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E-mail: ijsi@iscas.ac.cn
http://www.ijsi.org
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Geometric Measure of Quantum Entanglement for Multipartite Mixed States

Shenglong Hu[1], Liqun Qi[2], Yisheng Song[3], and Guofeng Zhang[4]

[1] (Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, China)
email: timhu@tju.edu.cn

[2] (Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong, China)
email: maqilq@polyu.edu.hk

[3] (School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China)
email: songyisheng1@gmail.com

[4] (Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong, China)
email: Guofeng.Zhang@polyu.edu.hk

Abstract The geometric measure of quantum entanglement of a pure state, defined by its distance to the set of pure separable states, is extended to multipartite mixed states. We characterize the nearest disentangled mixed state to a given mixed state with respect to the geometric measure by means of a system of equations. The entanglement eigenvalue for a mixed state is introduced. And we show that, for a given mixed state, its nearest disentangled mixed state is associated with its entanglement eigenvalue. Two numerical examples are used to demonstrate the effectiveness of the proposed method.
Key words: quantum entanglement; geometric measure; optimization

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1 Introduction

The quantum entanglement problem is regarded as a central problem in quantum physics and quantum information[8,9,14], and the geometric measure is one of the most important measures of quantum entanglement[1,9,13,15]. Geometric measure was first proposed by Shimony[13] and generalized to multipartite systems by Wei and Goldbart[15]. Since then it has become one of the widely used entanglement measures for multiparticle cases[2–5,10].

For a given pure state, the geometric measure is based on the geometric distance between the given pure state and the set of separable pure states, namely,

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Corresponding author: Guofeng Zhang, Email: Guofeng.Zhang@polyu.edu.hk
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product state. Based on this definition, the associated quantum eigenvalue problem is derived to characterize the nearest separable pure state in terms of the geometric measure[5,10,15]. This characterization is significant due to the fact that the eigenvalues are always real numbers and the largest one corresponds to the maximal overlap of the given pure state and the separable pure states.

Based on the convex roof construction, geometric measure is extended to the context of multipartite mixed states[15]. Although the extension is standard, analoguous characterizations for disentangled mixed states are not clear[6,15]. Instead of the convex roof extension, we propose in this paper a natural extension of the geometric measure from pure states to mixed states. Most interestingly, a characterization for the nearest disentangled mixed state studied in Refs. [6,15] still holds. We show that there is a system of equations associated to the proposed geometric measure for mixed states. The entanglement eigenvalue for a mixed state is introduced and it is proven to be an indicator of the proposed geometric measure. Moreover, the disentangled mixed state corresponding to the entanglement eigenvalue is shown to be the nearest disentangled mixed state to the given mixed state with respect to this measure.

The rest of this paper is organized as follows. Some preliminaries are presented in Section 2 to include some basic definitions. The geometric measure of mixed states is proposed in Section 3. In Section 4, the characterization for the nearest disentangled mixed state is investigated. Section 5 concludes this paper with some remarks.

2 Preliminaries

An m-partite pure state $|\Psi\rangle$ of a composite quantum system can be regarded as a normalized element in a Hilbert tensor product space $\mathcal{H} = \bigotimes_{k=1}^{m} \mathcal{H}_k$, where the dimension of $\mathcal{H}_k$ is d_k for $k = 1, \ldots, m$. Each Hilbert space $\mathcal{H}_k$ is armed with an underlying norm $\|\|$. A separable m-partite pure state $|\Phi\rangle \in \mathcal{H}$ can be described by $|\Phi\rangle = \bigotimes_{k=1}^{m} |\phi^{(k)}\rangle$ with $|\phi^{(k)}\rangle \in \mathcal{H}_k$ and $\||\phi^{(k)}\rangle\| = 1$ for $k = 1, \ldots, m$. Denote by $Separ(\mathcal{H})$ the set of all separable pure states in $\mathcal{H}$. A state is called entangled if it is not separable.

For a given m-partite pure state $|\Psi\rangle \in \mathcal{H}$, a geometric measure is then defined as[15]

$$\min_{|\Phi\rangle \in Separ(\mathcal{H})} \||\Psi\rangle - |\Phi\rangle\|, \tag{1}$$

or one may consider

$$\min_{|\Phi\rangle \in Separ(\mathcal{H})} \frac{1}{2}\||\Psi\rangle - |\Phi\rangle\|^2 = 1 - G(\Psi), \tag{2}$$

where $G(\Psi)$ is the maximal overlap:

$$G(\Psi) = \max_{|\Phi\rangle \in Separ(\mathcal{H})} |\langle\Psi|\Phi\rangle|. \tag{3}$$

Based on Eq. (2), the quantum eigenvalue problem is proposed and analyzed in Refs. [10,15]:

$$\begin{cases} \langle\Psi| \left(\bigotimes_{j\neq k} |\phi^{(j)}\rangle\right) = \lambda\langle\phi^{(k)}|, \\ \left(\bigotimes_{j\neq k}\langle\phi^{(j)}|\right) \Psi\rangle = \lambda|\phi^{(k)}\rangle, \\ \||\phi^{(k)}\rangle\| = 1, k = 1, \ldots, m. \end{cases} \tag{4}$$

Proposition 1. Let $|\Psi\rangle \in \mathcal{H}$ be a pure state and the corresponding quantum eigenvalue problem be Eq. (4). Then, λ is a real number and the maximal overlap in Eq. (3) is equal to the largest such λ.

Proof: See Ref. [15, Section II] or Ref. [10, Section 2] for the detailed proof.

The largest λ in Eq. (4), denoted by $\Lambda_{\max}$, is called the entanglement eigenvalue[10,15]. Consequently, the geometric measure in Eq. (2) equals $1 - \Lambda_{\max}$.

The entanglement problem for mixed states in $\mathcal{H}$ has attracted much attention as well[3,6,9,12,15,16]. Usually, a mixed state in $\mathcal{H}$ is represented by a density matrix ϱ of size $\prod_{k=1}^{m} d_k \times \prod_{k=1}^{m} d_k$[6,15,17]. Clearly, ϱ is Hermitian, positive semidefinite and trace one. There are several concepts on disentangled multipartite mixed states. We adopt the following one[9].

Definition 1. For a mixed state in $\mathcal{H}$ with density matrix ϱ, it is disentangled if

$$\varrho = \sum_k p_k |\Psi^{(k)}\rangle\langle\Psi^{(k)}|$$

for some pure separable states $|\Psi^{(k)}\rangle \in Separ(\mathcal{H})$, $p_k \geqslant 0$ and $\sum_k p_k = 1$.

Denote by $Disen(\mathcal{H})$ the set of all disentangled mixed states in $\mathcal{H}$.

The geometric measure for pure states can be extended to mixed states through the convex roof construction[15]:

$$E_C(\varrho) := \min_{\{p_i, \Psi^{(i)}\}} \sum_i p_i M(|\Psi^{(i)}\rangle) \tag{5}$$

where the minimum is taken over all decompositions $\varrho = \sum_i p_i |\Psi^{(i)}\rangle\langle\Psi^{(i)}|$ into pure states with the p_i forming a probability distribution, and the measure M for pure states can be chosen to be either the measure Eq. (2) or any other measures.

3 Geometric Measure for Mixed States

Although the geometric measure defined in Eq. (5) satisfies the criteria for entanglement monotone[14,15], the extension of Proposition 1 to mixed states is not clear and there lack characterizations of the nearest disentanglement mixed state to an arbitrary mixed state. In this section, instead of Eq. (5), we propose a geometric measure for mixed states which is a natural extension of Eq. (2).

Definition 2. For a mixed state in $\mathcal{H}$ with density matrix ϱ, its geometric measure is defined as:

$$E(\varrho) := \min_{\rho\in Disen(\mathcal{H}),\ \|\rho\|=\|\varrho\|} \|\varrho - \rho\|, \tag{6}$$

where the norm is the Frobenius norm of matrices.

To see that Definition 2 is well-defined, the following lemma is essential.

Lemma 1. Let ϱ be the density matrix of a mixed state in $\mathcal{H}$. Then the set $S(\varrho) := \{\rho \in Disen(\mathcal{H}) \mid \|\rho\| = \|\varrho\|\}$ is a nonempty compact set.

Proof: Let $n := \Pi_{k=1}^m d_k$ and $N := n^2 + 1$. The density matrix is an $n \times n$ Hermitian matrix. For any density matrix ϱ, its real part is a symmetric $n \times n$ matrix and its imaginary part is a skew-symmetric $n \times n$ matrix. Consequently, the real dimension of $Disen(\mathcal{H})$ is n^2. By the definition of $Disen(\mathcal{H})$, every $\rho \in Disen(\mathcal{H})$ can be represented as a convex combination of density matrices of pure separable states. By Caratheodory's theorem[11], the number of density matrices of pure separable states in such a combination can be chosen to be at most N. Consequently, we have

$$S(\varrho) = \left\{ \begin{array}{c} \rho = \sum_{k=1}^N p_k |\phi_1^{(k)}\rangle \cdots |\phi_m^{(k)}\rangle \\ \langle \phi_m^{(k)}| \cdots \langle \phi_1^{(k)}| \end{array} \left| \begin{array}{c} \left\| |\phi_i^{(k)}\rangle \right\|^2 = 1,\ i=1,\ldots,m,\ k=1,\ldots,N, \\ \sum_{r,s=1}^N p_r p_s \prod_{i=1}^m \langle \phi_i^{(r)} | \phi_i^{(s)} \rangle \langle \phi_i^{(s)} | \phi_i^{(r)} \rangle = \|\varrho\|^2, \\ \sum_{k=1}^N p_k = 1,\ p_k \geqslant 0,\ k=1,\ldots,N. \end{array} \right. \right\},$$

which is obviously bounded and closed. Since ϱ a positive semidefinite $n \times n$ matrix, we can assume that $\varrho = \sum_{k=1}^K \alpha_k |\Psi^{(k)}\rangle\langle\Psi^{(k)}|$ be the orthogonal eigenvalue decomposition. Then, $\sum_{k=1}^K \alpha_k^2 = \|\varrho\|^2$, and $\sum_{k=1}^K \alpha_k = 1$ as $\mathrm{Tr}(\varrho) = 1$. Since $K \leqslant n$, we can find $\{|\phi_1^{(k)}\rangle, \ldots, |\phi_m^{(k)}\rangle\}_{k=1}^K$ such that

$$\left\| |\phi_i^{(k)}\rangle \right\|^2 = 1,\ i = 1, \ldots, m,\ k = 1, \ldots, K,$$
$$\prod_{i=1}^m \langle \phi_i^{(r)} | \phi_i^{(s)} \rangle = 0,\ \forall r \neq s,\ r, s = 1, \ldots, K.$$

Consequently, $\rho := \sum_{k=1}^K \alpha_k |\phi_1^{(k)}\rangle \cdots |\phi_m^{(k)}\rangle\langle\phi_m^{(k)}| \cdots \langle\phi_1^{(k)}| \in S(\varrho)$. The result follows.

The following proposition concerns some properties of the measure Eq. (6).

Proposition 2. Let ϱ be the density matrix of a mixed state in $\mathcal{H}$ and $E(\varrho)$ be defined as Eq. (6). Then, we have

(a) $E(\varrho) \geqslant 0$ and $E(\varrho) = 0$ if and only if $\varrho \in Disen(\mathcal{H})$.

(b) Local unitary transformations on $Disen(\mathcal{H})$ do not change E.

Proof (a) By Eq. (6), $E(\varrho) \geqslant 0$ for any $\varrho \in Disen(\mathcal{H})$. If $\varrho \in Disen(\mathcal{H})$, then with $\rho := \varrho$, we get $\|\varrho - \rho\| = 0$. consequently, $0 \leqslant E(\varrho) \leqslant 0$ as desired. Now, suppose that $E(\varrho) = 0$, i.e., there exists $\rho \in Disen(\mathcal{H})$ such that $\|\varrho - \rho\| = 0$. Consequently, $\varrho = \rho \in Disen(\mathcal{H})$. The results follow.

(b) Denote by $\mathfrak{U}(\mathcal{H})$ the group of local unitary linear transformations of $\mathcal{H}$. By the definition of $Disen(\mathcal{H})$, it is obviously that $Disen(\mathcal{H})$ is $\mathfrak{U}(\mathcal{H})$-invariant. This, together with the fact that norm $\|\cdot\|$ is $\mathfrak{U}(\mathcal{H})$-invariant, implies that E is $\mathfrak{U}(\mathcal{H})$-invariant. $\square$

4 The Nearest Disentangled Mixed State

In this section, we establish an analogue of Proposition 1 for mixed states based on the geometric measure defined by Definition 2. Like in Refs. [10,15], where Eq. (2)

is considered instead of Eq. (1), we now, consider

$$\min_{\rho\in S(\varrho)} \frac{1}{2}\|\varrho-\rho\|^2 \tag{7}$$

instead of Eq. (6). Here $S(\varrho)$ is defined as that in Lemma 1. By the proof of Lemma 1, the optimization problem Eq. (7) can be parameterized as:

$$\begin{array}{ll}\min & \frac{1}{2}\left\|\varrho-\sum_{k=1}^{N}p_k|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle\langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\right\|^2\\ \text{s.t.} & \left\||\phi_i^{(k)}\rangle\right\|^2=1,\ i=1,\ldots,m,\ k=1,\ldots,N,\\ & \sum_{r,s=1}^{N}p_rp_s\prod_{i=1}^{m}\langle\phi_i^{(r)}|\phi_i^{(s)}\rangle\langle\phi_i^{(s)}|\phi_i^{(r)}\rangle=\|\varrho\|^2,\\ & \sum_{k=1}^{N}p_k=1,\ p_k\geqslant 0,\ k=1,\ldots,N.\end{array} \tag{8}$$

It is easy to see that Eq. (8) is equivalent to:

$$\begin{array}{ll}\max & \sum_{k=1}^{N}p_k\langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\varrho|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle\\ \text{s.t.} & \left\||\phi_i^{(k)}\rangle\right\|^2=1,\ i=1,\ldots,m,\ k=1,\ldots,N,\\ & \sum_{r,s=1}^{N}p_rp_s\prod_{i=1}^{m}\langle\phi_i^{(r)}|\phi_i^{(s)}\rangle\langle\phi_i^{(s)}|\phi_i^{(r)}\rangle=\|\varrho\|^2,\\ & \sum_{k=1}^{N}p_k=1,\ p_k\geqslant 0,\ k=1,\ldots,N.\end{array} \tag{9}$$

Proposition 3. The optimality conditions of maximization problem (9) are:

$$\begin{cases} p_k\langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\varrho\prod_{j\neq i}|\phi_j^{(k)}\rangle=\mu_{ik}\langle\phi_i^{(k)}|\\ \qquad +\lambda p_k\sum_{t=1}^{N}p_t\left(\prod_{j\neq i}|\langle\phi_j^{(k)}|\phi_j^{(t)}\rangle|\right)^2(\langle\phi_i^{(k)}|\phi_i^{(t)}\rangle)\langle\phi_i^{(t)}|,\\ \qquad i=1,\ldots,m,k=1,\ldots,N,\\ p_k\prod_{j\neq i}\langle\phi_j^{(k)}|\varrho|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle=\mu_{ik}|\phi_i^{(k)}\rangle\\ \qquad +\lambda p_k\sum_{t=1}^{N}p_t\left(\prod_{j\neq i}|\langle\phi_j^{(k)}|\phi_j^{(t)}\rangle|\right)^2(\langle\phi_i^{(t)}|\phi_i^{(k)}\rangle)|\phi_i^{(t)}\rangle,\\ \qquad i=1,\ldots,m,k=1,\ldots,N,\\ \langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\varrho|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle=\lambda\sum_{t=1}^{N}p_t\left(\prod_{i=1}^{m}|\langle\phi_i^{(k)}|\phi_i^{(t)}\rangle|\right)^2+\kappa-\tau_k,\\ \qquad k=1,\ldots,N,\\ \tau_k,p_k\geqslant 0,\tau_kp_k=0,\ k=1,\ldots,N,\\ \left\||\phi_i^{(k)}\rangle\right\|^2=1,\ i=1,\ldots,m,k=1,\ldots,N,\\ \sum_{r,s=1}^{N}p_rp_s\prod_{i=1}^{m}\langle\phi_i^{(r)}|\phi_i^{(s)}\rangle\langle\phi_i^{(s)}|\phi_i^{(r)}\rangle=\|\varrho\|^2,\\ \sum_{k=1}^{N}p_k=1.\end{cases} \tag{10}$$

Proof It follows from the Lagrange multiplier theorem and the concept of H-derivative in complex geometry[7]. □

Proposition 4. Let ϱ be the density matrix of a mixed state in $\mathcal{H}$ and $\{\lambda,p_k,\kappa,\tau_k,\mu_{ik},|\phi_i^{(k)}\rangle\}$ be a solution for Eq. (10). We have the following conclusions.

(a) $\mu_{1k}=\cdots=\mu_{mk}$ for any $k=1,\ldots,N$.

(b) Let $\mu_k := \mu_{1k} = \cdots = \mu_{mk}$. Then, $\kappa = \sum_{k=1}^N \mu_k$.

(c) $\lambda\|\varrho\|^2 + \kappa \in \mathbb{R}$ is a nonnegative real number and

$$\sum_{k=1}^N p_k \langle\phi_m^{(k)}| \cdots \langle\phi_1^{(k)}|\varrho|\phi_1^{(k)}\rangle \cdots |\phi_m^{(k)}\rangle = \lambda\|\varrho\|^2 + \sum_{k=1}^N \mu_k = \lambda\|\varrho\|^2 + \kappa. \quad (11)$$

Proof: (a) By the first equation of (10), we have that

$$\begin{aligned}\mu_{ik} = &\left[p_k\langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\varrho\prod_{j\neq i}|\phi_j^{(k)}\rangle - \lambda p_k\sum_{t=1}^N p_t\right.\\ &\left.\left(\prod_{j\neq i}|\langle\phi_j^{(k)}|\phi_j^{(t)}\rangle|\right)^2(\langle\phi_i^{(k)}|\phi_i^{(t)}\rangle)\langle\phi_i^{(t)}|\right]|\phi_i^{(k)}\rangle\\ =&p_k\left[\langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\varrho|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle - \lambda\sum_{t=1}^N p_t\left(\prod_{j=1}^m|\langle\phi_j^{(k)}|\phi_j^{(t)}\rangle|\right)^2\right], \quad (12)\end{aligned}$$

which is independent of index i. Then, the result follows.

(b) Let $\mu_k := \mu_{1k} = \cdots = \mu_{mk}$. Multiplying the first equation of (10) by $|\phi_i^{(k)}\rangle$ and then subtracting p_k times the third equation of (10), we get

$$\mu_k = p_k\kappa - p_k\tau_k.$$

This, together with the fourth and the last equations of Eq. (10), implies that

$$\sum_{k=1}^N \mu_k = \kappa.$$

(c) The result (b), together with the summation of the equations Eq. (12) from $k=1$ to N, implies Eq. (11). Now, the facts that $p_k \geqslant 0$ and ϱ is positive semidefinite imply that $\lambda\|\varrho\|^2 + \kappa \in \mathbb{R}$ is a nonnegative real number.

Similar to the entanglement eigenvalue for a pure state Eq. (4), we define the entanglement eigenvalue for a mixed state.

Definition 3. Let ϱ be the density matrix of a mixed state in $\mathcal{H}$.

$$\chi(\varrho) := \max\left\{\lambda\|\varrho\|^2 + \kappa \mid \{\lambda, p_k, \tau_k, \kappa, \mu_k, |\phi_i^{(k)}\rangle\} \text{ satisfies } (13)\right\}$$

is called the entanglement eigenvalue of ϱ. Here system Eq. (13) is defined as:

$$\begin{cases} p_k\langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\varrho\prod_{j\neq i}|\phi_j^{(k)}\rangle=\mu_k\langle\phi_i^{(k)}| \\ \quad +\lambda p_k\sum_{t=1}^N p_t\left(\prod_{j\neq i}|\langle\phi_j^{(k)}|\phi_j^{(t)}\rangle|\right)^2(\langle\phi_i^{(k)}|\phi_i^{(t)}\rangle)\langle\phi_i^{(t)}|, \\ \quad i=1,\ldots,m,k=1,\ldots,N, \\ p_k\prod_{j\neq i}\langle\phi_j^{(k)}|\varrho|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle=\mu_k|\phi_i^{(k)}\rangle \\ \quad +\lambda p_k\sum_{t=1}^N p_t\left(\prod_{j\neq i}|\langle\phi_j^{(k)}|\phi_j^{(t)}\rangle|\right)^2(\langle\phi_i^{(t)}|\phi_i^{(k)}\rangle)|\phi_i^{(t)}\rangle, \\ \quad i=1,\ldots,m,k=1,\ldots,N, \\ \langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|\varrho|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle=\lambda\sum_{t=1}^N p_t\left(\prod_{i=1}^m|\langle\phi_i^{(k)}|\phi_i^{(t)}\rangle|\right)^2+\kappa-\tau_k, \\ \quad k=1,\ldots,N, \\ \tau_k,p_k\geqslant 0,\tau_k p_k=0,\ k=1,\ldots,N, \\ \left\||\phi_i^{(k)}\rangle\right\|^2=1,\ i=1,\ldots,m,k=1,\ldots,N, \\ \sum_{r,s=1}^N p_r p_s\prod_{i=1}^m\langle\phi_i^{(r)}|\phi_i^{(s)}\rangle\langle\phi_i^{(s)}|\phi_i^{(r)}\rangle=\|\varrho\|^2, \\ \sum_{k=1}^N p_k=1. \end{cases} \tag{13}$$

Now, we have the following theorem.

Theorem 1. Let ϱ be the density matrix of a mixed state in $\mathcal{H}$. If $\chi(\varrho)$ is the entanglement eigenvalue of ϱ, then

$$\frac{1}{2}E(\varrho)^2=\|\varrho\|^2-\chi(\varrho). \tag{14}$$

Moreover, $\rho:=\sum_{k=1}^N p_k|\phi_1^{(k)}\rangle\cdots|\phi_m^{(k)}\rangle\langle\phi_m^{(k)}|\cdots\langle\phi_1^{(k)}|$ corresponding to $\chi(\varrho)$ is the nearest disentangled mixed state to ϱ.

Proof: It follows from Eq. (7) and Eq. (9), Proposition 4 and Definitions 2 and 3 immediately.

It is noted that $\chi(\varrho)$ is equal to the optimal value of problem Eq. (9) and Eq. (14) and can be reduced to $1-\Lambda_{\max}$ for a pure state.

We now compute the geometric measure defined in Eq. (7) for two examples. The computation is based on the maximization problem Eq. (9).

Example 1. In this example, we consider the following bipartite qubit mixed state

$$\begin{aligned}\varrho:=\alpha&\left(\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|11\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 00|+\frac{1}{\sqrt{2}}\langle 11|\right)\\ &+(1-\alpha)\left(\frac{1}{\sqrt{2}}|01\rangle+\frac{1}{\sqrt{2}}|10\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 01|+\frac{1}{\sqrt{2}}\langle 10|\right),\end{aligned}$$

where $\alpha\in[0,1]$. It is easy to see $\frac{1}{2}E(\varrho)^2=\frac{1}{2}$ when both $\alpha=0$ and $\alpha=1$, which correspond to pure states. For general $\alpha\in(0,1)$, we use Eq. (9) to compute $\frac{1}{2}E(\varrho)^2$. It can be seen that $n=4$ and $N=17$. Under the basis $\{|0\rangle,|1\rangle\}$, the corresponding maximization problem Eq. (9) can be transformed into a maximization problem only involving real variables. By parameterizing $|\phi_j^{(k)}\rangle:=\left(\mathbf{x}_1^{(k,j)}+i\mathbf{y}_1^{(k,j)}\right)|0\rangle+\left(\mathbf{x}_2^{(k,j)}+\right.$

$i\mathbf{y}_2^{(k,j)}\Big)\,|1\rangle$ for $j=1,2$ and $k=1,\ldots,17$, we have

$$\begin{aligned}
\max\quad & \textstyle\sum_{k=1}^{17} p_k \Big\{ \frac{\alpha}{2}\Big[\Big(\big(\mathbf{x}^{(k,1)}\big)^T\mathbf{x}^{(k,2)} - \big(\mathbf{y}^{(k,1)}\big)^T\mathbf{y}^{(k,2)}\Big)^2 \\
& + \Big(\big(\mathbf{x}^{(k,1)}\big)^T\mathbf{y}^{(k,2)} + \big(\mathbf{y}^{(k,1)}\big)^T\mathbf{x}^{(k,2)}\Big)^2\Big] \\
& + \tfrac{1-\alpha}{2}\Big[\Big(\mathbf{x}_2^{(k,1)}\mathbf{x}_1^{(k,2)} + \mathbf{x}_1^{(k,1)}\mathbf{x}_2^{(k,2)} - \mathbf{y}_1^{(k,1)}\mathbf{y}_2^{(k,2)} - \mathbf{y}_2^{(k,1)}\mathbf{y}_1^{(k,2)}\Big)^2 \\
& + \Big(\mathbf{y}_1^{(k,1)}\mathbf{x}_2^{(k,2)} + \mathbf{y}_2^{(k,2)}\mathbf{x}_1^{(k,1)} + \mathbf{y}_1^{(k,2)}\mathbf{x}_2^{(k,1)} + \mathbf{y}_2^{(k,1)}\mathbf{x}_1^{(k,2)}\Big)^2\Big]\Big\} \\
\text{s.t.}\quad & \big(\mathbf{x}^{(k,1)}\big)^T\mathbf{x}^{(k,1)} + \big(\mathbf{y}^{(k,1)}\big)^T\mathbf{y}^{(k,1)} = 1,\ k=1,\ldots,17, \\
& \big(\mathbf{x}^{(k,2)}\big)^T\mathbf{x}^{(k,2)} + \big(\mathbf{y}^{(k,2)}\big)^T\mathbf{y}^{(k,2)} = 1,\ k=1,\ldots,17, \\
& \textstyle\sum_{r,s=1}^{17} p_r p_s \prod_{i=1}^{2}\Big\{\Big[\big(\mathbf{x}^{(r,i)}\big)^T\mathbf{x}^{(s,i)} + \big(\mathbf{y}^{(r,i)}\big)^T\mathbf{y}^{(s,i)}\Big]^2 \\
& + \Big[\big(\mathbf{x}^{(r,i)}\big)^T\mathbf{y}^{(s,i)} - \big(\mathbf{x}^{(s,i)}\big)^T\mathbf{y}^{(r,i)}\Big]^2\Big\} = 1 + 2\alpha^2 - 2\alpha, \\
& \textstyle\sum_{k=1}^{17} p_k = 1,\ p_k \geqslant 0,\ k=1,\ldots,17.
\end{aligned} \tag{15}$$

Problem Eq. (15) is solved using MatLab Optimization ToolBox, which can always find a good local maximizer. The result is shown in Figure 1. For $\alpha=0.5$, the nearest ρ computed is

$$\rho = \sum_{k=1}^{4} p_k|\phi_1^{(k)}\rangle|\phi_2^{(k)}\rangle\langle\phi_2^{(k)}|\langle\phi_1^{(k)}|$$

with $|\phi_j^{(k)}\rangle := \Big(\mathbf{x}_1^{(k,j)} + i\mathbf{y}_1^{(k,j)}\Big)|0\rangle + \Big(\mathbf{x}_2^{(k,j)} + i\mathbf{y}_2^{(k,j)}\Big)|1\rangle$ for $j=1,2$ and the parameters being given in Table 1.

Table 1 Parameters for the nearest disentangled mixed state

k	p_k	$\mathbf{x}_1^{(k,1)}$	$\mathbf{x}_2^{(k,1)}$	$\mathbf{x}_1^{(k,2)}$	$\mathbf{x}_2^{(k,2)}$	$\mathbf{y}_1^{(k,1)}$	$\mathbf{y}_2^{(k,1)}$	$\mathbf{y}_1^{(k,2)}$	$\mathbf{y}_2^{(k,2)}$
1	0.0414	0.4495	0.4497	0.6749	0.6747	0.5458	0.5457	0.2113	0.2114
2	0.2163	0.5211	0.5210	0.1227	0.1227	0.4780	0.4781	0.6963	0.6964
3	0.5000	0.5572	−0.5572	−0.7061	0.7061	−0.4353	0.4353	0.0369	−0.0369
4	0.2423	0.6908	0.6909	0.3020	0.3020	0.1506	0.1508	0.6393	0.6395

We now consider a class of two-qubit mixed states with less symmetric structures.

Example 2. In this example, we consider the following two-qubit mixed state

$$\begin{aligned}
\varrho :=& \alpha\left(\gamma_1|00\rangle + \gamma_2|11\rangle\right)\left(\gamma_1\langle 00| + \gamma_2\langle 11|\right) \\
&+ (1-\alpha)\left(\gamma_3|01\rangle + \gamma_4|10\rangle\right)\left(\gamma_3\langle 01| + \gamma_4\langle 10|\right),
\end{aligned}$$

where $\alpha\in[0,1]$, $\gamma_1^2+\gamma_2^2=1$ and $\gamma_3^2+\gamma_4^2=1$. The optimization problem is similar to (15). For **Case I**: $\gamma_1=\gamma_3:=\frac{1}{\sqrt{3}}$ and $\gamma_2=\gamma_4:=\sqrt{\frac{2}{3}}$, and **Case II**: $\gamma_1:=\frac{1}{\sqrt{3}}$ and $\gamma_2:=\sqrt{\frac{2}{3}}$, and $\gamma_3:=\frac{1}{\sqrt{4}}$ and $\gamma_4:=\sqrt{\frac{3}{4}}$, the computational results are shown in Figure 2. We see that the curve of **Case II** is not symmetric with respect to $\alpha=0.5$, which agrees with the choice of parameters. The other cases for parameters α,γ have similar phenomena.

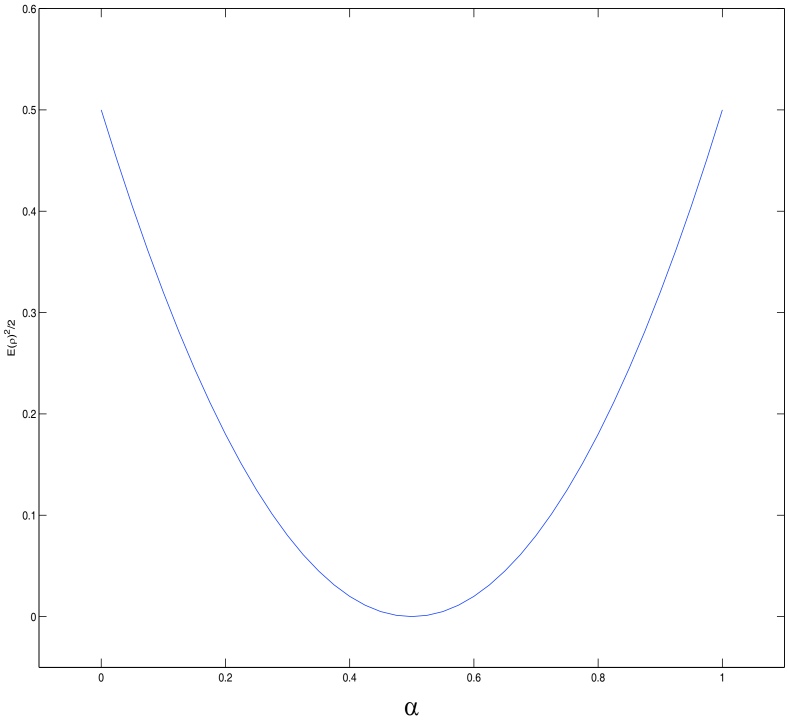

Figure 1. The measure $\frac{E(\varrho)^2}{2}$ for the mixed states in Example 1

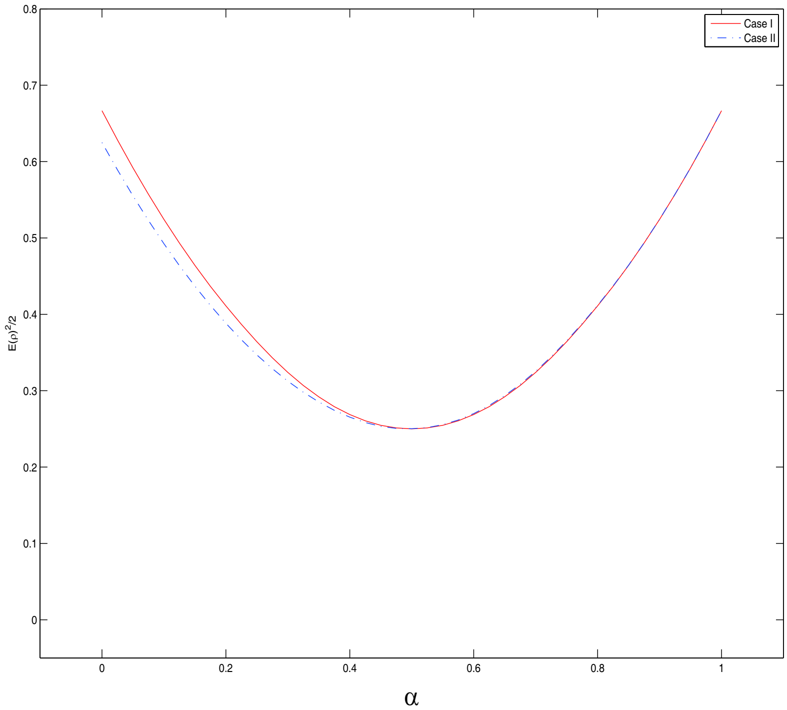

Figure 2. The measure $\frac{E(\varrho)^2}{2}$ for the mixed states in Example 2

5 Conclusion

We have extended the geometric measure to mixed states and established a characterization of the nearest disentangled mixed state of a given mixed state with respect to this measure. The analogue results for the quantum eigenvalue of a pure state are established for mixed states, namely, Proposition 4 and Theorem 1. Based on this geometric measure, further works on the analysis and the computation are desired.

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