of existence of periodic oscillations using Bendixson's theorem [21], and the output feedback stabilization problems [1]. For $n \leq 3$, the positive definiteness of the homogeneous polynomial form can be checked by a method based on the Sturm theorem [4]. For $n > 3$ and $m \geq 4$, it is difficult to determine a given even-order multivariate polynomial $f(x)$ is positive semi-definite or not because the problem is NP-hard. In [19], Qi pointed out that a multivariate polynomial $f(x)$ is positive definite if and only if the real symmetric tensor $\mathcal{A}$ in (3) is positive definite. However, it is also difficult to determine a given even-order symmetric tensor is positive definite or not because the problem is also NP-hard. For this case, recently, by introducing the definition of strong $\mathcal{H}$-tensor, Li et al. [14] provided the following theorem.

Theorem 1 *Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an even-order real symmetric tensor of order m dimension n with $a_{k \cdots k} > 0$ for all $k \in [n]$. If $\mathcal{A}$ is a strong $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive definite.*

Theorem 1 provides a method for identifying the positive definiteness of an even-order symmetric tensor by determining strong $\mathcal{H}$-tensors. But it is still difficult to determine a strong $\mathcal{H}$-tensor in practice by using the definition of strong $\mathcal{H}$-tensor because the conditions "there is an entrywise positive vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that for all $i \in [n]$, the Inequation (1) holds" in Definition 3 is unverifiable for there are an infinite number of positive vector in $\mathbb{R}^n$. Therefore, finding effective criteria to identify strong $\mathcal{H}$-tensor is interesting.

In the present paper, several new simple interesting criteria for strong $\mathcal{H}$-tensors are obtained. In Section 2, we give an equivalent condition for a strong $\mathcal{H}$-tensor. Via using only the elements of tensors, five criteria for identifying strong $\mathcal{H}$-tensor are obtained in Section 3. A direct algorithm for identifying strong $\mathcal{H}$-tensor is put forward in Section 4. Numerical examples are then presented in Section 5 which shows that our proposed algorithm are efficient. Finally, we conclude the paper in Section 6.

We adopt the following notation throughout this paper. The calligraphy letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{H}$, $\cdots$ denote the tensors; the capital letters A, B, D, $\cdots$ represent the matrices; the lowercase letters x, y, $\cdots$ refer to the vectors.

2 A sufficient and necessary condition for a strong $\mathcal{H}$-tensor

For the convenience of discussion, we start with the following definitions and lemmas.

Definition 5 [26] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a complex tensor of order m dimension n. $\mathcal{A}$ is diagonally dominant if for all $i \in [n]$,

$$|a_{ii\cdots i}| \geq \sum_{\substack{i_2,\ldots,i_m \in [n],\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|. \tag{4}$$

$\mathcal{A}$ is strictly diagonally dominant if the strict inequality holds in (4) for all i.

Definition 6 [10, 15] The product of an order m dimension n tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and a n-by-n matrix $X = (x_{ij})$ on mode-k is defined by

$$(\mathcal{A} \times_k X)_{i_1 \cdots j_k \cdots i_m} = \sum_{i_k=1}^{n} a_{i_1 \cdots i_k \cdots i_m} x_{i_k j_k}.$$

Remark 1 According to the Definition 6, we denote

$$\mathcal{A}X^{m-1} := \mathcal{A} \times_2 X \times_3 \cdots \times_m X.$$

Particularly, for $X = diag(x_1, x_2, \cdots, x_n)$, the product of the tensor $\mathcal{A}$ and the matrix X is given by:

$$(\mathcal{A}X^{m-1})_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}. \tag{5}$$

Lemma 1 [7] *The following conditions are equivalent:*

(i) *A tensor $\mathcal{A}$ is a strong $\mathcal{H}$-tensor;*
(ii) *There exists a positive diagonal matrix D such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant;*
(iii) *There exist two positive diagonal matrix D_1 and D_2 such that $D_1\mathcal{A}D_2^{m-1}$ is strictly diagonally dominant.*

The following is a sufficient and necessary condition for a tensor to be a strong $\mathcal{H}$-tensor.

Theorem 2 *Let $\mathcal{A}$ be a complex tensor of order m dimension n. Then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor if and only if $\mathcal{A}X^{m-1}$ is a strong $\mathcal{H}$-tensor, where X is an arbitrary positive diagonal matrix.*

Proof Let $X = diag(x_1, x_2, \cdots, x_n)$ is a positive diagonal matrix, and denote $\mathcal{B}_1 = (b^{(1)}_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$. Then from Equality (5), we have

$$b^{(1)}_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}, \forall i_j \in [n], j \in [m].$$

First, we show the necessity. Suppose that $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. By Definition 3, there exists an entrywise positive vector $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ such that for all $i \in [n]$,

$$|a_{ii\cdots i}| y_i^{m-1} > \sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| y_{i_2} \cdots y_{i_m}. \tag{6}$$

Let $D = diag(\frac{y_1}{x_1}, \frac{y_2}{x_2}, \cdots, \frac{y_n}{x_n})$. Obviously, D is a positive diagonal matrix. It follows from Inequality (6) that for each $i \in [n]$

$$\begin{aligned} |b^{(1)}_{ii\cdots i}| (\frac{y_i}{x_i})^{m-1} &= |a_{ii\cdots i} x_i^{m-1}| (\frac{y_i}{x_i})^{m-1} \\ &= |a_{ii\cdots i}| y_i^{m-1} \\ &> \sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| y_{i_2} \cdots y_{i_m} \\ &= \sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\ldots i_m}=0}} |b^{(1)}_{ii_2\cdots i_m}| (\frac{y_{i_2}}{x_{i_2}}) \cdots (\frac{y_{i_m}}{x_{i_m}}). \end{aligned} \tag{7}$$

This means that $\mathcal{B}_1 D^{m-1}$ is strictly diagonally dominant. Furthermore, by Lemma 1, $\mathcal{B}_1 = \mathcal{A}X^{m-1}$ is a strong $\mathcal{H}$-tensor.

Now, we show the sufficiency. Assume that $\mathcal{B}_1$ is a strong $\mathcal{H}$-tensor. Thus, there exists an entrywise positive vector $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n$ such that for each $i \in [n]$,

$$|b^{(1)}_{i\cdots i}| z_i^{m-1} > \sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\ldots i_m}=0}} |b^{(1)}_{ii_2\cdots i_m}| z_{i_2} \cdots z_{i_m}.$$

Let $D_1 = diag(x_1 z_1, x_2 z_2, \cdots, x_n z_n)$. By using the similar technique in Inequality (7), we obtain $\mathcal{A}D_1^{m-1}$ is strictly diagonally dominant. Thus, by Lemma 1, $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. □

Remark 2 Note that the sufficient condition of a strong $\mathcal{H}$-tensor in Theorem 2 is the Corollary 2.1 proposed by Wang, Zhou, and Caccetta in [24]. In fact, we prove here that this sufficient condition is also the necessary condition for a strong $\mathcal{H}$-tensor.

3 Criteria for identifying the strong $\mathcal{H}$-tensors

In this section, we give five criteria for identifying strong $\mathcal{H}$-tensors by making use of elements of tensors only. First, some notations and two lemmas for strong $\mathcal{H}$-tensors are given.

Assume that Λ denote an arbitrary nonempty subset of $[n]$, let

$$\Lambda^{m-1} := \{i_2 i_3 \cdots i_m : i_j \in \Lambda,\ j = 2, 3, \cdots, m\},$$

$$[n]^{m-1} \setminus \Lambda^{m-1} := \{i_2 i_3 \cdots i_m : i_2 i_3 \cdots i_m \in [n]^{m-1} \ and\ i_2 i_3 \cdots i_m \notin \Lambda^{m-1}\}.$$

Given an order m dimension n complex tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$, let

$$r_i(\mathcal{A}) := \sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{i i_2 \ldots i_m} = 0}} |a_{i i_2 \cdots i_m}| = \sum_{i_2,\ldots,i_m \in [n]} |a_{i i_2 \cdots i_m}| - |a_{i \cdots i}|,$$

$$\Lambda_1 := \{i \in [n] : |a_{i \cdots i}| > r_i(\mathcal{A})\},$$

$$\Lambda_2 := \{i \in [n] : |a_{i \cdots i}| \leq r_i(\mathcal{A})\}.$$

Lemma 2 [14] *Let $\mathcal{A}$ be a complex tensor of order m dimension n. If $\mathcal{A}$ is a strictly diagonally dominant tensor, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.*

Lemma 3 [14] *Let $\mathcal{A}$ be a complex tensor of order m dimension n. If $\mathcal{A}$ is a strong $\mathcal{H}$-tensor, then $\Lambda_1 \neq \emptyset$, that is, at least one $i \in [n]$ such that*

$$|a_{i \cdots i}| > r_i(\mathcal{A}).$$

Remark here that from Lemma 2, we have if $\Lambda_2 = \emptyset$ ($\mathcal{A}$ is a strictly diagonally dominant tensor), then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. In addition, by Lemma 3, for a strong $\mathcal{H}$-tensor, there exists at least one strict diagonally dominant row, i.e., $\Lambda_1 \neq \emptyset$. So we always assume that both Λ_1 and Λ_2 are not empty. We next give five criteria for identifying strong $\mathcal{H}$-tensors.

Theorem 3 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a complex tensor of order m dimension n. If*

$$|a_{i i \cdots i}| > \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj \cdots j}|} |a_{i i_2 \cdots i_m}|, \ \forall i \in \Lambda_2, \tag{8}$$

then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

Proof Let

$$\xi_i \equiv \frac{1}{\sum\limits_{i_2i_3\cdots i_m\in\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}|} \left\{ |a_{ii\cdots i}| - \sum_{\substack{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \right.$$

$$\left. - \sum_{i_2i_3\cdots i_m\in\Lambda_1^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}| \right\}, \ \forall i \in \Lambda_2. \tag{9}$$

If $\sum\limits_{i_2i_3\cdots i_m\in\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}| = 0$, we denote $\xi_i = +\infty$. From Inequality (8), we obtain $\xi_i > 0$ for all $i \in \Lambda_2$. Hence, there exists a positive number $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\min_{i\in\Lambda_2}\xi_i,\ 1 - \max_{j\in\Lambda_1}\frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|}\right\}. \tag{10}$$

Let the matrix $X = diag(x_1, x_2, \cdots, x_n)$, where

$$x_i = \begin{cases} \left(\varepsilon + \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i \in \Lambda_1, \\ 1, & i \in \Lambda_2. \end{cases}$$

By Inequality (10), we have $\left(\varepsilon + \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}} < 1$, for all $i \in \Lambda_1$. Because $\varepsilon \neq +\infty$, so $x_i \neq +\infty$, which implies that X is a diagonal matrix with positive entries.

Let $\mathcal{B}_2 = (b^{(2)}_{i_1i_2\cdots i_m}) = \mathcal{A}X^{m-1}$. From Equality (5), we obtain

$$b^{(2)}_{i_1i_2\cdots i_m} = a_{i_1i_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_{i_m}, \forall i_j \in [n], j \in [m].$$

Now, we prove that $\mathcal{B}_2$ is strictly diagonally dominant. Let us first consider $i \in \Lambda_2$. If $\sum\limits_{i_2i_3\cdots i_m\in\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}| = 0$, then by Inequalities (8) and (10), we have

$$\begin{aligned} &r_i(\mathcal{B}_2) \\ &= \sum_{\substack{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |b^{(2)}_{ii_2\cdots i_m}| + \sum_{i_2i_3\cdots i_m\in\Lambda_1^{m-1}} |b^{(2)}_{ii_2\cdots i_m}| \\ &= \sum_{\substack{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m} + \sum_{i_2i_3\cdots i_m\in\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m} \\ &= \sum_{\substack{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m} \\ &\le \sum_{\substack{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \\ &< |a_{ii\cdots i}| = |b^{(2)}_{ii\cdots i}|. \end{aligned} \tag{11}$$

If $\sum\limits_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \neq 0$, then by Inequalities (9) and (10), we have

$$
\begin{aligned}
& r_i(\mathcal{B}_2) \\
= & \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m} + \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m} \\
= & \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m} + \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \\
& \cdot \left(\varepsilon + \frac{r_{i_2}(\mathcal{A})}{|a_{i_2 i_2 \cdots i_2}|} \right)^{\frac{1}{m-1}} \cdots \left(\varepsilon + \frac{r_{i_m}(\mathcal{A})}{|a_{i_m i_m \cdots i_m}|} \right)^{\frac{1}{m-1}} \\
\leq & \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m} \\
& + \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \left(\varepsilon + \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{j j \cdots j}|} \right) \\
\leq & \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{j j \cdots j}|} \\
& + \varepsilon \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \\
< & |a_{i i \cdots i}| = |b_{i i \cdots i}^{(2)}|. \qquad (12)
\end{aligned}
$$

Finally, we consider $i \in \Lambda_1$. Since $|a_{i i \cdots i}| > r_i(\mathcal{A})$, we have

$$
|a_{i i \cdots i}| - \sum_{\substack{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| > 0, \qquad (13)
$$

and

$$
\sum_{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| + \sum_{\substack{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{j j \cdots j}|} - r_i(\mathcal{A}) \leq 0,
$$

which, together with Inequality (13) and $\varepsilon > 0$, yields

$$\varepsilon > \frac{1}{|a_{ii\cdots i}| - \sum\limits_{\substack{i_2i_3\cdots i_m\in\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|} \left\{ \sum_{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}| \right.$$

$$\left. + \sum_{\substack{i_2i_3\cdots i_m\in\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|} - r_i(\mathcal{A}) \right\}. \tag{14}$$

From Inequality (14), for each $i \in \Lambda_1$, we have

$$\begin{aligned}
&|b^{(2)}_{ii\cdots i}| - r_i(\mathcal{B}_2)\\
&= |a_{ii\cdots i}|\left(\varepsilon + \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|}\right) - \sum_{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m}\\
&\quad - \sum_{\substack{i_2i_3\cdots i_m\in\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\varepsilon + \frac{r_{i_2}(\mathcal{A})}{|a_{i_2i_2\cdots i_2}|}\right)^{\frac{1}{m-1}} \cdots \left(\varepsilon + \frac{r_{i_m}(\mathcal{A})}{|a_{i_mi_m\cdots i_m}|}\right)^{\frac{1}{m-1}}\\
&\geq |a_{ii\cdots i}|\left(\varepsilon + \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|}\right) - \sum_{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}|\\
&\quad - \sum_{\substack{i_2i_3\cdots i_m\in\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\varepsilon + \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|}\right)\\
&= \varepsilon\left(|a_{ii\cdots i}| - \sum_{\substack{i_2i_3\cdots i_m\in\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|\right) + r_i(\mathcal{A}) - \sum_{i_2i_3\cdots i_m\in[n]^{m-1}\setminus\Lambda_1^{m-1}} |a_{ii_2\cdots i_m}|\\
&\quad - \sum_{\substack{i_2i_3\cdots i_m\in\Lambda_1^{m-1},\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|} > 0.
\end{aligned} \tag{15}$$

Hence, from Inequalities (11), (12), and (15), we obtain $|b^{(2)}_{ii\cdots i}| > r_i(\mathcal{B}_2)$ for all $i \in [n]$, that is, $\mathcal{B}_2$ is strictly diagonally dominant; therefore, $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. □

Remark 3 If Λ_1 contain only one element, then Theorem 3 reduces to Lemma 12 of [14].

Remark 4 For a set Λ with finite elements, we use $|\Lambda|$ to denote the number of elements in the set Λ. From Inequation (8), we obtain the number of the basic arithmetic operations of Inequation (8) is $n^m - 2n + |\Lambda_2|(n^{m-1} - 2) + |\Lambda_2||\Lambda_1|^{m-1} + |\Lambda_1||\Lambda_2|$ (requiring $n^m - 2n + |\Lambda_2|(n^{m-1} - 2)$ additions and $|\Lambda_2||\Lambda_1|^{m-1} + |\Lambda_1||\Lambda_2|$ multiplications and divisions of numbers). Furthermore, it follows from $|\Lambda_1| < n$ and $|\Lambda_2| < n$ that $n^m - 2n + |\Lambda_2|(n^{m-1} - 2) + |\Lambda_2||\Lambda_1|^{m-1} + |\Lambda_1||\Lambda_2| < 3n^m + n^2 - 2n$. Thus, Inequation (8) of Theorem 3 can be checked in polynomial time.

Theorem 4 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a complex tensor of order m dimension n. If*

$$|a_{ii\cdots i}| > \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}|, \ \forall i \in \Lambda_2, \tag{16}$$

and

$$\sum_{j_2 j_3 \cdots j_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{j j_2 \cdots j_m}| = 0, \ \forall j \in \Lambda_1, \tag{17}$$

then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

Proof By Inequality (16), for each $i \in \Lambda_2$, there exists a positive number $\varsigma_i > 1$, such that

$$|a_{ii\cdots i}| > \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| + \frac{1}{\varsigma_i} \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|}. \tag{18}$$

Denote, $\varsigma \equiv \max\{\varsigma_i, i \in \Lambda_2\}$. By Inequality (18), we obtain

$$|a_{ii\cdots i}| > \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| + \frac{1}{\varsigma} \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|}, \ \forall i \in \Lambda_2. \tag{19}$$

Since $|a_{ii\cdots i}| \le r_i(\mathcal{A})$, for all $i \in \Lambda_2$ and Inequality (16), so

$$\sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| > 0, \forall i \in \Lambda_2. \tag{20}$$

Denote

$$\chi_i \equiv \frac{1}{\sum\limits_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}|} \left\{ |a_{i i \cdots i}| - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \right.$$

$$\left. - \frac{1}{\varsigma} \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|} |a_{i i_2 \cdots i_m}| \right\}, \ \forall i \in \Lambda_2.$$

From Inequalities (19) and (20), we have $\chi_i > 0$. Therefore, there exists a positive number $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ \min_{i \in \Lambda_2} \chi_i, 1 - \max_{j \in \Lambda_1} \frac{r_j(\mathcal{A})}{\varsigma |a_{jj\cdots j}|} \right\}.$$

Let the matrix $X = diag(x_1, x_2, \cdots, x_n)$, where

$$x_i = \begin{cases} \left(\varepsilon + \frac{r_i(\mathcal{A})}{\varsigma |a_{ii\cdots i}|} \right)^{\frac{1}{m-1}}, & i \in \Lambda_1, \\ 1, & i \in \Lambda_2. \end{cases}$$

Let $\mathcal{B}_3 = (b^{(3)}_{i_1 i_2 \cdots i_m}) = \mathcal{A} X^{m-1}$. Similar to the proof of Theorem 3, we can prove that $\mathcal{B}_3$ is strictly diagonally dominant. Then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. □

Remark 5 Using the same argument as Remark 4 in Section 3, the number of the basic arithmetic operations of Inequalities (16) and (17) is less than $2n^m - 2n$, respectively. Thus, Inequalities (16) and (17) of Theorem 4 can be checked in polynomial time.

Remark 6 There is no inclusion relation between the conditions of Theorem 3 and the conditions of Theorem 4, which can be seen from the following examples.

Example 1 Consider a tensor $\mathcal{A} = (a_{ijk})$ of order 3 dimension 3 defined as follows:

$$\mathcal{A} = [A(1, :, :), A(2, :, :), A(3, :, :)],$$

$$A(1, :, :) = \begin{pmatrix} 15 & 1 & 0 \\ 1 & 10 & 0 \\ 1 & 1 & 10 \end{pmatrix}, \ A(2, :, :) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 8 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$A(3, :, :) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

By calculation, we have

$$|a_{111}| = 15, r_1(\mathcal{A}) = 24, |a_{222}| = 8, r_2(\mathcal{A}) = 4, |a_{333}| = 10, r_3(\mathcal{A}) = 5,$$

$$\frac{r_2(\mathcal{A})}{|a_{222}|} = \frac{1}{2}, \frac{r_3(\mathcal{A})}{|a_{333}|} = \frac{1}{2},$$

and $\Lambda_1 = \{2, 3\}$, $\Lambda_2 = \{1\}$. Since

$$\sum_{\substack{jk\in[3]^2\setminus\Lambda_1^2,\\ \delta_{1jk}=0}} |a_{1jk}| + \sum_{jk\in\Lambda_1^2} \max_{t\in\{j,k\}} \frac{r_t(\mathcal{A})}{|a_{ttt}|}|a_{1jk}| = 13.5 < 15 = |a_{111}|,$$

we know that $\mathcal{A}$ satisfies the conditions of Theorem 3, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. But $\sum\limits_{jk\in[3]^2\setminus\Lambda_1^2} = 3 \neq 0$, so $\mathcal{A}$ does not satisfy the conditions of Theorem 4.

Example 2 Consider a tensor $\mathcal{A} = (a_{ijk})$ of order 3 dimension 3 defined as follows:

$$\mathcal{A} = [A(1, :, :), A(2, :, :), A(3, :, :)],$$

$$A(1, :, :) = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 10 & 0 \\ 1 & 1 & 10 \end{pmatrix}, \ A(2, :, :) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 8 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

$$A(3, :, :) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 10 \end{pmatrix}.$$

By calculation, we have

$$|a_{111}| = 4, r_1(\mathcal{A}) = 24, |a_{222}| = 8, r_2(\mathcal{A}) = 4, |a_{333}| = 10, r_3(\mathcal{A}) = 5,$$

$$\frac{r_2(\mathcal{A})}{|a_{222}|} = \frac{1}{2}, \frac{r_3(\mathcal{A})}{|a_{333}|} = \frac{1}{2},$$

and $\Lambda_1 = \{2, 3\}$, $\Lambda_2 = \{1\}$. Since

$$\sum_{\substack{jk\in[3]^2\setminus\Lambda_1^2,\\ \delta_{1jk}=0}} |a_{1jk}| = 3 < 4 = |a_{111}|,$$

and

$$\sum_{jk\in[3]^2\setminus\Lambda_1^2} |a_{2jk}| = 0, \sum_{jk\in[3]^2\setminus\Lambda_1^2} |a_{3jk}| = 0.$$

we have that $\mathcal{A}$ satisfies the conditions of Theorem 4, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. But

$$\sum_{\substack{jk\in[3]^2\setminus\Lambda_1^2,\\ \delta_{1jk}=0}} |a_{1jk}| + \sum_{jk\in\Lambda_1^2} \max_{t\in\{j,k\}} \frac{r_t(\mathcal{A})}{|a_{ttt}|}|a_{1jk}| = \frac{27}{2} > 4 = |a_{111}|,$$

so $\mathcal{A}$ does not satisfy the conditions of Theorem 3.

Theorem 5 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a complex tensor of order m dimension n and*

$$\alpha = \max_{i \in \Lambda_2} \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|}.$$

If

$$\left(r_i(\mathcal{A}) - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \alpha \right) \left(|a_{jj\cdots j}| - \sum_{\substack{j_2 j_3 \cdots j_m \in \Lambda_1^{m-1}, \\ \delta_{j j_2 \cdots j_m} = 0}} |a_{j j_2 \cdots j_m}| \right)$$
$$> \sum_{l_2 l_3 \cdots l_m \in \Lambda_1^{m-1}} |a_{i l_2 \cdots l_m}| \sum_{t_2 t_3 \cdots t_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{j t_2 \cdots t_m}| \alpha, \quad \forall i \in \Lambda_2, \ j \in \Lambda_1, \tag{21}$$

then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

Proof Let

$$\Theta_i \equiv \frac{r_i(\mathcal{A}) - \sum\limits_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \alpha}{\sum\limits_{l_2 l_3 \cdots l_m \in \Lambda_1^{m-1}} |a_{i l_2 \cdots l_m}|}, \forall i \in \Lambda_2,$$

and

$$\theta_j \equiv \frac{\sum\limits_{t_2 t_3 \cdots t_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{j t_2 \cdots t_m}| \alpha}{|a_{jj\cdots j}| - \sum\limits_{\substack{j_2 j_3 \cdots j_m \in \Lambda_1^{m-1}, \\ \delta_{j j_2 \cdots j_m} = 0}} |a_{j j_2 \cdots j_m}|}, \forall j \in \Lambda_1.$$

It follows from inequality $|a_{jj\cdots j|} > r_j(\mathcal{A})$ for each $j \in \Lambda_1$ that

$$|a_{jj\cdots j}| - \sum_{\substack{j_2 j_3 \cdots j_m \in \Lambda_1^{m-1}, \\ \delta_{j j_2 \cdots j_m} = 0}} |a_{j j_2 \cdots j_m}| > 0, \ \forall j \in \Lambda_1,$$

which, together with Inequality (21), yields

$$\Theta_i > \theta_j \geq 0, \ \forall i \in \Lambda_2, \ j \in \Lambda_1.$$

If $\sum\limits_{l_2 l_3 \cdots l_m \in \Lambda_1^{m-1}} |a_{i l_2 \cdots l_m}| = 0$, we denote $\Theta_i = +\infty$, then there exists a positive number $\varepsilon > 0$ such that

$$0 \leq \max_{j \in \Lambda_1} \theta_j < \max_{j \in \Lambda_1} \theta_j + \varepsilon < \min\{\min_{i \in \Lambda_2} \Theta_i, \ \alpha\}. \tag{22}$$

Let the matrix $X = diag(x_1, x_2, \cdots, x_n)$, where

$$x_i = \begin{cases} \left(\max\limits_{j\in\Lambda_1} \theta_j + \varepsilon\right)^{\frac{1}{m-1}}, & i \in \Lambda_1, \\ \alpha^{\frac{1}{m-1}}, & i \in \Lambda_2. \end{cases}$$

We have by Inequalities (22) that $\left(\max\limits_{i\in\Lambda_1} \theta_i + \varepsilon\right)^{\frac{1}{m-1}} < \alpha^{\frac{1}{m-1}}$. Because $\varepsilon \neq +\infty$, so $x_i \neq +\infty$, which shows that X is a diagonal matrix with positive entries. Let $\mathcal{B}_4 = (b^{(4)}_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$. From Equality (5), we have

$$b^{(4)}_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}, \forall i_j \in [n], j \in [m].$$

Now, we prove that $\mathcal{B}_4$ is strictly diagonally dominant.

For any $i \in \Lambda_2$, we have

$$\begin{aligned} & |b^{(4)}_{ii\cdots i}| - r_i(\mathcal{B}_4) \\ = & |a_{ii\cdots i}|\alpha - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m} \\ & - \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \left(\max_{j\in\Lambda_1} \theta_j + \varepsilon\right) \\ \geq & |a_{ii\cdots i}| \cdot \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|} - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}|\alpha \\ & - \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \left(\max_{j\in\Lambda_1} \theta_j + \varepsilon\right) \\ = & r_i(\mathcal{A}) - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}|\alpha \\ & - \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \left(\max_{j\in\Lambda_1} \theta_j + \varepsilon\right). \end{aligned} \tag{23}$$

If $\sum\limits_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| = 0$, then Inequality (23) and $\Theta_i > 0$ imply that

$$|b^{(4)}_{ii\cdots i}| - r_i(\mathcal{B}_4) \geq r_i(\mathcal{A}) - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}|\alpha > 0. \tag{24}$$

If $\sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \neq 0$, then by Inequalities (22) and (23), we have

$$\begin{aligned}
&|b^{(4)}_{ii\cdots i}| - r_i(\mathcal{B}_4) \\
\geq\ & r_i(\mathcal{A}) - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \alpha \\
&- \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \left(\max_{j \in \Lambda_1} \theta_j + \varepsilon \right) \\
>\ & r_i(\mathcal{A}) - \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \alpha - \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \Theta_i \\
=\ & 0. \qquad (25)
\end{aligned}$$

For any $i \in \Lambda_1$, we have

$$\begin{aligned}
&|b^{(4)}_{ii\cdots i}| - r_i(\mathcal{B}_4) \\
=\ & |a_{ii\cdots i}| \left(\max_{j \in \Lambda_1} \theta_j + \varepsilon \right) - \sum_{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m} \\
&- \left(\max_{j \in \Lambda_1} \theta_j + \varepsilon \right) \sum_{\substack{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \\
\geq\ & |a_{ii\cdots i}| \left(\max_{j \in \Lambda_1} \theta_j + \varepsilon \right) - \sum_{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \alpha \\
&- \left(\max_{j \in \Lambda_1} \theta_j + \varepsilon \right) \sum_{\substack{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \\
=\ & \left(|a_{ii\cdots i}| - \sum_{\substack{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \right) \left(\max_{j \in \Lambda_1} \theta_j + \varepsilon \right) \\
&- \sum_{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \alpha \\
>\ & \left(|a_{ii\cdots i}| - \sum_{\substack{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}, \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| \right) \theta_i - \sum_{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus \Lambda_1^{m-1}} |a_{i i_2 \cdots i_m}| \alpha \\
=\ & 0 \qquad (26)
\end{aligned}$$

Hence, from Inequalities (24), (25), and (26), we conclude that $|b^{(4)}_{ii\cdots i}| > r_i(\mathcal{B}_4)$ for all $i \in [n]$, that is, $\mathcal{B}_4$ is strictly diagonally dominant. As a result, $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. □

Remark 7 Using the same argument as Remark 4 in Section 3, the number of the basic arithmetic operations in Inequation (21) is less than $5(n^m)^2 - n^2$. Thus, Inequation (21) of Theorem 5 can be checked in polynomial time.

To give Theorem 6, we need the following definition and lemma.

Definition 7 [22] A complex tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ of order m dimension n is called reducible, if there exists a nonempty proper index subset $I \subset [n]$ such that

$$a_{i_1 i_2\cdots i_m} = 0 \ for\ all\ i_1 \in I,\ \ for\ all\ i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

Lemma 4 [14] *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor of order m dimension n. If $\mathcal{A}$ is irreducible,*

$$|a_{i\cdots i}| \geq r_i(\mathcal{A})\ i\ \ \in [n],$$

and strictly inequality holds for at least one i, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

Theorem 6 *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ be a complex tensor of order m dimension n. Define α be the number defined in Theorem 5. If $\mathcal{A}$ is irreducible, and*

$$\left(r_i(\mathcal{A}) - \sum_{\substack{i_2 i_3\cdots i_m \in [n]^{m-1}\setminus\Lambda_1^{m-1},\\ \delta_{i i_2\cdots i_m}=0}} |a_{i i_2\cdots i_m}|\alpha \right)\left(|a_{jj\cdots j}| - \sum_{\substack{j_2 j_3\cdots j_m \in \Lambda_1^{m-1},\\ \delta_{j j_2\cdots j_m}=0}} |a_{j j_2\cdots j_m}| \right)$$
$$\geq \sum_{l_2 l_3\cdots l_m \in \Lambda_1^{m-1}} |a_{i l_2\cdots l_m}| \sum_{t_2 t_3\cdots t_m \in [n]^{m-1}\setminus\Lambda_1^{m-1}} |a_{j t_2\cdots t_m}|\alpha,\ \forall i \in \Lambda_2,\ j \in \Lambda_1, \quad (27)$$

in addition, the strict inequality holds for at least one pair of indices $i \in \Lambda_2$ and $j \in \Lambda_1$. Then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

Proof Defining Θ_i and θ_j as in the proof of Theorem 5. From Inequality (27), we have $\min_{i\in\Lambda_2} \Theta_i \geq \max_{j\in\Lambda_1} \theta_j$. In addition, a strict inequality holds for at least onel pair of indices $i \in \Lambda_2$ and $j \in \Lambda_1$. Notice that $\mathcal{A}$ is irreducible, this implies

$$\sum_{t_2 t_3\cdots t_m \in [n]^{m-1}\setminus\Lambda_1^{m-1}} |a_{j t_2\cdots t_m}| > 0,\ \ j \in \Lambda_1,$$

which, together with the definition of θ_j, yields $\max_{j\in\Lambda_1}\theta_j > 0$. Let the matrix $X = diag(x_1, x_2, \cdots, x_n)$, where

$$x_i = \begin{cases} \left(\max_{j\in\Lambda_1}\theta_j\right)^{\frac{1}{m-1}}, & i \in \Lambda_1, \\ \alpha^{\frac{1}{m-1}}, & i \in \Lambda_2. \end{cases}$$

Let $\mathcal{B}_5 = (b^{(5)}_{i_1 i_2\cdots i_m}) = \mathcal{A}X^{m-1}$.

Adopting the same procedure as in the proof of Theorem 5, we conclude that $|b^{(5)}_{ii\cdots i}| \geq r_i(\mathcal{B}_5)$ for all $i \in [n]$. Because of $\Theta_i \geq \theta_j$, for all $i \in \Lambda_2$ and $j \in \Lambda_1$; moreover, the strict inequality holds for at least one pair of indices $i \in \Lambda_2$ and $j \in \Lambda_1$, thus, there exists at least an $i \in [n]$ such that $|b^{(5)}_{ii\cdots i}| > r_i(\mathcal{B}_5)$.

On the other hand, since $\mathcal{A}$ is irreducible and so is $\mathcal{B}_5$. Then by Lemma 4, we have that $\mathcal{B}_5$ is a strong $\mathcal{H}$-tensor. By Theorem 2, $\mathcal{A}$ is also a strong $\mathcal{H}$-tensor. □

Remark 8 Using the same argument as Remark 4 in Section 3. The number of the basic arithmetic operations in Inequation (27) is less than $5(n^m)^2 - n^2$. Thus, Inequation (27) of Theorem 6 can be checked in polynomial time.

Theorem 7 *Let $\mathcal{A}$ be a complex tensor of order m dimension n, and*

$$\begin{aligned} h_1(\mathcal{A}) &= r_1(\mathcal{A}), \\ h_i(\mathcal{A}) &= \sum_{i_2 i_3\cdots i_m\in[i-1]^{m-1}} |a_{ii_2\cdots i_m}| \left(\frac{h_{i_2}(\mathcal{A})}{|a_{i_2\cdots i_2}|}\right)^{\frac{1}{m-1}} \cdots \left(\frac{h_{i_m}(\mathcal{A})}{|a_{i_m\cdots i_m}|}\right)^{\frac{1}{m-1}} \\ &\quad + \sum_{\substack{i_2 i_3\cdots i_m\in[n]^{m-1}\setminus[i-1]^{m-1}, \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, \; i = 2, 3, \cdots, n. \end{aligned} \tag{28}$$

If

$$|a_{ii\cdots i}| > h_i(\mathcal{A}), \tag{29}$$

and for each $i_1 \in [n-1]$, there exists $j \in \{2, 3, \cdots, m\}$ such that $i_j > i_1$, and $|a_{i_1 i_2\cdots i_m}| \neq 0$, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

Proof Observe that, by hypothesis, for each $i_1 \in [n-1]$, there exists $i_j > i_1$, such that $|a_{i_1 i_2\cdots i_m}| \neq 0$. Then

$$\sum_{\substack{i_2 i_3\cdots i_m\in[n]^{m-1}\setminus[i-1]^{m-1}, \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| > 0, \forall i \in [n-1], \tag{30}$$

which implies

$$h_i(\mathcal{A}) > 0, \forall i \in [n-1],$$

which, together with (29), yields $0 < \left(\frac{h_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}} < 1$ for all $i \in [n-1]$, and there exists a positive number $\varepsilon > 0$ such that $0 < \frac{h_n(\mathcal{A})}{|a_{nn\cdots n}|} + \varepsilon < 1$. Let the matrix $X = diag(x_1, x_2, \cdots, x_n)$, where

$$x_i = \begin{cases} \left(\frac{h_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i \in [n-1], \\ \left(\frac{h_n(\mathcal{A})}{|a_{nn\cdots n}|} + \varepsilon\right)^{\frac{1}{m-1}}, & i = n, \end{cases}$$

and $\mathcal{B}_6 = (b^{(6)}_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$, then from Equality (5), we obtain

$$b^{(6)}_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}, \forall i_j \in [n], j \in [m].$$

Let us first consider $i \in [n-1]$. By Inequalities (28) and (30), we have

$$\begin{aligned} |b^{(6)}_{ii\cdots i}| &= |a_{ii\cdots i}|(x_i)^{m-1} = h_i(\mathcal{A}) \\ &= \sum_{i_2 i_3 \cdots i_m \in [i-1]^{m-1}} |a_{ii_2\cdots i_m}| \left(\frac{h_{i_2}(\mathcal{A})}{|a_{i_2 i_2 \cdots i_2}|}\right)^{\frac{1}{m-1}} \cdots \left(\frac{h_{i_m}(\mathcal{A})}{|a_{i_m i_m \cdots i_m}|}\right)^{\frac{1}{m-1}} \\ &\quad + \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus [i-1]^{m-1}, \\ \delta_{ii_2\cdots i_m} = 0}} |a_{ii_2\cdots i_m}| \\ &> \sum_{\substack{i_2, i_3, \cdots, i_m \in [n], \\ \delta_{ii_2\cdots i_m} = 0}} |a_{ii_2\cdots i_m}| x_{i_2} \cdots x_{i_m} = \sum_{\substack{i_2, i_3, \cdots, i_m \in [n], \\ \delta_{ii_2\cdots i_m} = 0}} |b^{(6)}_{ii_2\cdots i_m}|. \end{aligned} \tag{31}$$

Finally, we consider $i = n$. By Inequalities (28) and (29), we have

$$\begin{aligned} |b^{(6)}_{nn\cdots n}| &= |a_{nn\cdots n}| \left(\frac{h_n(\mathcal{A})}{|a_{nn\cdots n}|} + \varepsilon\right) = h_n(\mathcal{A}) + \varepsilon |a_{nn\cdots n}| \\ &= \sum_{i_2 i_3 \cdots i_m \in [n-1]^{m-1}} |a_{ni_2\cdots i_m}| \left(\frac{h_{i_2}(\mathcal{A})}{|a_{i_2 i_2 \cdots i_2}|}\right)^{\frac{1}{m-1}} \cdots \left(\frac{h_{i_m}(\mathcal{A})}{|a_{i_m i_m \cdots i_m}|}\right)^{\frac{1}{m-1}} \\ &\quad + \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus [n-1]^{m-1}, \\ \delta_{ni_2\cdots i_m} = 0}} |a_{ni_2\cdots i_m}| + \varepsilon |a_{nn\cdots n}| \\ &> \sum_{i_2 i_3 \cdots i_m \in [n-1]^{m-1}} |a_{ni_2\cdots i_m}| \left(\frac{h_{i_2}(\mathcal{A})}{|a_{i_2 i_2 \cdots i_2}|}\right)^{\frac{1}{m-1}} \cdots \left(\frac{h_{i_m}(\mathcal{A})}{|a_{i_m i_m \cdots i_m}|}\right)^{\frac{1}{m-1}} \\ &\quad + \sum_{\substack{i_2 i_3 \cdots i_m \in [n]^{m-1} \setminus [n-1]^{m-1}, \\ \delta_{ni_2\cdots i_m} = 0}} |a_{ni_2\cdots i_m}| \\ &\geq \sum_{\substack{i_2, i_3, \cdots, i_m \in [n], \\ \delta_{ni_2\cdots i_m} = 0}} |a_{ni_2\cdots i_m}| x_{i_2} \cdots x_{i_m} = \sum_{\substack{i_2, i_3, \cdots, i_m \in [n], \\ \delta_{ni_2\cdots i_m} = 0}} |b^{(6)}_{ni_2\cdots i_m}|. \end{aligned} \tag{32}$$

Hence, form Inequalities (31) and (32), we conclude that $|b^{(6)}_{ii\cdots i}| > r_i(\mathcal{B}_6)$ for all $i \in [n]$, that is, $\mathcal{B}_6$ is strictly diagonally dominant. Consequently, $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. □

Remark 9 Using the same argument as Remark 4 in Section 3. The number of the basic arithmetic operations in Inequality (28) is less than $(2m-1)n^m - 2n$.

4 An algorithm for identifying strong $\mathcal{H}$-tensors

In this section, we present an algorithm for identifying strong $\mathcal{H}$-tensors on the basis of the results in the above section.

Algorithm 1

Step 0. Set $k_1 := 0$, $k_2 := 0$, $k_3 := 0$ and $s := 50$.

Step 1. Given a complex tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ with $a_{i\cdots i} \neq 0$ for all $i \in [n]$. If $k_3 = s$, then output k_1 and k_2, stop. Otherwise,

Step 2. Compute $|a_{i\cdots i}|$ and $r_i(\mathcal{A})$ for all $i \in [n]$,

Step 3. If $\Lambda_1 = [n]$, then print "$\mathcal{A}$ is a strong $\mathcal{H}$-tensor." and go to step 4. Otherwise, go to step 5.

Step 4. Replace k_1 by $k_1 + 1$ and replace k_3 by $k_3 + 1$, and go to step 1.

Step 5. If $\Lambda_1 = \emptyset$, then print "$\mathcal{A}$ is a not strong $\mathcal{H}$-tensor." and go to step 6. Otherwise, go to step 7.

Step 6. Replace k_2 by $k_2 + 1$ and replace k_3 by $k_3 + 1$. Go to Step 1.

Step 7. Compute

$$\sum_{i_2 i_3\cdots i_m \in \Lambda_1^{m-1}} |a_{i i_2\cdots i_m}|, \quad \sum_{\substack{i_2 i_3\cdots i_m \in [n]^{m-1}\setminus \Lambda_1^{m-1},\\ \delta_{i i_2\cdots i_m}=0}} |a_{i i_2\cdots i_m}|,$$

$$\text{and} \quad \sum_{i_2 i_3\cdots i_m \in \Lambda_1^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|} |a_{i i_2\cdots i_m}|, \quad \text{for all } i \in \Lambda_2.$$

Step 8. If Inequality (8) holds, then print "$\mathcal{A}$ is a strong $\mathcal{H}$-tensor." and go to step 4. Otherwise,

Step 9. Compute

$$\sum_{\substack{j_2 j_3\cdots j_m \in \Lambda_1^{m-1},\\ \delta_{j j_2\cdots j_m}=0}} |a_{j j_2\cdots j_m}| \ \text{and} \sum_{j_2 j_3\cdots j_m \in [n]^{m-1}\setminus \Lambda_1^{m-1}} |a_{j j_2\cdots j_m}|, \ \text{for all } j \in \Lambda_1.$$

Step 10. If Inequalities (16) and (17) hold, then print "$\mathcal{A}$ is a strong $\mathcal{H}$-tensor." and go to step 4. Otherwise,

Step 11. Compute α. If Inequality (21) holds, then print "$\mathcal{A}$ is a strong $\mathcal{H}$-tensor." and go to step 4. Otherwise,

Step 12. Compute $h_i(\mathcal{A})$, $\forall i \in [n]$. If Inequality (29) holds and for each $i_1 \in [n-1]$, there exists $j \in \{2, 3, \cdots, m\}$ such that $i_j > i_1$, and $|a_{i_1 i_2\cdots i_m}| \neq 0$, then print "$\mathcal{A}$ is a strong $\mathcal{H}$-tensor.", and go to step 4. Otherwise,

Step 13. Print "Whether $\mathcal{A}$ is a strong $\mathcal{H}$-tensor is not checkable by using Lemmas 2 and 3, Theorems 3-5 and 7.", replace k_3 by $k_3 + 1$. Go to Step 1.

Remark 10 (i) Note that s denotes the total number of tensors. The output parameter k_1 is the number of tensors which are strong $\mathcal{H}$-tensor and the output parameter k_2 is the number of tensors which are not strong $\mathcal{H}$-tensor.

(ii) Algorithm 1 is a direct method for identifying strong $\mathcal{H}$-tensor and the calculations only depend on the elements of tensor. Therefore, Algorithm 1 stops after finitely steps.

(iii) For some tensors, we are unable to identify whether they are strong $\mathcal{H}$-tensor or not by using Algorithm 1, because the conditions of Lemma 2 and Theorems 3–5 and 7 are sufficient but not necessary for a strong $\mathcal{H}$-tensor. It is easy to obtain that the number of tensors which are not checkable by using Algorithm 1 is $s - k_1 - k_2$.

5 Numerical example

Example 3 In the implementation of Algorithm 1. Randomly generate 50 tensors of order m dimension n such that the elements of each tensor satisfying

$$a_{i_1 i_2 \cdots i_m} \in \begin{cases} (-n^m \times 0.6, n^m \times 0.6), & if\ i_1 = i_2 = \cdots = i_m; \\ (-1, 1), & otherwise. \end{cases}$$

We determine whether they are strong $\mathcal{H}$-tensor or not by using Algorithm 1. The numerical results are reported in Table 1. In this table, m and n specify the order and the dimension of the randomly generated tensor, respectively. In the "k_1" column, we show the number of tensors which are strong $\mathcal{H}$-tensor. In the "k_2" column, we show the number of tensors which are not strong $\mathcal{H}$-tensor. In the "$s - k_1 - k_2$" column, we give the number of tensors that whether they are strong $\mathcal{H}$-tensor are not checkable by using Algorithm 1. The results reported in Table 1 show that Algorithm 1 can identifying some tensors whether are strong $\mathcal{H}$-tensors or not.

We remark here that the randomly generated tensors in Example 3 satisfy $\Lambda_1 \neq \emptyset$, therefore $k_2 = 0$.

The following example shows that Algorithm 1 also can be used to testing the positive definiteness of the multivariate form $f(x)$ in (3) for some cases.

Example 4 Consider the following 6th-degree homogeneous polynomial

$$f(x) = \mathcal{A}x^6, \tag{33}$$

Table 1 The numbers of strong $\mathcal{H}$-tensors in the 50 randomly generated tensors

m(order)	n(dimension)	k_1	k_2	$s-k_1-k_2$
4	10	31	0	19
4	11	30	0	20
4	12	35	0	15
4	13	32	0	18
4	14	33	0	17
4	15	34	0	16
5	10	29	0	21
5	11	29	0	21
5	12	33	0	17
5	13	29	0	21
5	14	31	0	19
5	15	23	0	27
6	10	33	0	17
6	11	35	0	15
6	12	26	0	24
6	13	31	0	19
6	14	26	0	24
6	15	29	0	21

where $x = (x_1, \cdots, x_6)^T$ and $\mathcal{A} = (a_{i_1\cdots i_6})$ is a symmetric tensor of order 6 dimension 6 with elements defined as follows:

$$\begin{gathered}
a_{111111}=4, a_{222222}=18, a_{333333}=35, a_{444444}=16, a_{555555}=1, a_{666666}=1\\
a_{122222}=a_{212222}=a_{221222}=a_{222122}=a_{222212}=a_{222221}=-1,\\
a_{133333}=a_{313333}=a_{331333}=a_{333133}=a_{333313}=a_{333331}=-2,\\
a_{144444}=a_{414444}=a_{441444}=a_{444144}=a_{444414}=a_{444441}=-1,\\
a_{233333}=a_{323333}=a_{332333}=a_{333233}=a_{333323}=a_{333332}=-2,\\
a_{244444}=a_{424444}=a_{442444}=a_{444244}=a_{444424}=a_{444442}=-1,\\
a_{344444}=a_{434444}=a_{443444}=a_{444344}=a_{444434}=a_{444443}=-1,\\
a_{222333}=a_{223233}=a_{223323}=a_{223332}=a_{232233}=a_{232323}=a_{232332}=-1\\
a_{233223}=a_{233232}=a_{233322}=a_{333222}=a_{332322}=a_{332232}=a_{332223}=-1\\
a_{323322}=a_{323232}=a_{323223}=a_{322332}=a_{322323}=a_{322233}=-1, other\ a_{i_1\cdots i_6}=0.
\end{gathered}$$

In Algorithm 1, set $s := 1$, we obtain that $\mathcal{A}$ is a strong $\mathcal{H}$-tensor with $a_{i\cdots i} > 0$ for all $i \in \{1, \cdots, 6\}$. It follows from Theorem 1 that $\mathcal{A}$ is positive definite, that is, the $f(x)$ in (33) is positive definite.

6 Conclusions

In this paper, we give some criterions for identifying the strong $\mathcal{H}$-tensor which only depend on the elements of tensor. We also present an algorithm for identifying the strong $\mathcal{H}$-tensor based on these criterions.

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