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Isotropic polynomial invariants of the Hall tensor*

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Abstract
The Hall tensor emerges from the study of the Hall effect, an important magnetic effect observed in electric conductors and semiconductors. The Hall tensor is third order and three dimensional, whose first two indices are skew-symmetric. In this paper, we investigate the isotropic polynomial invariants of the Hall tensor by connecting it with a second order tensor via the third order Levi-Civita tensor. We propose a minimal isotropic integrity basis with 10 invariants for the Hall tensor. Furthermore, we prove that this minimal integrity basis is also an irreducible isotropic function basis of the Hall tensor.

Key words isotropic polynomial invariants, irreducibility, function basis, integrity basis, the Hall tensor

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Nomenclature

- $\mathbf{I}$ second order identify tensor
- $\boldsymbol{\varepsilon}$ permutation tensor(i.e., Levi-Civita tensor) with components ε_{ijk} in three dimensions
- $\mathbf{Q}$ orthogonal tensor with components q_{ij}
- $\langle \mathbf{Q} \rangle \mathbf{A}$ a second order tensor $\mathbf{A}$ under an orthogonal transformation
- $\mathcal{K}$ a Hall tensor with components k_{ijk}
- $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ orthonormal base in three dimensions
- $\operatorname{tr} \mathbf{A}$, $\mathbf{A}^\top$ trace and transpose, respectively, of a second order tensor $\mathbf{A}$
- $\det \mathbf{A}$ determinant of a second order tensor $\mathbf{A}$
- $\boldsymbol{\varepsilon}\mathcal{K}$ second order tensor with components $\varepsilon_{kli}k_{klj}$ in three dimensions
- $\boldsymbol{\varepsilon}\mathbf{A}$ third order tensor with components $\varepsilon_{ijl}a_{lk}$ in three dimensions
- $\mathbb{R}^n$ the real number field with dimension n
- $\mathbb{R}^{m\times n}$ m−by−n matrix on the real number field

1 Introduction

Tensor function representation theory is an essential topic in continuum mechanics, which focuses on the tensor invariants under coordinate transformations. Since tensor invariants

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often reveal more intrinsic information of materials than tensor components, the complete and irreducible representation for invariant tensor functions plays a key role in modeling nonlinear constitutive equations in both theoretical and applied physics. Such representation prescribes the number and the type of scalar variables required in the constitutive equations. These representations are efficient in the process of describing the physical behavior of anisotropic materials, because the invariant conditions are dominant and no other simple methods are able to determine such information. There are plenty of research works on this topic from the last century[1–11]. For instance, the minimal integrity basis and irreducible function basis of a second order tensor in three dimensions were well studied by Wang[7], Smith[4], Boehler[1], Pennisi and Trovato[3], and Zheng[8]. Furthermore, Boehler et al.[2] investigate polynomial invariants for the elasticity tensor in 1993. Some very recent works are devoted to minimal integrity bases and irreducible function bases for third order and fourth order tensors[12,13].

The tensor function representation theory is also closely related to the classical invariant theory in algebraic geometry[14–17]. One of the most famous approaches for computing the complete invariant basis was first introduced by Hilbert[15]. In 2017, Olive, Kolev, and Auffray[18] studied the minimal integrity basis with 297 invariants of the fourth order elasticity tensors successfully via the approaches from the algebraic geometry.

The Hall effect is an important magnetic effect observed in electric conductors and semiconductors [19]. It was discovered in 1879 by and named after Edwin Hall [20]. When an electric current density $\mathbf{J}$ is flowing through a plate and the plate is simultaneously immersed in a magnetic field $\mathbf{H}$ with a component transverse to the current, the electric field strength $\mathbf{E}$ is proportional to current density and magnetic field strength

$$E_i = k_{ijk} J_j H_k,$$

where the third order tensor $\mathcal{K}$ with components k_{ijk} is called the Hall tensor [19]. We note that the representation of the Hall tensor under any orthonormal basis satisfies $k_{ijk} = -k_{jik}$ for all $i, j, k = 1, 2, 3$, since the Onsager relation for transport processes with time reversal is valid. The Hall tensor is essential for describing the electromagnetic induction. Therefore, it is significant to investigate the minimal integrity basis and irreducible function basis of the Hall tensor. In physics, there are other tensors which are third order three dimensional tensors whose first two indices are antisymmetric. For example, the tensors in the Faraday effect [19].

This paper is devoted to the invariants of the Hall tensor and organized as follows. We first briefly review some basic definitions and the relationship between the integrity basis and the function basis of a tensor in Section 2. Then we build a connection between the invariants of a Hall tensor and that of a second order tensor, which is important for the subsequent contents. Furthermore, we propose a minimal isotropic integrity basis with 10 isotropic invariants of the Hall tensor in Section 3. In Section 4, we proved that the minimal integrity basis with 10 invariants of the Hall tensor is also its irreducible function basis. Finally, we draw some concluding remarks and raise one further question in Section 5.

2 Preliminaries

Denote $\mathcal{A}$ as an mth order tensor represented by $a_{i_1 i_2 \cdots i_m}$ under some orthonormal coordinate. A scalar-valued tensor function $f(\mathcal{A})$ is called an isotropic invariant of $\mathcal{A}$ if it is invariant under any orthogonal transformations, including rotations and reflections, i.e.,

$$f(\langle \mathbf{Q} \rangle \mathcal{A}) = f(\mathcal{A}),$$

or equivalently expressed by

$$f(q_{i_1 j_1} q_{i_2 j_2} \dots q_{i_m j_m} a_{j_1 j_2 \dots j_m}) = f(a_{i_1 i_2 \dots i_m}),$$

where $\mathbf{Q}$ is an orthogonal tensor ($\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$) with components q_{ij}. If $f(\mathcal{A})$ is only invariant under rotations, i.e., $f(\langle\mathbf{Q}\rangle\mathcal{A}) = f(\mathcal{A})$ for any orthogonal tensor $\mathbf{Q}$ with $\det\mathbf{Q} = 1$, then it is called a hemitropic invariant of tensor $\mathcal{A}$. Furthermore, if $f(\mathcal{A})$ is a polynomial, then it is called a polynomial invariant of $\mathcal{A}$. In the subsequence, invariants always stand for polynomial invariants unless specific remarks are made there.

For any second order tensor, the hemitropic invariants and the isotropic invariants are equivalent, since it keeps unaltered under the central inversion $-\mathbf{I}$ [9]. Nevertheless, any isotropic polynomial invariant of a third order tensor has to be the summation of several even order degree polynomials.

We then briefly review the definitions and properties of (minimal) integrity bases and (irreducible) function bases of a tensor.

Definition 1 (Integrity basis). *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_r\}$ be a set of isotropic (or hemitropic, respectively) invariants of a tensor $\mathcal{A}$.*

(1) *Ψ is said to be* **polynomial irreducible** *if none of $\psi_1, \psi_2, \ldots, \psi_r$ can be expressed by a polynomial of the remainders;*

(2) *Ψ is called an isotropic (or hemitropic, respectively)* **integrity basis** *if any isotropic (or hemitropic, respectively) invariant of $\mathcal{A}$ is expressible by a polynomial of $\psi_1, \psi_2, \ldots, \psi_r$;*

(3) *Ψ is called an isotropic (or hemitropic, respectively)* **minimal integrity basis** *if it is polynomial irreducible and an isotropic (or hemitropic, respectively) integrity basis.*

Definition 2 (Function basis). *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_r\}$ be a set of isotropic (or hemitropic, respectively) invariants of a tensor $\mathcal{A}$.*

(1) *An invariant in Ψ is said to be* **functionally irreducible** *if it cannot be expressed by a single-valued function of the remainders, Ψ is said to be functionally irreducible if all of $\psi_1, \psi_2, \ldots, \psi_r$ are functionally irreducible;*

(2) *Ψ is called an isotropic (or hemitropic, respectively)* **function basis** *if any isotropic (or hemitropic, respectively) invariant of $\mathcal{A}$ is expressible by a function of $\psi_1, \psi_2, \ldots, \psi_r$;*

(3) *Ψ is called an isotropic (or hemitropic, respectively)* **irreducible function basis** *if it is functionally irreducible and is an isotropic (or hemitropic, respectively) function basis.*

It is straightforward to verify by definitions that an isotropic (or hemitropic, respectively) integrity basis is an isotropic (or hemitropic, respectively) function basis, but the converse is incorrect in general. Thus, the number of invariants in an isotropic (or hemitropic, respectively) irreducible function basis is no greater than the number of invariants in an isotropic (or hemitropic, respectively) minimal integrity basis. For instance, the number of irreducible function basis of a third order traceless symmetric tensor is 11 which is less than 13, the number of invariants in its minimal integrity basis[13].

Particularly, it has been proved that the number of invariants of each degree in an isotropic (or hemitropic, respectively) minimal integrity basis is always fixed [18]. Nevertheless, it is still unclear whether the number of invariants of an irreducible function basis is fixed.

3 Minimal integrity basis of the Hall tensor

Let $\mathcal{K}$ be a Hall tensor represented by k_{ijk} under an orthonormal basis $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$. Define a second order tensor $\mathbf{A}$ accordingly, with components a_{ij} under this orthonormal basis, by the tensor product operation

$$\mathbf{A} := \tfrac{1}{2}\boldsymbol{\varepsilon}\mathcal{K},$$

or equivalently

$$(a_{ij})\mathbf{e}_i \otimes \mathbf{e}_j = (\tfrac{1}{2}\varepsilon_{kli}k_{klj})\mathbf{e}_i \otimes \mathbf{e}_j,$$

where $\boldsymbol{\varepsilon}$ is the third order Levi-Civita tensor. Conversely, the Hall tensor can also be expressed with this second order tensor by

$$\mathcal{K} := \boldsymbol{\varepsilon}\mathbf{A},$$

or equivalently

$$(k_{ijk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = (\varepsilon_{ijl}a_{lk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k.$$

There are nine independent components in a Hall tensor $\mathcal{K}$ due to the anti-symmetry of the first two indices of the components in a Hall tensor. Without loss of generality, denote the nine independent components of the Hall tensor $\mathcal{K}$ as:

$$k_{121}, k_{122}, k_{123}, k_{131}, k_{132}, k_{133}, k_{231}, k_{232}, k_{233}.$$

Then under a right-handed coordinate, the representation of the associated second order tensor can be written into a matrix form mathematically:

$$\begin{pmatrix} k_{231} & k_{232} & k_{233} \\ -k_{131} & -k_{132} & -k_{133} \\ k_{121} & k_{122} & k_{123} \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

The following theorem reveals the relationship between the invariants of the Hall tensor and the ones of its associated second order tensor.

Theorem 1. *Let $\mathcal{K}$ be a Hall tensor with components k_{ijk}. We use $\mathbf{A}(\mathcal{K})$ to denote the associated second order tensor of $\mathcal{K}$.*

(1) *Any isotropic invariant of $\mathcal{K}$ is an isotropic invariant of $\mathbf{A}(\mathcal{K})$;*
(2) *Any isotropic invariant of $\mathbf{A}(\mathcal{K})$ with even degree is an isotropic invariant of $\mathcal{K}$, and any isotropic invariant of $\mathbf{A}(\mathcal{K})$ with odd degree is an hemitropic invariant of $\mathcal{K}$*

Proof. (1) An isotropic invariant $f(\mathcal{K})$ of the Hall tensor $\mathcal{K}$ is also a polynomial function of its associated second order tensor $\mathbf{A}(\mathcal{K})$, denoted by $g(\mathbf{A}) := f(\boldsymbol{\varepsilon}\mathbf{A})$. Now we need to show that $g(\mathbf{A})$ is an isotropic invariant of $\mathbf{A}(\mathcal{K})$. Let $\mathbf{Q}$ be any orthogonal tensor. According to the definition of isotropic invariants, we have

$$f(\langle\mathbf{Q}\rangle\mathcal{K}) = f(\mathcal{K}) = f(\boldsymbol{\varepsilon}\mathbf{A}) = g(\mathbf{A}).$$

Make use of the equality $\langle\mathbf{Q}\rangle\boldsymbol{\varepsilon} = (\det\mathbf{Q})\boldsymbol{\varepsilon}$, then

$$f(\langle\mathbf{Q}\rangle\mathcal{K}) = f(\langle\mathbf{Q}\rangle(\boldsymbol{\varepsilon}\mathbf{A})) = f(\langle\mathbf{Q}\rangle\boldsymbol{\varepsilon}\langle\mathbf{Q}\rangle\mathbf{A}) = f((\det\mathbf{Q})\boldsymbol{\varepsilon}\langle\mathbf{Q}\rangle\mathbf{A}) = g((\det\mathbf{Q})\langle\mathbf{Q}\rangle\mathbf{A}).$$

Since an isotropic invariant of a third order tensor must be an even function, thus

$$g((\det\mathbf{Q})\langle\mathbf{Q}\rangle\mathbf{A}) = g(\langle\mathbf{Q}\rangle\mathbf{A}).$$

Hence, $g(\langle\mathbf{Q}\rangle\mathbf{A}) = g(\mathbf{A})$, i.e., $g(\mathbf{A})$ is an isotropic invariant of $\mathbf{A}$.

(2) Denote an invariant of $\mathbf{A}(\mathcal{K})$ as $g(\mathbf{A})$. It is also a polynomial of the Hall tensor $\mathcal{K}$, denoted by $f(\mathcal{K}) := g(\frac{1}{2}\boldsymbol{\varepsilon}\mathcal{K})$. For any orthogonal tensor $\mathbf{Q}$, since $g(\mathbf{A})$ is an invariant, we know

$$g(\langle\mathbf{Q}\rangle\mathbf{A}) = g(\mathbf{A}) = g(\boldsymbol{\varepsilon}\mathcal{K}) = f(\mathcal{K}).$$

Recall that $\langle\mathbf{Q}\rangle\boldsymbol{\varepsilon} = (\det\mathbf{Q})\boldsymbol{\varepsilon}$. Then

$$g(\langle\mathbf{Q}\rangle\mathbf{A}) = g(\langle\mathbf{Q}\rangle(\boldsymbol{\varepsilon}\mathcal{K})) = g(\langle\mathbf{Q}\rangle\boldsymbol{\varepsilon}\langle\mathbf{Q}\rangle\mathcal{K}) = g((\det\mathbf{Q})\boldsymbol{\varepsilon}\langle\mathbf{Q}\rangle\mathcal{K} = f((\det\mathbf{Q})\langle\mathbf{Q}\rangle\mathcal{K}).$$

Hence, when $g(\mathbf{A})$ is an invariant of even degree, we have $f(\langle\mathbf{Q}\rangle\mathcal{K}) = f(\mathcal{K})$ for any orthogonal tensor $\mathbf{Q}$. That is, $f(\mathcal{K})$ is an isotropic invariant of the Hall tensor $\mathcal{K}$.

When $g(\mathbf{A})$ is an invariant of odd degree, only for orthogonal tensor $\mathbf{Q}$ satisfying $\det\mathbf{Q} = 1$, it holds that $f(\langle\mathbf{Q}\rangle\mathcal{K}) = f(\mathcal{K})$, which means that $f(\mathcal{K})$ is an hemitropic invariant of the Hall tensor $\mathcal{K}$. The proof is completed. □

Hence, we can construct an integrity basis for a Hall tensor from the integrity basis of its associated second order tensor. For the associated second order tensor $\mathbf{A}(\mathcal{K})$, we split it into $\mathbf{A}(\mathcal{K}) = \mathbf{T} + \mathbf{W}$, where $\mathbf{T}$ is symmetric with components $t_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $\mathbf{W}$ is skew-symmetric with components $w_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$. It is well known that 7 invariants $\operatorname{tr}\mathbf{T}$, $\operatorname{tr}\mathbf{T}^2$, $\operatorname{tr}\mathbf{T}^3$, $\operatorname{tr}\mathbf{W}^2$, $\operatorname{tr}\mathbf{T}\mathbf{W}^2$, $\operatorname{tr}\mathbf{T}^2\mathbf{W}^2$, $\operatorname{tr}\mathbf{T}^2\mathbf{W}^2\mathbf{T}\mathbf{W}$ form a minimal integrity basis of $\mathbf{A}(\mathcal{K})$ and also an irreducible function basis as well. We denote the invariants of $\mathbf{A}(\mathcal{K})$ as follows:

$$\begin{array}{llll} I_1 := \operatorname{tr}\mathbf{T}, & I_2 := \operatorname{tr}\mathbf{T}^2, & J_2 := \operatorname{tr}\mathbf{W}^2, & I_3 := \operatorname{tr}\mathbf{T}^3, \\ J_3 := \operatorname{tr}\mathbf{T}\mathbf{W}^2, & I_4 := \operatorname{tr}\mathbf{T}^2\mathbf{W}^2, & I_6 := \operatorname{tr}\mathbf{T}^2\mathbf{W}^2\mathbf{T}\mathbf{W}. & \end{array}$$

The following theorem shows the way to obtain a minimal integrity basis of $\mathcal{K}$ from this particular minimal integrity basis of $\mathbf{A}(\mathcal{K})$.

Theorem 2. *Let $\mathcal{K}$ be a Hall tensor with components k_{ijk}, and $\mathbf{A}(\mathcal{K})$ be its associated second order tensor with components a_{ij}. Denote $K_2 := I_1^2$, $J_4 := I_1 I_3$, $K_4 := I_1 J_3$, $J_6 := I_3^2$, $K_6 := J_3^2$, $L_6 := I_3 J_3$. Then the invariant set*

$$\{I_2, J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\} \tag{1}$$

is a minimal integrity basis of $\mathcal{K}$.

Proof. By Theorem 1, any isotropic invariant of $\mathcal{K}$ is also an invariant of $\mathbf{A}(\mathcal{K})$, thus can be expressed by a polynomial $p(I_1, I_2, J_2, I_3, J_3, I_4, I_6)$. Moreover, any isotropic invariant of an even order tensor consists of several even degree monomials. Each even degree monomial containing I_1, I_3, J_3 should be a polynomial of $I_1^2, I_1 I_3, I_1 J_3, I_3^2, I_3 J_3, J_3^2$. Therefore, the isotropic invariant $p(I_1, I_2, J_2, I_3, J_3, I_4, I_6)$ can also be written into a polynomial of the invariants in (1). That is, (1) is an integrity basis of $\mathcal{K}$.

Next, we need to verify the polynomial irreducibility of this integrity basis. A natural observation is that these isotropic invariants are homogenous polynomials of the 9 independent components in the Hall tensor $\mathcal{K}$. A similar approach as the method proposed by Chen et al.[12] is employed in this part.

(i) There are exactly 3 degree-2 isotropic invariants I_2, J_2, K_2 in this integrity basis. Take I_2 for example. If it is not polynomial irreducible with the other 9 invariants in this basis, then it has to be a linear combination of the other 2 degree-2 invariants J_2, K_2. Therefore, if I_2, J_2, K_2 are polynomial irreducible, then the unique triple of (c_1, c_2, c_3) such that

$$c_1 I_2 + c_2 J_2 + c_3 K_2 = 0 \tag{2}$$

is $c_1 = c_2 = c_3 = 0$. Note that (2) holds for an arbitrary Hall tensor. Thus when we generate n points $y_1, \cdots, y_n \in \mathbb{R}^9$, where $\mathbb{R}^9$ is the real number field with dimension 9, c_1, c_2, c_3 must be the solution to the linear system of equations

$$\begin{pmatrix} I_2(y_1) & J_2(y_1) & K_2(y_1) \\ I_2(y_2) & J_2(y_2) & K_2(y_2) \\ \vdots & \vdots & \vdots \\ I_2(y_n) & J_2(y_n) & K_2(y_n) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3}$$

The coefficient matrix of System (3) is denoted by $M2$, and denote $r(M2)$ as the rank of the coefficient matrix $M2$. Then $r(M2)$ shows the number of polynomial irreducible invariants in these three isotropic invariants. Take $n = 3$ and

- $y_1 = (-2, 3, 5, 0, -5, -4, -5, 2, -2)$,
- $y_2 = (-3, 0, 1, 1, 2, -4, 3, 0, 3)$,

- $y_3 = (-2, 0, -1, 2, 1, -3, 5, 2, 3)$.

By numerical calculations, we can determine that $r(M2) = 3$. Hence, the only solution for System (3) is $c_1 = c_2 = c_3 = 0$, which implies that these three invariants of degree 2 are polynomial irreducible.

(ii) For the invariants of degree 4, we need to consider the following linear equation

$$c_1(I_2)^2 + c_2(J_2)^2 + c_3(K_2)^2 + c_4 I_2 J_2 + c_5 I_2 K_2 + c_6 J_2 K_2 + c_7 I_4 + c_8 J_4 + c_9 K_4 = 0, \quad (4)$$

where $c_1, \cdots, c_9$ are scalars. If the unique $(c_1, c_2, \ldots, c_9)$ such that (4) holds for any Hall tensor is $(0, 0, \ldots, 0)$, then all the 3 degree-4 invariants I_4, J_4, K_4 are polynomial irreducible. We generate n points $y_1, \cdots, y_n \in \mathbb{R}^9$ and consider the following linear system:

$$\begin{pmatrix} I_2^2(y_1) & \cdots & I_4(y_1) & J_4(y_1) & K_4(y_1) \\ I_2^2(y_2) & \cdots & I_4(y_2) & J_4(y_2) & K_4(y_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_2^2(y_n) & \cdots & I_4(y_n) & J_4(y_n) & K_4(y_n) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (5)$$

The coefficient matrix of System (5) is denoted by $M4$. Take $n = 9$ and

- $y_1 = (4, 1, -3, 1, -4, -2, -1, 0, -5)$,
- $y_2 = (1, 5, 4, 0, -1, -5, -3, 5, -2)$,
- $y_3 = (-4, 4, -4, 1, -5, -2, 2, 3, 4)$,
- $y_4 = (-4, -5, 5, 5, -2, 3, 5, -1, 2)$,
- $y_5 = (0, 4, 3, 3, 1, -2, 3, 5, -4)$,
- $y_6 = (5, -3, 3, 3, -4, -2, 3, 5, -5)$,
- $y_7 = (-3, -2, 2, 4, -4, 1, 4, 2, 0)$,
- $y_8 = (-5, -3, 4, -1, 1, -2, -2, -3, 0)$,
- $y_9 = (0, -2, -2, 1, 5, 3, 4, 0, 0)$.

We can verify that the rank of $M4$ is $r(M4) = 9$, which implies that these three degree-4 invariants cannot be polynomial represented by other invariants of degree-4 and degree-2.

(iii) Similarly, in the case of degree 6, the verification linear equation is

$$c_1(I_2)^3 + c_2(J_2)^3 + \cdots + c_{19} K_2 K_4 + c_{20} I_6 + \cdots + c_{23} L_6 = 0. \quad (6)$$

Thus we generate n points $y_1, \cdots, y_n \in \mathbb{R}^9$. Consider a linear system similar with system (5). Its coefficient matrix is denoted as $M6$, and its rank is denoted by $r(M6)$. Take $n = 23$ and

- $y_1 = (3, -5, 1, 4, 2, 3, 3, 1, -3)$,
- $y_2 = (-5, -1, 2, -5, -2, 3, 3, 4, -1)$,
- $y_3 = (-4, 2, 1, -3, -2, -2, 1, 4, -1)$,
- $y_4 = (-2, 0, 3, 2, -2, -2, -5, 5, 2)$,
- $y_5 = (-2, -5, -5, -4, 3, -5, -3, 2, -3)$,
- $y_6 = (5, -4, 1, 3, -4, 1, -1, 4, 0)$,
- $y_7 = (-3, 3, 5, -3, -3, 1, 2, -2, -3)$,
- $y_8 = (2, 2, -5, 4, 4, -1, -5, 4, -5)$,
- $y_9 = (-2, -1, 2, 3, -2, -1, -2, -2, 5)$,
- $y_{10} = (-4, -3, -4, -2, -5, -5, 5, -2, -3)$,
- $y_{11} = (3, 2, -2, -5, 5, -3, 0, -2, -5)$,

- $y_{12} = (4, -4, -1, 4, -4, 0, 1, 3, -1)$,
- $y_{13} = (3, 0, -5, 0, 2, -5, -5, 4, 1)$,
- $y_{14} = (-4, 5, -5, 2, -1, -4, -5, -2, -5)$,
- $y_{15} = (2, -5, -5, 5, 0, 2, 2, 3, 4)$,
- $y_{16} = (1, 4, 4, -1, -5, -3, 4, -5, 1)$,
- $y_{17} = (-2, 5, -5, 1, -2, 1, 0, -5, 4)$,
- $y_{18} = (0, -4, -5, 0, -5, -2, -2, -2, 2)$,
- $y_{19} = (1, 2, 1, -1, 3, -4, -5, 4, 5)$,
- $y_{20} = (3, -3, 1, -3, -5, 3, 5, 1, 1)$,
- $y_{21} = (0, -1, 3, 0, -3, 5, 3, 0, 3)$,
- $y_{22} = (1, -5, -4, -1, 0, -1, -5, -5, 2)$,
- $y_{23} = (-4, -2, 3, 4, 5, -3, 4, 3, 3)$.

Then $r(M6) = 23$, which implies that these four invariants with degree 6 are polynomial irreducible in the integrity basis.

Therefore, we have shown that (1) is a minimal integrity basis of $\mathcal{K}$. □

In the above discussion, we fix the inducing initial, i.e., a particular minimal integrity basis of the second order tensor. Nevertheless, the minimal integrity basis is generally not unique. We can also start from another minimal integrity basis of the second order tensor, denoted by

$$\{\tilde{I}_1, \tilde{I}_2, \tilde{J}_2, \tilde{I}_3, \tilde{J}_3, \tilde{I}_4, \tilde{I}_6\}.$$

Construct another integrity basis $\{\tilde{I}_2, \tilde{J}_2, \tilde{K}_2, \tilde{I}_4, \tilde{J}_4, \tilde{K}_4, \tilde{I}_6, \tilde{J}_6, \tilde{K}_6, \tilde{L}_6\}$ of the Hall tensor in the same way, where

$$\tilde{K}_2 := \tilde{I}_1^2,\ \tilde{J}_4 := \tilde{I}_1\tilde{I}_3,\ \tilde{K}_4 := \tilde{I}_1\tilde{J}_3,\ \tilde{J}_6 := \tilde{I}_3^2,\ \tilde{K}_6 := \tilde{J}_3^2,\ \tilde{L}_6 := \tilde{I}_3\tilde{J}_3.$$

Since this integrity basis has already got the same number of invariants as the minimal integrity basis (1), it must also be a minimal integrity basis. Therefore, we have the following corollary.

Corollary 1. *Let $\mathcal{K}$ be a Hall tensor with components k_{ijk}, and $\mathbf{A}(\mathcal{K})$ be its associated second order tensor with components a_{ij}. Let $\{\tilde{I}_1, \tilde{I}_2, \tilde{J}_2, \tilde{I}_3, \tilde{J}_3, \tilde{I}_4, \tilde{I}_6\}$ be any minimal integrity basis of the second order tensor $\mathbf{A}(\mathcal{K})$. Denote $\Psi := \{\tilde{I}_2, \tilde{J}_2, \tilde{K}_2, \tilde{I}_4, \tilde{J}_4, \tilde{K}_4, \tilde{I}_6, \tilde{J}_6, \tilde{K}_6, \tilde{L}_6\}$ with $\tilde{K}_2 := \tilde{I}_1^2$, $\tilde{J}_4 := \tilde{I}_1\tilde{I}_3$, $\tilde{K}_4 := \tilde{I}_1\tilde{J}_3$, $\tilde{J}_6 := \tilde{I}_3^2$, $\tilde{K}_6 := \tilde{J}_3^2$, $\tilde{L}_6 := \tilde{I}_3\tilde{J}_3$. Then Ψ is a minimal integrity basis of the Hall tensor $\mathcal{K}$.*

4 Irreducible function basis

Since a minimal integrity basis for a tensor is also a function basis, the number of invariants in an irreducible function basis consisting of polynomial invariants is no more than that of a minimal integrity basis. Moreover, the number of invariants in a minimal integrity basis of a tensor can be very big. For example, the number of minimal integrity basis of an elasticity tensor is 297[18]. However, from an experimental point of view, it will be easier to detect all the values of the invariants in an irreducible function basis of a tensor. Hence, it is meaningful to study the irreducible function basis of a tensor. For a symmetric third order tensor, one of its irreducible function base contains 11 invariants, while its minimal integrity basis contains 13 invariants[13].

In this section, we shall show that the minimal integrity basis given in Section 3 is also an irreducible function basis of the Hall tensor $\mathcal{K}$. According to the method proposed by Pennisi and Trovato[3] in 1987, to show a given function basis of a tensor is functionally irreducible, for each invariant in this basis, we need to find two different sets of independent variables in the tensor, denoted by V and $V^{'}$, such that this invariant takes different values in V and $V^{'}$ while all the remainders are the same in V and $V^{'}$. The following theorem is proved in this spirit.

Theorem 3 *The set* $\{I_2, J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$ *is an irreducible function basis of the Hall tensor.*

Proof. It can be verified by definitions that an integrity basis of a tensor is a function basis of the tensor. We have proved in Section 3 that these ten invariants form a minimal integrity basis of the Hall tensor. Thus this basis is also a function basis.

Denote $V = \{k_{121}, k_{122}, k_{123}, k_{131}, k_{132}, k_{133}, k_{231}, k_{232}, k_{233}\}$ and $V^{'} = \{k^{'}_{121}, k^{'}_{122}, k^{'}_{123}, k^{'}_{131}, k^{'}_{132}, k^{'}_{133}, k^{'}_{231}, k^{'}_{232}, k^{'}_{233}\}$ as two different sets of independent variables of the Hall tensor $\mathcal{K}$. Then we shall find ten pairs of $\{V, V^{'}\}$ to show that all the ten isotropic invariants in (1) is functionally irreducible.

(1) For I_2, in V, let $k_{121} = k_{122} = k_{123} = k_{132} = k_{133} = k_{231} = k_{233} = 0$, $k_{131} = -1, k_{232} = 1$. Then in $V^{'}$, let $k^{'}_{121} = k^{'}_{122} = k^{'}_{123} = k^{'}_{132} = k^{'}_{133} = k^{'}_{231} = k^{'}_{233} = 0$, $k^{'}_{131} = -2, k^{'}_{232} = 2$. We have that $I_2 = 2$ and $I^{'}_2 = 8$, while other invariants: $\{J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$, and $\{J^{'}_2, K^{'}_2, I^{'}_4, J^{'}_4, K^{'}_4, I^{'}_6, J^{'}_6, K^{'}_6, L^{'}_6\}$ are all equal to 0. This means that I_2 is functionally irreducible in the function basis (1).

(2) For J_2, in V, let $k_{121} = k_{122} = k_{123} = k_{132} = k_{133} = k_{231} = k_{233} = 0$, $k_{131} = 1, k_{232} = 1$. Then in $V^{'}$, let all the variables be 0. We have that $J_2 = 2$ and $J^{'}_2 = 0$, while other invariants: $\{I_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$, and $\{I^{'}_2, K^{'}_2, I^{'}_4, J^{'}_4, K^{'}_4, I^{'}_6, J^{'}_6, K^{'}_6, L^{'}_6\}$ are all equal to 0. This means that J_2 is functionally irreducible in the function basis (1).

(3) For K_2, in V, let $k_{121} = k_{122} = k_{131} = k_{133} = k_{232} = k_{233} = 0$, and $k_{123} = -\sqrt{\frac{2+\sqrt[3]{4}}{2}}, k_{132} = 0, k_{231} = \sqrt{\frac{2+\sqrt[3]{4}}{2}}$. In $V^{'}$, let $k^{'}_{121} = k^{'}_{122} = k^{'}_{131} = k^{'}_{133} = k^{'}_{232} = k^{'}_{233} = 0$, and $k^{'}_{123} = 1, k^{'}_{132} = \sqrt[3]{2}, k^{'}_{231} = 1$. We have $K_2 = 0$. It is not equal to $K^{'}_2 = (2 - \sqrt[3]{2})^2$, while other invariants: $I_2 = I^{'}_2 = 2 + \sqrt[3]{4}$, and $J_2 = J^{'}_2 = I_4 = I^{'}_4 = J_4 = J^{'}_4 = K_4 = K^{'}_4 = I_6 = I^{'}_6 = J_6 = J^{'}_6 = K_6 = K^{'}_6 = L_6 = L^{'}_6 = 0$. This means that K_2 is functionally irreducible.

(4) For I_4, in V, let $k_{121} = -2$, $k_{122} = 0$, $k_{123} = 1$, $k_{131} = 1$, $k_{132} = 1$, $k_{133} = 0$, $k_{231} = 0$, $k_{232} = 1$, and $k_{233} = 2$. In $V^{'}$, let $k^{'}_{121} = -\sqrt{3}$, $k^{'}_{122} = -\sqrt{2}$, $k^{'}_{123} = 1$, $k^{'}_{131} = 0$, $k^{'}_{132} = 1$, $k^{'}_{133} = -\sqrt{2}$, $k^{'}_{231} = 0$, $k^{'}_{232} = 0$, and $k^{'}_{233} = \sqrt{3}$. We have $I_4 = 5$. It is not equal to $I^{'}_4 = 7$, while $I_2 = I^{'}_2 = 2$, $J_2 = J^{'}_2 = 10$, $K_6 = K^{'}_6 = 9$, and others are all equal to 0. This means that I_4 is functionally irreducible.

(5) For J_4, assume that $s = 4 + \sqrt{14}$, and $t = 4 - \sqrt{14}$. In V, let $k_{121} = k_{122} = k_{131} = k_{133} = k_{232} = k_{233} = 0$, and

$$k_{123} = 1, k_{132} = 1, k_{231} = \frac{\sqrt[3]{2t}}{2} + \sqrt[3]{\frac{s}{4}}.$$

In $V^{'}$, let $k^{'}_{121} = k^{'}_{122} = k^{'}_{131} = k^{'}_{133} = k^{'}_{232} = k^{'}_{233} = 0$, and

$$k^{'}_{123} = 2 - \frac{\sqrt[3]{2t}}{2} - \frac{\sqrt[3]{2s}}{2} - \frac{\sqrt[6]{2}}{2}\sqrt{2\sqrt[3]{4} + 8\sqrt[3]{t} + \sqrt[3]{2t^2} + 8\sqrt[3]{s} + \sqrt[3]{2s^2}},$$
$$k^{'}_{132} = -1 + \frac{\sqrt[3]{2t}}{4} + \frac{\sqrt[3]{2s}}{4} - \frac{\sqrt[6]{2}}{4}\sqrt{2\sqrt[3]{4} + 8\sqrt[3]{t} + \sqrt[3]{2t^2} + 8\sqrt[3]{s} + \sqrt[3]{2s^2}},$$

and $k^{'}_{231} = 0$.

We have $J_4 = -J_4^{'} = \frac{3}{8}\left(-4+\sqrt[3]{2t}+\sqrt[3]{2s}\right)\left(\sqrt[3]{2t}+\sqrt[3]{2s}\right)$. Meanwhile,

$$\begin{array}{lll} I_2 & = I_2^{'} & = 2+\frac{\left(\sqrt[3]{2t}+\sqrt[3]{2s}\right)^2}{4}, \\ K_2 & = K_2^{'} & = \frac{1}{4}\left(-4+\sqrt[3]{2t}+\sqrt[3]{2s}\right)^2, \\ J_6 & = J_6^{'} & = \frac{9}{16}\left(\sqrt[3]{2t}+\sqrt[3]{2s}\right)^2, \end{array}$$

and others are all equal to 0. This shows that J_4 is functionally irreducible.

(6) For K_4, in V, let

$$\begin{array}{lll} k_{121} = -\frac{1}{2}\sqrt{\frac{-12+6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, & k_{122} = \frac{1}{2}, & k_{123} = -1, \\ k_{131} = 0, & k_{132} = -\frac{\sqrt[3]{9}}{2}, & k_{133} = \frac{1}{2}, \\ k_{231} = -\frac{1}{2}, & k_{232} = 0, & k_{233} = \frac{1}{2}\sqrt{\frac{-12+6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}. \end{array}$$

In $V^{'}$, let

$$\begin{array}{lll} k_{121}^{'} = 0, & k_{122}^{'} = -\frac{\sqrt[6]{3}}{2}\sqrt{\frac{9+5\sqrt[3]{3}-6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, & k_{123}^{'} = 1, \\ k_{131}^{'} = -\frac{1}{2}\sqrt{\frac{22-12\sqrt[3]{3}-2\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, & k_{132}^{'} = \frac{\sqrt[3]{9}}{2}, & k_{133}^{'} = -\frac{\sqrt[6]{3}}{2}\sqrt{\frac{9+5\sqrt[3]{3}-6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, \\ k_{231}^{'} = \frac{1}{2}, & k_{232}^{'} = -\frac{1}{2}\sqrt{\frac{22-12\sqrt[3]{3}-2\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, & k_{233}^{'} = 0. \end{array}$$

We have

$$K_4 = -K_4^{'} = \frac{6+21\sqrt[3]{3}-17\sqrt[3]{9}}{-256+48\sqrt[3]{3}+48\sqrt[3]{9}}.$$

Meanwhile,

$$\begin{array}{lll} I_2 & = I_2^{'} & = \frac{5+3\sqrt[3]{3}}{4}, \\ J_2 & = J_2^{'} & = \frac{4-3\sqrt[3]{3}+3\sqrt[3]{9}}{-32+6\sqrt[3]{3}+6\sqrt[3]{9}}, \\ K_2 & = K_2^{'} & = \frac{(-3+\sqrt[3]{9})^2}{4}, \\ I_4 & = I_4^{'} & = \frac{-23+36\sqrt[3]{3}+9\sqrt[3]{9}}{-256+48\sqrt[3]{3}+48\sqrt[3]{9}}, \\ K_6 & = K_6^{'} & = \frac{-47+78\sqrt[3]{3}-31\sqrt[3]{9}}{64(-16+3\sqrt[3]{3}+3\sqrt[3]{9})^2}, \end{array}$$

and others are all equal to 0. This shows that K_4 is functionally irreducible.

(7) For I_6, in V, let $k_{121} = -1$, $k_{122} = -1$, $k_{123} = 1$, $k_{131} = 1$, $k_{132} = 1$, $k_{133} = -1$, $k_{231} = 0$, $k_{232} = 1$, and $k_{233} = 1$.
In $V^{'}$, let $k_{121}^{'} = -1$, $k_{122}^{'} = -1$, $k_{123}^{'} = 1$, $k_{131}^{'} = -1$, $k_{132}^{'} = 1$, $k_{133}^{'} = -1$, $k_{231}^{'} = 0$, $k_{232}^{'} = -1$, and $k_{233}^{'} = 1$.
We have $I_6 = -I_6^{'} = 2$. Meanwhile, $I_2 = I_2^{'} = 2$, $J_2 = J_2^{'} = 6$, $I_4 = I_4^{'} = 4$, and others are all equal to 0. This means that I_6 is functionally irreducible.

(8) For J_6, in V, let $k_{121} = k_{122} = k_{131} = k_{133} = k_{232} = k_{233} = 0$, and $k_{123} = -\sqrt{3}\sqrt[3]{2}, k_{132} = 0, k_{231} = \sqrt{3}\sqrt[3]{2}$.
In $V^{'}$, let $k_{121}^{'} = k_{122}^{'} = k_{131}^{'} = k_{133}^{'} = k_{232}^{'} = k_{233}^{'} = 0$, and $k_{123}^{'} = \sqrt[3]{2}, k_{132}^{'} = 2\sqrt[3]{2}, k_{231}^{'} = \sqrt[3]{2}$.
We have $J_6 = 0 \neq J_6^{'} = 144$. Meanwhile, $I_2 = I_2^{'} = 6\sqrt[3]{4}$, and others are all equal to 0. This means that J_6 is functionally irreducible.

(9) For K_6, in V, let $k_{121} = \frac{1}{2}$, $k_{122} = 1$, $k_{123} = 0$, $k_{131} = \frac{3}{2}$, $k_{132} = 0$, $k_{133} = 1$, $k_{231} = 0$, $k_{232} = \frac{3}{2}$, and $k_{233} = \frac{1}{2}$.

In V', let $k'_{121} = -\frac{1}{2}$, $k'_{122} = \frac{1}{2}$, $k'_{123} = 0$, $k'_{131} = \sqrt{3}$, $k'_{132} = 0$, $k'_{133} = \frac{1}{2}$, $k'_{231} = 0$, $k'_{232} = \sqrt{3}$, and $k'_{233} = -\frac{1}{2}$.

We have $K_6 = \frac{9}{4} \neq K'_6 = \frac{3}{4}$. Meanwhile, $I_2 = I'_2 = \frac{1}{2}$, $J_2 = J'_2 = -\frac{13}{2}$, $I_4 = I'_4 = -\frac{13}{16}$, and others are all equal to 0. This means that K_6 is functionally irreducible.

(10) For L_6, in V, let $k_{121} = -1$, $k_{122} = \frac{1}{2}$, $k_{123} = -1$, $k_{131} = 0$, $k_{132} = 2$, $k_{133} = \frac{1}{2}$, $k_{231} = 3$, $k_{232} = 0$, and $k_{233} = 1$.

In V', let $k'_{121} = 0$, $k'_{122} = -\frac{1}{2}\sqrt{\frac{5}{2}}$, $k'_{123} = 1$, $k'_{131} = -\frac{1}{2}\sqrt{\frac{5}{2}}$, $k'_{132} = -2$, $k'_{133} = -\frac{1}{2}\sqrt{\frac{5}{2}}$, $k'_{231} = -3$, $k'_{232} = -\frac{1}{2}\sqrt{\frac{5}{2}}$, and $k'_{233} = 0$.

We have $L_6 = -L'_6 = -\frac{45}{2}$. Meanwhile, $I_2 = I'_2 = 14$, $J_2 = J'_2 = -\frac{5}{2}$, $I_4 = I'_4 = -\frac{45}{4}$, $J_6 = J'_6 = 324$, $K_6 = K'_6 = \frac{25}{16}$, and others are all equal to 0. This means that L_6 is functionally irreducible.

Therefore, this particular minimal integrity basis $\{I_2, J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$ is also an irreducible function basis of the Hall tensor $\mathcal{K}$. □

In the above proof, the examples V and V' in the cases (1), (2), (4) and (7) are based on related sets in Pennisi and Trovato[3], while the examples V and V' in the case (5) are suggested by Dr. Yannan Chen.

5 Conclusions and A Further Question

In this paper, we investigate isotropic invariants of the Hall tensor. For this purpose, we connect the invariants of the Hall tensor $\mathcal{K}$ with the ones of its associated second order tensor $\mathbf{A}(\mathcal{K})$. $\mathbf{A}(\mathcal{K})$ can be split into a second order symmetric tensor $\mathbf{T}$ and a second order skew-symmetric tensor $\mathbf{W}$. Then $\{I_1 := \operatorname{tr}\mathbf{T}, I_2 := \operatorname{tr}\mathbf{T}^2, J_2 := \operatorname{tr}\mathbf{W}^2, I_3 := \operatorname{tr}\mathbf{T}^3, J_3 := \operatorname{tr}\mathbf{T}\mathbf{W}^2, I_4 := \operatorname{tr}\mathbf{T}^2\mathbf{W}^2, I_6 := \operatorname{tr}\mathbf{T}^2\mathbf{W}^2\mathbf{T}\mathbf{W}\}$ is the minimal integrity basis of $\mathbf{A}(\mathcal{K})$ as in the previous sections. It is also an irreducible function basis of $\mathbf{A}(\mathcal{K})$. We prove in this paper the following statements:

(i) $\{I_1^2, I_2, J_2, I_4, I_1I_3, I_1J_3, I_6, I_3^2, J_3^2, I_3J_3\}$ is an isotropic minimal integrity basis of the Hall tensor $\mathcal{K}$.

(ii) $\{I_1^2, I_2, J_2, I_4, I_1I_3, I_1J_3, I_6, I_3^2, J_3^2, I_3J_3\}$ is also an isotropic irreducible function basis of the Hall tensor $\mathcal{K}$ as well.

Apart from this particular selection, we can also begin with any minimal integrity basis of the second order tensor and use the same approach to construct an invariant basis of the Hall tensor. We prove in the paper that such basis of the Hall tensor is a minimal integrity basis.

A further question is whether there exists an irreducible function basis consisting of less than ten polynomial invariants.

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