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The proof of a conjecture on largest Laplacian and signless Laplacian H-eigenvalues of uniform hypergraphs ☆

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ABSTRACT

Let $\mathcal{A}(G)$, $\mathcal{L}(G)$ and $\mathcal{Q}(G)$ be the adjacency tensor, Laplacian tensor and signless Laplacian tensor of uniform hypergraph G, respectively. Denote by $\lambda(\mathcal{T})$ the largest H-eigenvalue of tensor $\mathcal{T}$. Let H be a uniform hypergraph, and H' be obtained from H by inserting a new vertex with degree one in each edge. We prove that $\lambda(\mathcal{Q}(H')) \leq \lambda(\mathcal{Q}(H))$. Denote by G^k the kth power hypergraph of an ordinary graph G with maximum degree $\Delta \geq 2$. We prove that $\{\lambda(\mathcal{Q}(G^k))\}$ is a strictly decreasing sequence, which implies Conjecture 4.1 of Hu, Qi and Shao in [4]. We also prove that $\lambda(\mathcal{Q}(G^k))$ converges to Δ when k goes to infinity. The definition of kth power hypergraph G^k has been generalized as $G^{k,s}$. We also prove some eigenvalues properties about $\mathcal{A}(G^{k,s})$, which generalize

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some known results. Some related results about $\mathcal{L}(G)$ are also mentioned.

1. Introduction

Let G be an ordinary graph, and $A(G)$ be the adjacency matrix of G. We denote the set $\{1, 2, \cdots, n\}$ by $[n]$. Hypergraph is a natural generalization of ordinary graph (see [1]). A hypergraph $G = (V(G), E(G))$ on n vertices is a set of vertices, say $V(G) = \{1, 2, \cdots, n\}$ and a set of edges, say $E(G) = \{e_1, e_2, \cdots, e_m\}$, where $e_i = \{i_1, i_2, \cdots, i_l\}$, $i_j \in [n]$, $j = 1, 2, \cdots, l$. If $|e_i| = k$ for any $i = 1, 2, \cdots, m$, then G is called a k-uniform hypergraph. In particular, the 2-uniform hypergraphs are exactly the ordinary graphs. For a vertex $v \in V(G)$ the degree $d_G(v)$ is defined as $d_G(v) = |\{e_i : v \in e_i \in E(G)\}|$. Vertex with degree one is called pendent vertex in this paper.

An order k dimension n tensor $\mathcal{T} = (\mathcal{T}_{i_1 i_2 \cdots i_k}) \in \mathbb{C}^{n \times n \times \cdots \times n}$ is a multidimensional array with n^k entries, where $i_j \in [n]$ for each $j = 1, 2, \cdots, k$.

To study the properties of uniform hypergraphs by algebraic methods, adjacency matrix, signless Laplacian matrix and Laplacian matrix of graph are generalized to adjacency tenor, signless Laplacian tensor and Laplacian tensor of uniform hypergraph.

Definition 1. (See [6,11].) Let $G = (V(G), E(G))$ be a k-uniform hypergraph on n vertices. The adjacency tensor of G is defined as the k-th order n-dimensional tensor $\mathcal{A}(G)$ whose $(i_1 \cdots i_k)$-entry is:

$$(\mathcal{A}(G))_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1, i_2, \cdots, i_k\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(G)$ be a k-th order n-dimensional diagonal tensor, with its diagonal entry $\mathcal{D}_{ii \cdots i}$ the degree of vertex i, for all $i \in [n]$. Then $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ is the signless Laplacian tensor of the uniform hypergraph G, and $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ is the Laplacian tensor of the uniform hypergraph G.

The following general product of tensors, was defined in [12] by Shao, which is a generalization of the matrix case.

Definition 2. Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times n_3 \times \cdots \times n_{k+1}}$ be order $m \geq 2$ and $k \geq 1$ tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ with entries:

$$\mathcal{C}_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\cdots,i_m\in[n_2]} \mathcal{A}_{ii_2\cdots i_m}\mathcal{B}_{i_2\alpha_1}\cdots\mathcal{B}_{i_m\alpha_{m-1}}, \tag{1}$$

where $i \in [n], \alpha_1, \cdots, \alpha_{m-1} \in [n_3] \times \cdots \times [n_{k+1}]$.

Let $\mathcal{T}$ be an order k dimension n tensor, let $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension n. Then by (1) $\mathcal{T}x$ is a vector in $\mathbb{C}^n$ whose ith component is as the following

$$(\mathcal{T}x)_i = \sum_{i_2,\cdots,i_k=1}^{n} \mathcal{T}_{ii_2\cdots i_k}x_{i_2}\cdots x_{i_k}. \tag{2}$$

Let $x^{[k]} = (x_1^k, \cdots, x_n^k)^T$. Then (see [2,11]) a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $\mathcal{T}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$\mathcal{T}x = \lambda x^{[k-1]}, \tag{3}$$

and in this case, x is called an eigenvector of $\mathcal{T}$ corresponding to eigenvalue λ.

An eigenvalue of $\mathcal{T}$ is called an H-eigenvalue, if there exists a real eigenvector corresponding to it (see [11]). In this paper we will focus on the largest H-eigenvalue of tensor $\mathcal{T}$, denoted by $\lambda(\mathcal{T})$.

The concept of power hypergraphs was introduced in [4].

Definition 3. Let $G = (V(G), E(G))$ be an ordinary graph. For every $k \geq 2$, the kth power of G, $G^k := (V(G^k), E(G^k))$ is defined as the k-uniform hypergraph with the edge set

$$E(G^k) := \{e \cup \{i_{e,1}, \cdots, i_{e,k-2}\} \mid e \in E(G)\}$$

and the vertex set

$$V(G^k) := V(G) \cup (\cup_{e\in E(G)}\{i_{e,1}, \cdots, i_{e,k-2}\}).$$

For convenience here $G^2 = G$. In [4], the k-th power of path, and cycle is called loose path, and loose cycle, respectively. Denote by S_m the star with m edges. The k-th power of star is called sunflower in [4], or hyperstar in [10].

Definition 4. (See [4].) Let $G = (V, E)$ be a k-uniform hypergraph. If there is a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \cdots \cup V_d$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_d| = k - 1$, and $E = \{V_0 \cup V_i \mid i \in [d]\}$, then G is called a sunflower. The degree d of the vertex in V_0, which is called the heart, is the size of the sunflower. Denote by S_d^k the k-uniform sunflower of size d.

For even k, when G is a cycle or star, Hu, Qi and Shao proved that $\{\lambda(\mathcal{Q}(G^k))\}$ is a strictly decreasing sequence in [4]. They believed that it is true for any graph G, see Conjecture 4.1 of [4]. This phenomena was also observed in [16] when G is a path (namely, G^k is a loose path).

Conjecture 5. *(See [4].) Let G be an ordinary graph, $k = 2r$ be even and G^k be the k-th power hypergraph of G. Then $\{\lambda(\mathcal{L}(G^k)) = \lambda(\mathcal{Q}(G^k))\}$ is a strictly decreasing sequence.*

For $t \geq 1$ let tS_1^k be t disjoint union of S_1^k. We may point out that when $G = tS_1^2$, we have $\lambda(\mathcal{L}(G^k)) = \lambda(\mathcal{Q}(G^k)) = 2$ for any $k \geq 2$. Namely, in this case Conjecture 5 is false.

Let H be a uniform hypergraph and $H \neq tS_1^k$, and H' be obtained from H by inserting a new pendent vertex in each edge. In Section 3, we will prove that $\lambda(\mathcal{Q}(H')) < \lambda(\mathcal{Q}(H))$ in Theorem 10. So for any ordinary graph $G \neq tS_1$ (maximum degree $\Delta \geq 2$), $\{\lambda(\mathcal{Q}(G^k))\}$ is a strictly decreasing sequence, which affirm Conjecture 5 for $\lambda(\mathcal{Q}(G^k))$. We also determine the value $\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k))$ in Theorem 20.

For an ordinary graph G, the definition for k-th power hypergraph G^k has been generalized by Khan and Fan in [7].

Definition 6. Let $G = (V, E)$ be an ordinary graph. For any $k \geq 3$ and $1 \leq s \leq k/2$. For each $v \in V$ (and $e \in E$), let V_v (and V_e) be a new vertex set with s (and $k - 2s$) elements such that all these new sets are pairwise disjoint. Then the generalized power of G, denoted by $G^{k,s}$, is defined as the k-uniform hypergraph with the vertex set

$$V(G^{k,s}) = \left(\bigcup_{v\in V} V_v\right) \bigcup \left(\bigcup_{e\in E} V_e\right)$$

and edge set

$$E(G^{k,s}) = \{V_u \cup V_v \cup V_e : e = \{u, v\} \in E\}.$$

If $s = 1$, then $G^{k,s}$ is exactly the kth power hypergraph G^k. The eigenvalues properties about $\mathcal{A}(G^k)$ was discussed in [17], and $\mathcal{A}(G^{k,k/2})$ was discussed in [7] and [8]. In Section 4, we will prove some eigenvalues properties about $\mathcal{A}(G^{k,s})$, which generalize some known results.

2. Auxiliary results for nonnegative tensors and H-spectrum of hypergraphs

In [3], the weak irreducibility of nonnegative tensors was defined. It was proved in [3] and [15] that a k-uniform hypergraph G is connected if and only if its adjacency tensor $\mathcal{A}(G)$ (and so $\mathcal{Q}(G)$) is weakly irreducible.

Let $\mathcal{T}$ be a kth-order n-dimensional nonnegative tensor. The spectral radius of $\mathcal{T}$ is defined as (see [10,7,8])

$$\rho(\mathcal{T}) = \max\{|\mu| : \mu \text{ is an eigenvalue of } \mathcal{T}\}.$$

Part of Perron–Frobenius theorem for nonnegative tensors is stated in the following for reference.

Theorem 7. *(See [2,14].) Let $\mathcal{T}$ be a nonnegative tensor. Then we have the following statements.*

(1) *$\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$ with a nonnegative eigenvector x corresponding to it.*
(2) *If $\mathcal{T}$ is weakly irreducible, then x is positive, and for any eigenvalue μ with nonnegative eigenvector, $\mu = \rho(\mathcal{T})$ holding.*
(3) *The nonnegative eigenvector x corresponding to $\rho(\mathcal{T})$ is unique up to a constant multiple.*

In virtue of (1) of Theorem 7, the largest H-eigenvalue of $\mathcal{A}(G)$ (or $\mathcal{Q}(G)$) is exactly the spectral radius of $\mathcal{A}(G)$ (or $\mathcal{Q}(G)$). For weakly irreducible nonnegative $\mathcal{T}$ of order k, the positive eigenvector x with $||x||_k = 1$ corresponding to $\rho(\mathcal{T})$ (i.e., the largest H-eigenvalue) is called the principal eigenvector of $\mathcal{T}$ in this paper.

Lemma 8. *(See [7].) Suppose that $\mathcal{T}$ is a weakly irreducible nonnegative tensor of order k. If there exists a nonnegative vector y such that $\mathcal{T}y \leq \mu y^{[k-1]}$ and $(\mathcal{T}y)_i < \mu y_i^{k-1}$ holding for some i, then $\lambda(\mathcal{T}) < \mu$.*

The H-spectrum of a real tensor $\mathcal{T}$, denoted by $Hspec(\mathcal{T})$, is defined to be the set of distinct H-eigenvalues of $\mathcal{T}$ [13]. Namely,

$$Hspec(\mathcal{T}) = \{\mu \mid \mu \text{ is an H-eigenvalue of } \mathcal{T}\}.$$

Lemma 9. *Let $G = \cup_{i=1}^{t} G_i$, where G_i is a connected uniform hypergraph. Then*

$$Hspec(\mathcal{L}(G)) = \bigcup_{i=1}^{t} Hspec(\mathcal{L}(G_i)), \tag{4}$$

and so

$$\lambda(\mathcal{L}(G)) = \max_{1 \leq i \leq t}\{\lambda(\mathcal{L}(G_i))\}.$$

Proof. Without loss of the generality, we may assume that the vertices of G are ordered in such a way that if $i < j$, then any vertex in G_i precedes any vertex in G_j. Let x be a column vector of dimension $|V(G)|$. We write x in the following block form

$$x = (x_1^T, x_2^T, \cdots, x_t^T)^T, \tag{5}$$

where x_i is a column vector corresponding to the vertices of G_i. Then it is not difficult to see that

$$\mathcal{L}(G)x = ((\mathcal{L}(G_1)x_1)^T, (\mathcal{L}(G_2)x_2)^T, \cdots, (\mathcal{L}(G_t)x_t)^T)^T. \tag{6}$$

Now we prove Eq. (4). If $\lambda \in Hspec(\mathcal{L}(G))$ with a real eigenvector x as in (5), where $x_j \neq 0$. Then by $\mathcal{L}(G)x = \lambda x^{[k-1]}$ and Eq. (6) we have

$$\mathcal{L}(G_j)x_j = \lambda x_j^{[k-1]}.$$

Thus

$$\lambda \in Hspec(\mathcal{L}(G_j)) \subseteq \bigcup_{i=1}^{t} Hspec(\mathcal{L}(G_i)). \tag{7}$$

On the other hand, if $\lambda \in \bigcup_{i=1}^{t} Hspec(\mathcal{L}(G_i))$, say, $\lambda \in Hspec(\mathcal{L}(G_j))$ for some $1 \leq j \leq t$ with a real eigenvector x_j. Take $x_i = 0$ for all $i \neq j$ and take x as in (5). Then by Eq. (6) we can verify that $\mathcal{L}(G)x = \lambda x^{[k-1]}$, thus $\lambda \in Hspec(\mathcal{L}(G))$. Combining these two aspects, we obtain (4).

Particularly, we have

$$\lambda(\mathcal{L}(G)) = \max_{1 \leq i \leq t} \{\lambda(\mathcal{L}(G_i))\}.$$

Similarly, we may prove that these results are also true for $\mathcal{Q}(G)$ and $\mathcal{A}(G)$. □

3. Largest H-eigenvalue of signless Laplacian tensor of G^k

In this section we will prove that for any graph G with maximum degree $\Delta \geq 2$, $\{\lambda(\mathcal{Q}(G^k))\}$ is a strictly decreasing sequence. First we will prove a more general result by constructing a new vector and using Lemma 8.

Theorem 10. *Let H be a k-uniform ($k \geq 2$) hypergraph, and H' be obtained from H by inserting a new pendent vertex in each edge. Then $\lambda(\mathcal{Q}(H')) \leq \lambda(\mathcal{Q}(H))$, equality holding if and only if $H = tS_1^k$ for some t.*

Proof. If $H = tS_1^k$ for some t, then $\lambda(\mathcal{Q}(H')) = \lambda(\mathcal{Q}(H)) = 2$. We suppose that $H \neq tS_1^k$ for any t.

Denote by

$$E(H) = \{e_1, e_2, \cdots, e_m\},$$

and let v_i be the new pendent vertex inserted in e_i for any $1 \leq i \leq m$, i.e.,

$$V(H') = V(H) \cup \{v_1, v_2, \cdots, v_m\}.$$

(1) First suppose that H is a connected, so $\mathcal{Q}(H)$ is weakly irreducible. Let x be the principal eigenvector to $\lambda(\mathcal{Q}(H))$, namely,

$$\mathcal{Q}(H)x = \lambda(\mathcal{Q}(H))x^{[k-1]}.$$

Now we construct a new vector y (of dimension $|V(H')|$) from x by adding m components. If $w \in V(H)$, set $y_w = x_w$; if $w = v_i$, i.e., w is a new pendent vertex inserted in e_i, set $y_w = \min\{x_u \mid u \in e_i\}$.

Now we will show $\mathcal{Q}(H')y \le \lambda(\mathcal{Q}(H))y^{[k]}$.

For any vertex $w \in V(H)$ we have

$$\begin{aligned}
(\mathcal{Q}(H')y)_w &= d_{H'}(w)y_w^k + \sum_{\{w,v_i,t_2,\cdots,t_k\}\in E(H')} y_{v_i}y_{t_2}\cdots y_{t_k} \\
&\le d_H(w)x_w^k + \sum_{\{w,v_i,t_2,\cdots,t_k\}\in E(H')} x_w x_{t_2}\cdots x_{t_k} \qquad (8)\\
&= x_w\Big[d_H(w)x_w^{k-1} + \sum_{\{w,t_2,\cdots,t_k\}\in E(H)} x_{t_2}\cdots x_{t_k}\Big] \\
&= x_w(\mathcal{Q}(H)x)_w \\
&= x_w\lambda(\mathcal{Q}(H))x_w^{k-1} \\
&= \lambda(\mathcal{Q}(H))x_w^k \\
&= \lambda(\mathcal{Q}(H))y_w^k.
\end{aligned}$$

Ineq. (8) is due to the fact $y_{v_i} \le x_w$. Furthermore, if $x_w > y_{v_i}$, namely, $x_w > \min\{x_u \mid u \in e_i\}$ for some edge e_i containing w, Ineq. (8) becomes strict.

For $w = v_i$ for some $1 \le i \le m$, i.e., w is a new pendent vertex inserted in e_i, we suppose $e_i = \{w_1, w_2, \cdots, w_k\}$ and $x_{w_1} = \min\{x_{w_1}, x_{w_2}, \ldots, x_{w_k}\}$. Then we have

$$\begin{aligned}
(\mathcal{Q}(H')y)_w &= y_w^k + y_{w_1}y_{w_2}\cdots y_{w_k} \\
&= x_{w_1}^k + x_{w_1}x_{w_2}\cdots x_{w_k} \\
&\le x_{w_1}\Big[d_H(w_1)x_{w_1}^{k-1} + x_{w_2}\cdots x_{w_k}\Big] \qquad (9)\\
&\le x_{w_1}(\mathcal{Q}(H)x)_{w_1} \\
&= \lambda(\mathcal{Q}(H))x_{w_1}^k \\
&= \lambda(\mathcal{Q}(H))y_w^k.
\end{aligned}$$

Ineq. (9) is due to the fact $d_H(w_1) \ge 1$. Furthermore, if $d_H(w_1) > 1$, Ineq. (9) becomes strict. So we have proved $\mathcal{Q}(H')y \le \lambda(\mathcal{Q}(H))y^{[k]}$.

If there exists $\{w, w'\} \subseteq e_i$ for some $e_i \in E(H)$ such that $x_w > x_{w'}$, then $x_w > \min\{x_u \mid u \in e_i\}$, and then Ineq. (8) becomes strict. Now we suppose all the vertices in each edge have the equal corresponding component in x. Furthermore, since H is connected, we know that all the components of x are equal, thus H is regular. We suppose H is a d-regular. Since $H \neq S_1^k$, we have $d \geq 2$. Now for some edge e_i, set $x_{w_1} = \min\{x_u \mid u \in e_i\}$, then $d_H(w_1) = d > 1$, and Ineq. (9) becomes a strict one.

Thus we have proved that $(\mathcal{Q}(H')y)_i < \lambda(\mathcal{Q}(H))y_i^{[k]}$ for some i, therefore, $\lambda(\mathcal{Q}(H')) < \lambda(\mathcal{Q}(H))$ by Lemma 8.

(2) If $H = H_1 \cup H_2 \cup \cdots \cup H_t$, where H_i is a connected component of H for $1 \leq i \leq t$ and $t \geq 2$, then

$$H' = H_1' \cup H_2' \cup \cdots \cup H_t'.$$

Since $H \neq tS_1^k$, we may suppose that $H_i \neq S_1^k$ for $1 \leq i \leq t' \leq t$, and $t' \geq 1$. Then for $1 \leq i \leq t'$ we have $\lambda(\mathcal{Q}(H_i')) < \lambda(\mathcal{Q}(H_i))$ by the above arguments. It is obvious that for $1 \leq i \leq t'$ we have $\lambda(\mathcal{Q}(H_i)) > \lambda(\mathcal{Q}(S_1^k))$ and $\lambda(\mathcal{Q}(H_i')) > \lambda(\mathcal{Q}(S_1^{k+1}))$. By Lemma 9 we have

$$\lambda(\mathcal{Q}(H')) = \max_{1 \leq i \leq t'}\{\lambda(\mathcal{Q}(H_i'))\} < \max_{1 \leq i \leq t'}\{\lambda(\mathcal{Q}(H_i))\} = \lambda(\mathcal{Q}(H)).$$

The proof is completed. □

By Theorem 10, we have the following result for $\{\lambda(\mathcal{Q}(G^k))\}$, which affirm Conjecture 5 for $\mathcal{Q}(G^k)$.

Theorem 11. *Let G be an ordinary graph with maximum $\Delta \geq 2$. When $k \geq 2$ we have $\lambda(\mathcal{Q}(G^{k+1})) < \lambda(\mathcal{Q}(G^k))$.*

The notion of odd-bipartite even-uniform hypergraphs was introduced in [5].

Definition 12. (See [5].) Let k be even and $G = (V, E)$ be a k-uniform hypergraph. It is called odd-bipartite if either it is trivial (i.e., $E = \emptyset$) or there is a disjoint partition of the vertex set V as $V = V_1 \cup V_2$ such that $V_1, V_2 \neq \emptyset$ and every edge in E intersects V_1 with exactly an odd number of vertices.

For even uniform odd-bipartite hypergraph, the following result was proved in [6] (see Theorem 5.8 of [6]), or in [13] (see Theorem 2.2 of [13]).

Lemma 13. *(See [6,13].) Let G be a connected even uniform odd-bipartite hypergraph. Then $\lambda(\mathcal{L}(G)) = \lambda(\mathcal{Q}(G))$.*

In fact Lemma 13 is also true for general even uniform odd-bipartite hypergraph G, see Lemma 14.

Lemma 14. *Let G be an even uniform odd-bipartite hypergraph. Then $\lambda(\mathcal{L}(G)) = \lambda(\mathcal{Q}(G))$.*

Proof. By Lemma 13, we only need to consider the case that G is not connected. Set $G = \cup_{i=1}^{t} G_i$, where G_i is a connected component of G for $1 \le i \le t$ and $t \ge 2$. Since G is even uniform and odd-bipartite, each G_i is connected even uniform and odd-bipartite. Thus by Lemma 13 we have $\lambda(\mathcal{L}(G_i)) = \lambda(\mathcal{Q}(G_i))$ $(i = 1, \cdots, t)$, and moreover, by Lemma 9 we have

$$\lambda(\mathcal{L}(G)) = \max_{1\le i\le t}\{\lambda(\mathcal{L}(G_i))\} = \max_{1\le i\le t}\{\lambda(\mathcal{Q}(G_i))\} = \lambda(\mathcal{Q}(G)).$$

The proof is completed. □

Remark 15. We can mention here that the condition $\lambda(\mathcal{L}(G)) = \lambda(\mathcal{Q}(G))$ does not imply that G is an even uniform odd-bipartite hypergraph. In fact take $G = G_1 \cup G_2$, where G_1 is not odd-bipartite and G_2 is a sunflower with size Δ satisfying

$$\Delta > \lambda(\mathcal{Q}(G_1)) \ge \lambda(\mathcal{L}(G_1)).$$

From Proposition 3.2 of [6] and Lemma 13, we know that

$$\lambda(\mathcal{L}(G_2)) = \lambda(\mathcal{Q}(G_2)) > \Delta.$$

Then G is not odd-bipartite (since G_1 is not), but we have

$$\lambda(\mathcal{L}(G)) = \max_{1\le i\le 2}\{\lambda(\mathcal{L}(G_i))\} = \lambda(\mathcal{L}(G_2)) = \lambda(\mathcal{Q}(G_2)) = \max_{1\le i\le 2}\{\lambda(\mathcal{Q}(G_i))\} = \lambda(\mathcal{Q}(G)).$$

Obviously, when k is even and $k \ge 4$, the k-th power hypergraph G^k is odd-bipartite. Then Lemma 14 and Theorem 11 imply that Conjecture 5 is true for $\mathcal{L}(G^k)$.

Theorem 16. *Let $G = (V, E)$ be an ordinary graph with maximum $\Delta \ge 2$, $k = 2r$ be even and G^k be the k-power hypergraph of G. Then $\{\lambda(\mathcal{L}(G^k)) = \lambda(\mathcal{Q}(G^k))\}$ is a strictly decreasing sequence.*

The value $\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k))$ was determined for a regular graph G by Zhou et al. in [17].

Lemma 17. *(See [17].) For any d-regular graph G with $d \ge 2$, we have $\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k)) = d$.*

Now we will prove that when k goes to infinity, $\lambda(\mathcal{Q}(S_d^k))$ converges to d, the maximum degree of S_d^k.

Lemma 18. *When $k \ge 2, d \ge 2$ we have $\lim_{k\to\infty} \lambda(\mathcal{Q}(S_d^k)) = d$.*

Proof. Write $\lambda_k = \lambda(\mathcal{Q}(S_d^k))$ for short. Let x be the principal eigenvector of $\mathcal{Q}(S_d^k)$ corresponding to λ_k. Let a be the component of x corresponding to the heart of S_d^k.

By the symmetry of the pendent vertices in the same edge, we see that they have the same component in x. Furthermore, by the uniqueness of x (see (3) of Theorem 7), we see that all the pendent vertices in S_d^k have the same component in x, say b. Then λ_k satisfies the following equations

$$\begin{cases} \lambda_k a^{k-1} &= da^{k-1} + db^{k-1}, \\ \lambda_k b^{k-1} &= b^{k-1} + ab^{k-2}. \end{cases}$$

By eliminations of a and b, we obtain

$$(\lambda_k - d)(\lambda_k - 1)^{k-1} - d = 0.$$

Set

$$f_k(\lambda) = (\lambda - d)(\lambda - 1)^{k-1} - d,$$

then λ_k is the largest real root of the equation $f_k(\lambda) = 0$.

Particularly,

$$(\lambda_{k+1} - d)(\lambda_{k+1} - 1)^{k} = d.$$

Since $f_k(d) = -d < 0$, and $\lim_{\lambda \to +\infty} f_k(\lambda) = +\infty$, we have $\lambda_k > d$.

So $\{\lambda_k\}$ is a strictly decreasing sequence and $\lambda_k > d$ when $d \geq 2$. Thus $\lim_{k\to\infty} \lambda_k$ exists. From

$$(\lambda_k - d)(\lambda_k - 1)^{k-1} = d,$$

we have

$$\lim_{k\to\infty} \lambda_k - d = \frac{d}{\lim_{k\to\infty}(\lambda_k - 1)^{k-1}} = 0,$$

thus $\lim_{k\to\infty} \lambda_k = d$, i.e., $\lim_{k\to\infty} \lambda(\mathcal{Q}(S_d^k)) = d$ holds. $\square$

To determine the value $\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k))$ for a general graph G, we first cite a result just for a graph due to Köing.

Lemma 19. *(See [9].) Every graph G of maximum degree Δ is an induced subgraph of some Δ-regular graph.*

Theorem 20. *Let G be an ordinary graph with maximum degree $\Delta \geq 2$. Then we have $\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k)) = \Delta$.*

Proof. We have proved that $\{\lambda(\mathcal{Q}(G^k))\}$ is a strictly decreasing sequence by Theorem 11. Obviously, $\lambda(\mathcal{Q}(G^k)) > 0$ for any $k \geq 2$. Thus $\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k))$ exists.

It is known that if F' is a sub-hypergraph of F, then $\lambda(\mathcal{Q}(F')) \leq \lambda(\mathcal{Q}(F))$ (see Proposition 4.5 in [6]).

Since G has maximum degree Δ, G contains star S_Δ as a sub-graph, and then G^k contains the sunflower S_Δ^k as a sub-hypergraph. Thus $\lambda(\mathcal{Q}(G^k)) \geq \lambda(\mathcal{Q}(S_\Delta^k))$. Furthermore Lemma 18 implies that

$$\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k)) \geq \lim_{k\to\infty} \lambda(\mathcal{Q}(S_\Delta^k)) = \Delta.$$

On the other hand, by Lemma 19, G is a subgraph of some Δ-regular graph F. Then G^k is a sub-hypergraph of F^k. Thus $\lambda(\mathcal{Q}(G^k)) \leq \lambda(\mathcal{Q}(F^k))$. Furthermore by Lemma 17 we have

$$\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k)) \leq \lim_{k\to\infty} \lambda(\mathcal{Q}(F^k)) = \Delta.$$

So we obtain

$$\lim_{k\to\infty} \lambda(\mathcal{Q}(G^k)) = \Delta. \qquad \square$$

4. Largest H-eigenvalue of adjacency tensor of $G^{k,s}$

In [17], it was proved that $\lambda(\mathcal{A}(G^k)) = \lambda(A(G))^{\frac{2}{k}}$; in [7] it was proved that $\lambda(\mathcal{A}(G^{k,k/2})) = \lambda(A(G))$. By using the technique provided in [17], we will prove a general case.

Theorem 21. *If $\mu > 0$ is an eigenvalue of the adjacency matrix $A(G)$ of graph G having a nonnegative eigenvector, then $\mu^{\frac{2s}{k}}$ is an eigenvalue of the adjacency tensor $\mathcal{A}(G^{k,s})$ having a nonnegative eigenvector. Moreover $\lambda(\mathcal{A}(G^{k,s})) = \lambda(A(G))^{\frac{2s}{k}}$.*

Proof. Suppose that x is a nonnegative eigenvector of the eigenvalue $\mu > 0$ of $A(G)$. As shown in Definition 6, for any edge $e = \{u, v\}$ denote by $V_u \cup V_v \cup V_e$ the corresponding edge of $G^{k,s}$.

Now we construct a new nonnegative vector y (of dimension $|V(G^{k,s})|$) from x by adding components. Set

$$y_w = \begin{cases} (x_v)^{\frac{2}{k}} & \text{if } w \in V_v \text{ for some } v, \\ (\mu^{-1} x_u x_v)^{\frac{1}{k}} & \text{if } w \in V_e \text{ for some edge } e = \{u, v\}. \end{cases}$$

Now we will show $\mathcal{A}(G^{k,s})y = \mu^{\frac{2s}{k}} y^{[k-1]}$ holding.

For any $w \in V_v$ for some v, by the formula

$$\sum_{\{u,v\}\in E(G)} x_u = \mu x_v,$$

we have

$$\begin{aligned}(\mathcal{A}(G^{k,s})y)_w &= \sum_{\{u,v\}\in E(G)} (x_v)^{\frac{2(s-1)}{k}}(x_u)^{\frac{2s}{k}}(\mu^{-1}x_u x_v)^{\frac{k-2s}{k}} \\ &= \mu^{\frac{2s}{k}-1}(x_v)^{\frac{k-2}{k}} \sum_{\{u,v\}\in E(G)} x_u \\ &= \mu^{\frac{2s}{k}}(x_v)^{\frac{2(k-1)}{k}} \\ &= \mu^{\frac{2s}{k}} y_w^{k-1}. \end{aligned} \tag{10}$$

For $w \in V_e$ for any edge $e = \{u, v\}$, we have

$$\begin{aligned}(\mathcal{A}(G^{k,s})y)_w &= (x_u)^{\frac{2s}{k}}(x_v)^{\frac{2s}{k}}(\mu^{-1}x_u x_v)^{\frac{k-2s-1}{k}} \\ &= \mu^{\frac{2s}{k}}(\mu^{-1}x_u x_v)^{\frac{k-1}{k}} \\ &= \mu^{\frac{2s}{k}} y_w^{k-1}. \end{aligned}$$

Hence $\mu^{\frac{2s}{k}}$ is an eigenvalue of $\mathcal{A}(G^{k,s})$ with a nonnegative eigenvector y.

If G is connected and $\mu = \lambda(A(G))$, then we may choose x as a positive eigenvector of $\lambda(A(G))$ by Perron–Frobenius theorem for irreducible nonnegative matrix. In this case y is a positive eigenvector of the eigenvalue $\lambda(A(G))^{\frac{2s}{k}}$ of the tensor $\mathcal{A}(G^{k,s})$. In virtue of (2) of Theorem 7 (or see Lemma 15 of [17]), we have

$$\lambda(\mathcal{A}(G^{k,s})) = \lambda(A(G))^{\frac{2s}{k}}.$$

If $G = G_1 \cup G_2 \cup \cdots \cup G_t$, where G_i is a connected component of G for $1 \le i \le t$ and $t \ge 2$, then by Lemma 9

$$\lambda(\mathcal{A}(G^{k,s})) = \max_{1\le i\le t}\{\lambda(\mathcal{A}(G_i^{k,s}))\} = \max_{1\le i\le t}\{\lambda(A(G_i))^{\frac{2s}{k}}\} = \lambda(A(G))^{\frac{2s}{k}}.$$

The proof is completed. □

Take $s = 1$, or $s = k/2$ for even k, and noting that $G^k = G^{k,1}$, we have Corollary 22 and Corollary 23.

Corollary 22. *(See [17].) If $\mu > 0$ is an eigenvalue of the adjacency matrix $A(G)$ of graph G having a nonnegative eigenvector, then $\mu^{\frac{2}{k}}$ is an eigenvalue of the adjacency tensor $\mathcal{A}(G^k)$ of the hypergraph G^k having a nonnegative eigenvector. Moreover $\lambda(\mathcal{A}(G^k)) = \lambda(A(G))^{\frac{2}{k}}$.*

Corollary 23. *(See [7].) Let G be a connected ordinary graph, and let $x > 0$ be vector defined on $V(G)$. Let $y > 0$ be a vector defined on $V(G^{k,k/2})$ such that $y_u = x_v^{\frac{2}{k}}$ for each vertex $u \in V_v$. Then x is an eigenvector of $A(G)$ corresponding to $\lambda(A(G))$ if and only if y is an eigenvector of $\mathcal{A}(G^{k,k/2})$ corresponding to $\lambda(\mathcal{A}(G^{k,k/2}))$. Hence $\lambda(\mathcal{A}(G^{k,k/2})) = \lambda(A(G))$.*

Corollary 24. *Let G be an ordinary graph with maximum $\Delta \geq 2$, $s \geq 1$ be a fixed integer. Then $\lambda(\mathcal{A}(G^{k+1,s})) < \lambda(\mathcal{A}(G^{k,s}))$, and $\lim_{k\to\infty} \lambda(\mathcal{A}(G^{k,s})) = 1$.*

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