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Algebraic connectivity of an even uniform hypergraph

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Abstract We generalize Laplacian matrices for graphs to Laplacian tensors for even uniform hypergraphs and set some foundations for the spectral hypergraph theory based upon Laplacian tensors. Especially, algebraic connectivity of an even uniform hypergraph based on Z-eigenvalues of the corresponding Laplacian tensor is introduced and its connections with edge connectivity and vertex connectivity are discussed.

Keywords Tensor · Hypergraph · Z-eigenvalue · Algebraic connectivity

1 Introduction

Like graphs, hypergraphs have many applications in various fields (Berge 1973; Lim 2007; Rota Bulò 2009; Rota Bulò and Pelillo 2009). As we know, many problems associated to graphs are combinatorial optimization problems which turn to be NP-hard or NP-complete problems (Chung 1997; Nemhauser and Wolsey 1988; Rota Bulò 2009). Nevertheless, many continuous characterizations of these NP-hard or NP-complete problems were developed in the past several decades. Among them, spectral graph theory plays a fundamental role (Chung 1997; Nemhauser and Wolsey 1988). So, the corresponding spectral hypergraph theory becomes the focus of many researchers in recent years. As graphs are related to matrices, hypergraphs are related to tensors (Berge 1973; Lim 2007; Qi 2005; Rota Bulò 2009) which could

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reveal more higher order structures than matrices. The spectral theory for matrices hence serves as a foundation for developing spectral graph theory (Chung 1997; Fiedler 1973). Unlike matrices, spectral theory for tensors was only developed recently in Lim (2005), Qi (2005, 2007). Several kinds of eigenvalues and singular values for tensors have been proposed. Based on these, it is natural to develop spectral hypergraph theory. The recent work (Lim 2007; Rota Bulò 2009; Rota Bulò and Pelillo 2009) is just in this stream. Nonetheless, these results are based on the so called adjacency tensor of a uniform hypergraph (Lim 2007; Rota Bulò 2009; Rota Bulò and Pelillo 2009). This kind of tensors only involves elements which have pairwise different indices. Hence, it is hard to obtain useful properties for adjacency tensors such as positive semidefiniteness like that in spectral graph theory (Chung 1997), and many properties developed in spectral graph theory (Chung 1997) are still mysterious for hypergraphs (Berge 1973; Lim 2007; Rota Bulò 2009).

In this paper, we introduce Laplacian tensors for even uniform hypergraphs. By even uniform hypergraphs, we mean r-uniform hypergraphs with even $r \geq 4$. The reason why we restrict our discussion to even uniform hypergraphs is that positive semidefiniteness of the proposed Laplacian tensors is crucial for the main results (e.g., Theorems 12 and 16) while there is no nontrivial odd order tensor which is positive semidefiniteness. We present the results only for 4-uniform hypergraphs for the sake of simplicity ((4) in Definition 2 and the proof for Lemma 19 would be more complicated for $r > 4$). Nonetheless, all the results could be extended to the content of r-uniform hypergraphs with even $r \geq 6$.

We show in the next section that the Laplacian tensor of an even hypergraph is symmetric, positive semidefinite and has a zero Z-eigenvalue with the normalized vector of all ones as a Z-eigenvector. We introduce the *algebraic connectivity* of an even hypergraph as the second smallest Z-eigenvalue of the Laplacian tensor like that for graphs (Chung 1997; Fiedler 1973), and show that the algebraic connectivity of an even hypergraph is larger than zero if and only if the hypergraph is connected. We also show that the number of connected components of an even hypergraph is actually the dimension of the set of Z-eigenvectors of the Laplacian tensor corresponding to the zero Z-eigenvalue. We characterize algebraic connectivity of an even hypergraph by a generalized Courant-Fischer Theorem (Horn and Johnson 1985) for the Laplacian tensor. Hence, computing the algebraic connectivity of an even hypergraph is transformed into computing the smallest Z-eigenvalue of another tensor resulted by multilinear transformation (Lim 2007) of its Laplacian tensor. We also point out the relationships between the Z-eigenvalue problems for Laplacian tensors and the generalized Laplace-Beltrami operators (Chung 1997). Two other foundational lemmas concerned algebraic connectivity are established at the end of Sect. 2, while some applications of them that involve the connections of algebraic connectivity with *edge connectivity* and *vertex connectivity* of an even hypergraph are discussed in Sect. 3. Some final remarks are given in the last section.

We add a comment on the notation that is used. Scalars are written as lowercase letters ($\lambda, \alpha, a, b, \ldots$), vectors are written as bold lowercase letters ($\mathbf{x}, \mathbf{y}, \ldots$), the i-th entry of a vector $\mathbf{x}$ is denoted by x_i, matrices and tensors correspond to italic capitals ($A, L, T, \ldots$), sets correspond to blackboard bold letters ($\mathbb{E}, \mathbb{X}, \mathbb{V}, \ldots$), the usual symbol $\otimes$ is used to denote the outer product of tensors, and $\mathbf{e}$ and I are reserved for the vector of all ones and the identity matrix, respectively. In this paper, tensors refer to fourth order tensors (Lim 2007; Qi 2005, 2007). For a tensor T with entries T_{ijkl} and a vector $\mathbf{x} \in \Re^n$, we associate them a scalar, denoted $T\mathbf{x}^4$, as $\sum_{i,j,k,l=1}^n T_{ijkl}x_i x_j x_k x_l$, and a vector, denoted $T\mathbf{x}^3$, as its i-th element being $\sum_{j,k,l=1}^n T_{ijkl}x_j x_k x_l$.

1.1 Preliminaries

Throughout the sequel discussion, we focus on 4-uniform hypergraphs. By a 4-uniform hypergraph (we will abbreviate it graph in the sequel if there is no confusion), we mean a hypergraph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ with vertices set $\mathbb{V} = \{1, \ldots, n\}$ of size $n \geq 4$ and edges set $\mathbb{E} = \{\mathbb{E}_1, \ldots, \mathbb{E}_m\}$ with size m and $|\mathbb{E}_i| = 4$ for every $i \in \{1, \ldots, m\}$. Here $|\cdot|$ means the cardinality of a set. A finite *path* from vertex i to vertex j is a finite sequence of vertices with its start vertex i and its end vertex j such that from each of its vertices there is an edge to the next vertex. Two vertices are called connected if there is a finite path between them. *A connected component* $\mathbb{X}$ of $\mathcal{G}$ is a subset of $\mathbb{V}$ such that any two vertices in $\mathbb{X}$ are connected and no other vertex in $\mathbb{V} \setminus \mathbb{X}$ is connected to any vertex in $\mathbb{X}$.

Definition 1 For every $i \in \mathbb{V}$, the *degree* of vertex i, denoted as d_i, is defined as the cardinality of the set $\mathbb{D} := \{\mathbb{E}_p \in \mathbb{E} \mid i \in \mathbb{E}_p\}$. Vertex i is called *isolated* if $d_i = 0$.

Let L be the Laplacian matrix of a 2-uniform graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, then for any $\mathbf{x} \in \Re^n$ (Merris 1994)

$$\mathbf{x}^T L\mathbf{x} = \sum_{\{i,j\} \in \mathbb{E}} (x_i - x_j)^2. \tag{1}$$

So, L is positive semidefinite with $\mathbf{e}$ its eigenvector corresponding to zero eigenvalue. A natural generalization of (1) to fourth order is as follows: for a 4-uniform hypergraph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, its Laplacian tensor T corresponds to the following quartic form:

$$T\mathbf{x}^4 := \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)\mathbf{x}^4, \quad \forall \mathbf{x} \in \Re^4 \tag{2}$$

with

$$L(\mathbb{E}_p)\mathbf{x}^4 = \frac{1}{84}\big[(x_i + x_j + x_k - 3x_l)^4 + (x_i + x_j + x_l - 3x_k)^4 + (x_i + x_k + x_l - 3x_j)^4 + (x_j + x_k + x_l - 3x_i)^4\big], \tag{3}$$

here $L(\mathbb{E}_p)$ is a tensor associated to edge $\mathbb{E}_p$. It is easy to see that $T_{iiii} = d_i$ for all $i \in \mathbb{V}$ as those for 2-uniform graphs (Merris 1994). This is one of reasons why $\frac{1}{84}$ is multiplied in (3).

Now, we collect the above idea into the following formal definitions.

Definition 2 Given any nonempty subset $\mathbb{I} \subseteq \mathbb{V}$, we associate it an n dimensional tensor $L(\mathbb{I})$, called the *core tensor with respect to* $\mathbb{I}$, as:

$$[L(\mathbb{I})]_{ijkl} := \begin{cases} 1 & i=j=k=l \in \mathbb{I}; \\ -\frac{1}{3} & \{i,j,k,l\} \subseteq \mathbb{I}, \text{ three of them equal, but not all;} \\ \frac{5}{21} & \{i,j,k,l\} \subseteq \mathbb{I}, \text{ two different pairs of them equal;} \\ \frac{1}{21} & \{i,j,k,l\} \subseteq \mathbb{I}, \text{ one pair equal, three of them different;} \\ -\frac{1}{7} & \{i,j,k,l\} \subseteq \mathbb{I}, \text{ pairwise different;} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We call $L(\mathbb{V})$ the *core tensor* of graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, denoted by L.

Definition 3 Given a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, we associate it an n dimensional nonnegative integer tensor K, called the *degree tensor* of $\mathcal{G}$, as K_{ijkl} being the cardinality of the set $\mathbb{D} := \{\mathbb{E}_p \in \mathbb{E} \mid \{i,j,k,l\} \subseteq \mathbb{E}_p\}$. It is easy to see that $K_{iiii} = d_i$ for all $i \in \{1, \ldots, n\}$.

Definition 4 Given a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, let K be the degree tensor of $\mathcal{G}$, and L be its core tensor. The *Laplacian tensor* T of $\mathcal{G}$ is defined as tensor $K * L$. Here $*$ represents the Hadamard product of tensors, i.e., the componentwise product.

It is a direct computation to see that the Lapalcian tensor T of a graph defined by Definition (4) indeed satisfies (2). A tensor T is called symmetric if $T_{ijkl} = T_{i_1 i_2 i_3 i_4}$ for arbitrary permutation (i, j, k, l) of (i_1, i_2, i_3, i_4). A tensor T is called positive semidefinite if $T\mathbf{x}^4 \geq 0$ for any $\mathbf{x} \in \Re^n$.

Definition 5 The *symmetric rank* r of a symmetric tensor T is the minimum nonnegative integer k such that T has the following representation:

$$T = \sum_{j=1}^{k} \alpha_j \mathbf{u}^j \otimes \mathbf{u}^j \otimes \mathbf{u}^j \otimes \mathbf{u}^j,$$

here $\alpha_j \in \Re$ and $\mathbf{u}^j \in \Re^n$ for all $j \in \{1, \ldots, k\}$.

2 Algebraic connectivity

In this section, we introduce *algebraic connectivity* of a graph and discuss some of its properties.

Lemma 6 *For any* $\mathbb{E}_p = \{i, j, k, l\}$ *with* $1 \leq i, j, k, l \leq n$, *let* $L(\mathbb{E}_p)$ *be the core tensor with respect to* $\mathbb{E}_p$ *and* $\mathbf{x} \in \Re^n$. *We have*

$$L(\mathbb{E}_p) = \frac{1}{84} \sum_{s=1}^{4} \mathbf{u}^s_{\mathbb{E}_p} \otimes \mathbf{u}^s_{\mathbb{E}_p} \otimes \mathbf{u}^s_{\mathbb{E}_p} \otimes \mathbf{u}^s_{\mathbb{E}_p} \tag{5}$$

with $\mathbf{u}^1_{\mathbb{E}_p} := \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k - 3\mathbf{e}_l$, $\mathbf{u}^2_{\mathbb{E}_p} := \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l - 3\mathbf{e}_k$, $\mathbf{u}^3_{\mathbb{E}_p} := \mathbf{e}_i + \mathbf{e}_k + \mathbf{e}_l - 3\mathbf{e}_j$ *and* $\mathbf{u}^4_{\mathbb{E}_p} := \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l - 3\mathbf{e}_i$, *where* $\mathbf{e}_t$ *is the* t*-th coordinate vector for any* $t \in \{i, j, k, l\}$. *So*, $L(\mathbb{E}_p)$ *is positive semidefinite.*

Proof It is easy to see that (5) follows from (3) and Definition 2 directly. The positive semidefiniteness of $L(\mathbb{E}_p)$ follows directly from (3). □

Proposition 7 *For any graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, *its associated Laplacian tensor* T *is symmetric, positive semidefinite with symmetric rank at most* $4m$ *with* $m = |\mathbb{E}|$.

Proof By Definitions 2 and 3, the core tensor L and the degree tensor K of a graph are both symmetric, then their Hadamard product T is symmetric as well. Actually,

$$T = K * L = \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p) \tag{6}$$

by Definitions 2 and 3.

Now, for any $\mathbf{x} \in \Re^n$

$$\begin{aligned} T\mathbf{x}^4 &= \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)\mathbf{x}^4 \\ &= \frac{1}{84} \sum_{\mathbb{E}_p \in \mathbb{E}} \sum_{s=1}^{4} (\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{x})^4 \\ &\geq 0 \end{aligned}$$

with $\mathbf{u}^s_{\mathbb{E}_p}$'s are defined in (5) and $\bullet$ the usual inner product in $\Re^n$. Hence, T is positive semidefinite.

The rank estimation follows from the above representations (6) and (5) directly. □

The concept of Z-eigenvalues is important for the sequel analysis, which is defined as follows.

Definition 8 For a tensor T, a pair $(\lambda, \mathbf{x})$ is a Z-eigenpair of T if the follows hold:

$$\begin{cases} T\mathbf{x}^3 = \lambda \mathbf{x}, \\ \lambda \in \Re, \quad \mathbf{x} \in \Re^n, \quad \mathbf{x}^T\mathbf{x} = 1. \end{cases} \tag{7}$$

λ is called a Z-eigenvalue and $\mathbf{x}$ is the associated Z-eigenvector (Qi 2005, 2007).

From Definition 8 and the fact that the gradient of $T\mathbf{x}^4$ with respect to $\mathbf{x}$ is $4T\mathbf{x}^3$ when T is symmetric, the following theorem is easy to get. See also the proofs for (Qi 2005, Theorems 3 and 5).

Theorem 9 *The Z-eigenvectors of a symmetric tensor T and the critical points of the following minimization problem have a one to one correspondence*:

$$\begin{aligned} &\min\ T\mathbf{x}^4 \\ &\text{s.t.}\quad \|\mathbf{x}\|_2 = 1, \quad \mathbf{x} \in \Re^n. \end{aligned} \tag{8}$$

Here $\|\cdot\|_2$ represents 2-norm in $\Re^n$. Furthermore, if $\mathbf{x}$ is a Z-eigenvector of T, then the corresponding Z-eigenvalue is $T\mathbf{x}^4$.

Since the minimization problem (8) is minimizing a continuous function on a compact set, it must have at least one critical point. Hence, there is at least one Z-eigenpair for a symmetric tensor.

Theorem 10 *For any graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, let T be its Laplacian tensor. Then, $\frac{\mathbf{e}}{\|\mathbf{e}\|_2}$ is a Z-eigenvector of T with the corresponding Z-eigenvalue zero.*

Proof For any $\{i, j, k, l\} = \mathbb{E}_p \in \mathbb{E}$, we have $L(\mathbb{E}_p)\mathbf{e}^4 = L(\mathbb{E}_p)(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l)^4 = 0$ by Lemma 6. So,

$$T\mathbf{e}^4 = \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)\mathbf{e}^4 = \sum_{\{i,j,k,l\} = \mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l)^4 = 0.$$

This, together with Proposition 7, implies that $\frac{\mathbf{e}}{\|\mathbf{e}\|_2}$ is a global minimizer of problem (8). By Theorem 9, $\frac{\mathbf{e}}{\|\mathbf{e}\|_2}$ is a Z-eigenvector of T with Z-eigenvalue zero. □

Lemma 11 *Let $\{i, j, k, l\} = \mathbb{E}_p \in \mathbb{E}$, then $L(\mathbb{E}_p)\mathbf{x}^4 = 0$ if and only if $x_i = x_j = x_k = x_l$.*

Proof By Lemma 6, we have that $L(\mathbb{E}_p)\mathbf{x}^4 = 0$ if and only if

$$\begin{aligned} &x_i + x_j + x_k = 3x_l, \quad x_i + x_j + x_l = 3x_k, \\ &x_i + x_k + x_l = 3x_j, \quad \text{and} \quad x_k + x_j + x_l = 3x_i. \end{aligned}$$

It is easy to see that the latter is equivalent to $x_i = x_j = x_k = x_l$. □

Theorem 12 *Given a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, let T be its Laplacian tensor. Let*

$$\mathbb{S}_0 := \left\{ \mathbf{x} \in \Re^n \setminus \{\mathbf{0}\} \,\middle|\, \frac{\mathbf{x}}{\|\mathbf{x}\|_2} \text{ is a Z-eigenvector of } T \text{ with Z-eigenvalue } 0 \right\} \cup \{\mathbf{0}\}.$$

Then, $\mathbb{S}_0$ is a linear subspace of $\Re^n$, and graph $\mathcal{G}$ has exactly $\mathrm{Dim}(\mathbb{S}_0)$ *connected components.*

Proof Suppose that $\{\mathbb{I}_1, \ldots, \mathbb{I}_q\}$ are the connected components of graph $\mathcal{G}$. For any $\mathbf{x} \in \Re^n$, $s \in \{1, \ldots, q\}$, denote by $\mathbf{x}_{\mathbb{I}_s} \in \Re^n$ a vector with its r-th element being x_r if $r \in \mathbb{I}_s$ and zero otherwise.

For every $\mathbf{y} := \mathbf{e}_{\mathbb{I}_s}$, we have that $T\mathbf{y}^4 = 0$ by Lemma 6 and (6). Hence, by Theorem 9 and Proposition 7, $\frac{\mathbf{e}_{\mathbb{I}_s}}{\|\mathbf{e}_{\mathbb{I}_s}\|_2}$ is a Z-eigenvector of T with Z-eigenvalue zero. So, $\mathbf{e}_{\mathbb{I}_s} \in \mathbb{S}_0$ for every $s \in \{1, \ldots, q\}$. Obviously, the set of vectors $\{\mathbf{e}_{\mathbb{I}_1}, \ldots, \mathbf{e}_{\mathbb{I}_q}\}$ is linearly independent. By Theorem 9 and Lemma 11, every nonzero linear combination of $\{\mathbf{e}_{\mathbb{I}_1}, \ldots, \mathbf{e}_{\mathbb{I}_q}\}$ is in $\mathbb{S}_0 \setminus \{\mathbf{0}\}$.

Now, for any $\mathbf{x} \in \mathbb{S}_0 \setminus \{\mathbf{0}\}$, by Theorem 9, we have

$$0 = T\mathbf{x}^4 = \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)\mathbf{x}^4.$$

By Lemma 6, every $L(\mathbb{E}_p)$ is positive semidefinite. Hence, $L(\mathbb{E}_p)\mathbf{x}^4 = 0$ for every $\mathbb{E}_p \in \mathbb{E}$. Thus, by Lemma 11, x_i's are a constant for all $i \in \mathbb{I}_s$ for every $s \in \{1, \ldots, q\}$. This, together with the fact that $\mathbf{x} \neq \mathbf{0}$, implies that $\mathbf{x} = \alpha_1 \mathbf{e}_{\mathbb{I}_1} + \cdots + \alpha_q \mathbf{e}_{\mathbb{I}_q}$ for some $\alpha \in \Re^q$ satisfying $\sum_{i=1}^q \alpha_i^2 > 0$.

So, $\mathbb{S}_0$ is a linear space of dimension q, i.e., $\mathrm{Dim}(\mathbb{S}_0) = q$, which is the exact number of connected components of graph $\mathcal{G}$. □

Like that in the linear algebra setting (Horn and Johnson 1985), $\mathrm{Dim}(\mathbb{S}_0)$ is called the *geometrical multiplicity* of the zero Z-eigenvalue of T. By Theorems 10 and 12, we get the following result.

Corollary 13 *Graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is connected if and only if its geometrical multiplicity of the zero Z-eigenvalue of its Laplacian tensor is one.*

By Proposition 7, the Laplacian tensor T of a graph $\mathcal{G}$ is positive semidefinite. By (Qi 2005, Theorem 5), T is positive semidefinite if and only if all its Z-eigenvalues are nonnegative. Thus, using these and Theorem 10, we could order all the Z-eigenvalues of T with multiplicity as:

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_b.$$

It is easy to see that

$$\begin{aligned} \lambda_b = \max \quad & T\mathbf{x}^4 \\ \text{s.t.} \quad & \|\mathbf{x}\|_2 = 1 \end{aligned} \tag{9}$$

by Theorem 9 since (8) and (9) have the same critical points. By Cartwright and Sturmfels (2011), Ni et al. (2007), we know that

$$1 \leq b \leq \frac{3^n - 1}{2}.$$

So it is not vacuous to talk about λ_1. As in the literature (Chung 1997; Fiedler 1973), we introduce the following concept.

Definition 14 We call λ_1 the *algebraic connectivity* of graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, denoted as $\alpha(\mathcal{G})$.

Corollary 15 *For a graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, $\alpha(\mathcal{G}) > 0$ *if and only if* $\mathrm{Dim}(\mathbb{S}_0) = 1$.

Here we give a variational characterization of $\alpha(\mathcal{G})$.

Theorem 16 *For any graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, *we have that*

$$\begin{aligned} \alpha(\mathcal{G}) := \lambda_1 = \min\ & T\mathbf{x}^4 \\ \text{s.t.}\quad & \|\mathbf{x}\|_2 = 1, \quad \mathbf{e}^T\mathbf{x} = 0. \end{aligned} \tag{10}$$

Proof The result is true when $\mathcal{G}$ is disconnected. Since then $\alpha(\mathcal{G}) = 0$ by Corollary 15. Let $\mathbb{X} \subset \mathbb{V}$ be one of the connected components of graph $\mathcal{G}$, and $\mathbf{y} := \sum_{i\in\mathbb{X}} \mathbf{e}_i$. We have an orthogonal decomposition of $\mathbf{y}$ as $\mathbf{y} = \beta\mathbf{e} + \mathbf{x}$ such that $\mathbf{e}^T\mathbf{x} = 0$. Actually, $\beta = \frac{|\mathbb{X}|}{n}$ and $\mathbf{x} = (\sum_{i\in\mathbb{X}} \frac{n-|\mathbb{X}|}{n}\mathbf{e}_i - \sum_{i\notin X} \frac{|X|}{n}\mathbf{e}_i)$. Now, $T\mathbf{x}^4 = T\mathbf{y}^4 = 0$ by Lemmas 6 and 11. This, together with the positive semidefiniteness of T by Proposition 7, implies that the optimal value of minimization problem (10) is actually $\alpha(\mathcal{G}) = 0$.

In the following, we assume that $\mathcal{G}$ is connected.

We first show that a global minimizer $\mathbf{x}$ of minimization problem (10) is indeed a Z-eigenvector of T. By the first order necessary optimality condition, the minimizer $\mathbf{x}$ of (10) satisfies $\|\mathbf{x}\|_2 = 1$ and

$$T\mathbf{x}^3 = \kappa\mathbf{x} + \nu\mathbf{e}$$

with some $\kappa \in \Re$ and $\nu \in \Re$. Taking inner products of the both sides with $\mathbf{e}$, we get

$$\begin{aligned} n\nu &= \kappa\mathbf{x} \bullet \mathbf{e} + \nu\mathbf{e} \bullet \mathbf{e} \\ &= \mathbf{e} \bullet T\mathbf{x}^3 \\ &= \mathbf{e} \bullet \left[\sum_{\mathbb{E}_p\in\mathbb{E}} L(\mathbb{E}_p)\mathbf{x}^3\right] \\ &= \mathbf{e} \bullet \left[\frac{1}{84}\sum_{\mathbb{E}_p\in\mathbb{E}}\sum_{s=1}^{4}(\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{x})^3\mathbf{u}^s_{\mathbb{E}_p}\right] \\ &= \frac{1}{84}\sum_{\mathbb{E}_p\in\mathbb{E}}\sum_{s=1}^{4}(\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{x})^3(\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{e}) \\ &= 0. \end{aligned}$$

Here the first equality follows from the fact that $\mathbf{x} \bullet \mathbf{e} = 0$, the fourth from Lemma 6, and the last from the fact that $\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{e} = 0$ by the definition of $\mathbf{u}^s_{\mathbb{E}_p}$ in Lemma 6. Hence, $\nu = 0$, and then $T\mathbf{x}^3 = \kappa\mathbf{x}$. So, $\mathbf{x}$ is a Z-eigenvector of T with Z-eigenvalue $\kappa = p^*$. Here we denote by p^* the optimal value of the minimization problem (10). Furthermore, by the hypothesis of that $\mathcal{G}$ is connected, Theorem 10 and Corollary 13, we get that $p^* > 0$.

Then, we prove that if $\mathbf{y} \in \Re^n$ with $\mathbf{y}^T\mathbf{y} = 1$ is a Z-eigenvector of T with Z-eigenvalue $\lambda > 0$, then $\lambda \geq p^*$. Hence, by the definition of algebraic connectivity of graph $\mathcal{G}$, $\alpha(\mathcal{G}) = \lambda_1 = p^*$.

To this end, suppose that $\mathbf{y} \in \Re^n$ with $\mathbf{y}^T\mathbf{y} = 1$ is a Z-eigenvector of T with Z-eigenvalue $\lambda > 0$. We have an orthogonal decomposition of $\mathbf{y}$ as $\mathbf{y} = \beta\mathbf{e} + \mathbf{x}$ for some $\beta \in \Re$ and $\mathbf{x} \in \Re^n$ with $\mathbf{x}^T\mathbf{e} = 0$ and $\mathbf{x} \neq 0$ by Corollary 13 and the assumption $\lambda > 0$. Moreover, we have

$$\begin{aligned}
T\mathbf{y}^3 &= \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)\mathbf{y}^3 \\
&= \sum_{\mathbb{E}_p \in \mathbb{E}} \sum_{s=1}^{4} \left(\mathbf{u}^s_{\mathbb{E}_p} \otimes \mathbf{u}^s_{\mathbb{E}_p} \otimes \mathbf{u}^s_{\mathbb{E}_p} \otimes \mathbf{u}^s_{\mathbb{E}_p}\right)\mathbf{y}^3 \\
&= \sum_{\mathbb{E}_p \in \mathbb{E}} \sum_{s=1}^{4} \left(\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{y}\right)^3 \mathbf{u}^s_{\mathbb{E}_p} \\
&= \sum_{\mathbb{E}_p \in \mathbb{E}} \sum_{s=1}^{4} \left(\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{x}\right)^3 \mathbf{u}^s_{\mathbb{E}_p},
\end{aligned}$$

and $T\mathbf{y}^3 = \lambda(\beta\mathbf{e} + \mathbf{x})$. Taking inner products of the both sides with $\mathbf{e}$, we get $0 = \lambda\beta n + 0$ since $\mathbf{u}^s_{\mathbb{E}_p} \bullet \mathbf{e} = 0$ by the definition of $\mathbf{u}^s_{\mathbb{E}_p}$ in Lemma 6. So, $\beta = 0$ as $\lambda > 0$. Hence, $\mathbf{y} = \mathbf{x}$ and $\mathbf{x}^T\mathbf{e} = 0$. That is to say $\mathbf{y}$ is feasible for minimization problem (10). By the fact that $\lambda = T\mathbf{y}^4$, we conclude that $\lambda \geq p^*$. □

Remark 17 Here are several remarks.

- Similar results for Theorem 16 are true for Laplacian matrices, namely the Courant-Fischer Theorem (Horn and Johnson 1985). Nevertheless, Theorem 16 is not true for general tensors, even for general positive semidefinite tensors. One reason why Theorem 16 is true is that the Z-eigenvalue problem (7) has the property of orthogonally transformational invariance (Qi 2005, 2007). It is worth to note that another eigenvalue problem (H-eigenvalue) introduced in Qi (2005, 2007) does not have the property of orthogonally transformational invariance. Hence, there are fundamental differences between the spectral theory of tensors and that of matrices.
- For usual graphs, similar results of Theorem 16 (Chung 1997) imply

$$\alpha(\mathcal{G}) = \inf_{\mathbf{x}\perp\mathbf{e}, \mathbf{x}\neq 0} \frac{M\mathbf{x}^2}{\|\mathbf{x}\|_2^2} = \inf_{\mathbf{x}\perp\mathbf{e}, \mathbf{x}\neq 0} \frac{\sum_{\{i,j\}=\mathbb{E}_p\in\mathbb{E}}(x_i - x_j)^2}{\sum_{i=1}^n x_i^2} \tag{11}$$

with $M = D(\mathcal{G}) - A(\mathcal{G})$ as the Laplacian matrix (Merris 1994) and $M\mathbf{x}^2 := \mathbf{x}^T M\mathbf{x}$, which corresponds to the eigenvalue problem of the Laplace-Beltrami operator in

Riemannian manifolds of the following form:

$$\lambda_M := \inf \frac{\int_M |\nabla h|^2}{\int_M |h|^2}, \tag{12}$$

where h ranges over functions satisfying $\int_M h = 0$. Here the measure on edges $\mathbb{E}_p \in \mathbb{E}$ and vertices $i \in \mathbb{V}$ is 1. In an equivalent form,

$$\lambda_M := \inf \int_M |\nabla h|^2,$$

where h ranges over functions satisfying $\int_M h = 0$ and $\int_M |h|^2 = 1$. One of the generalizations to fourth order is:

$$\lambda_T := \inf \int_T |\nabla h|^4,$$

where h ranges over functions satisfying $\int_T h = 0$ and $\int_T |h|^2 = 1$. When it is discrete, the resulting problem is actually (10). This is one of our motivations to define the core tensors in Definition 2.

- If we modify the constraint $\int_T |h|^2 = 1$ into $\int_T |h|^4 = 1$, the resulting eigenvalue problem is actually the H-eigenvalue problem: a pair $(\lambda, \mathbf{x})$ is an H-eigenpair of T if the follows hold:

$$\begin{cases} T\mathbf{x}^3 = \lambda \mathbf{x}^{[3]}, \\ \lambda \in \Re, \quad \mathbf{x} \in \Re^n \setminus \{\mathbf{0}\}. \end{cases} \tag{13}$$

  λ is called an H-eigenvalue and $\mathbf{x}$ is the associated H-eigenvector. Here $\mathbf{x}^{[3]} := (x_1^3, \ldots, x_n^3)^T$.
- As mentioned before, H-eigenvalues and H-eigenvectors are not orthogonal transformation invariant (Qi 2005, 2007) even for Laplacian tensors which have special structure. Nevertheless, (13) is a more natural generalization of (11) and (12) to higher order case. Maybe, it has a closer relationship with similar operators in Finsler geometry like (11) with the Laplace-Beltrami operator in Riemannian manifolds.

Lemma 18 *Let T be the Laplacian tensor of graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ and $\alpha(\mathcal{G})$ be the algebraic connectivity of $\mathcal{G}$. We have*

$$\alpha(\mathcal{G}) \le \frac{2n^2}{n^2 - 2} \min_{1 \le i \le n} d_i. \tag{14}$$

Proof Denote by the feasible solution set of (10) as $\mathbb{F} := \{\mathbf{x} \in \Re^n \mid \|\mathbf{x}\|_2 = 1, \mathbf{e}^T\mathbf{x} = 0\}$. Now, for any $\mathbf{y} \in \mathcal{S}^{n-1} := \{\mathbf{y} \in \Re^n \mid \|\mathbf{y}\|_2 = 1\}$, we could get a decomposition of $\mathbf{y}$ as $\mathbf{y} = c_1\mathbf{e} + c_2\mathbf{x}$ for some $c_1, c_2 \in \Re$ and $\mathbf{x} \in \mathbb{F}$. So,

$$\left[2T - \alpha(\mathcal{G})\left(I \otimes I - \frac{2}{n^2}\mathbf{e} \otimes \mathbf{e} \otimes \mathbf{e} \otimes \mathbf{e}\right)\right]\mathbf{y}^4$$

$$
\begin{aligned}
&= 2T(c_2\mathbf{x})^4 - \alpha(\mathcal{G})\left[(nc_1^2 + c_2^2\|\mathbf{x}\|_2^2)^2 - 2c_1^4 n^2\right] \\
&\geq 2T(c_2\mathbf{x})^4 - \alpha(\mathcal{G})\left[2(n^2c_1^4 + c_2^4\|\mathbf{x}\|_2^4) - 2c_1^4 n^2\right] \\
&= 2T(c_2\mathbf{x})^4 - 2c_2^4\alpha(\mathcal{G}) \\
&\geq 0
\end{aligned}
$$

for any $y \in \mathcal{S}^{n-1}$. Here the first inequality follows from the facts that $\alpha(\mathcal{G}) \geq 0$ and $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b \in \Re$, and the second from the fact that $\mathbf{x} \in \mathbb{F}$ and Theorem 16. Hence, tensor $W := 2T - \alpha(\mathcal{G})(I \otimes I - \frac{2}{n^2}\mathbf{e} \otimes \mathbf{e} \otimes \mathbf{e} \otimes \mathbf{e})$ is positive semidefinite. Especially, the diagonal elements of tensor W are nonnegative. So,

$$
\min_{1 \leq i \leq n} W_{iiii} = 2 \min_{1 \leq i \leq n} T_{iiii} - \alpha(\mathcal{G})\left(1 - \frac{2}{n^2}\right) \geq 0
$$

which, together with Definition 1, implies (14) directly. □

Lemma 19 *Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a graph, and $\mathcal{G}'$ be a graph by removing a vertex from $\mathcal{G}$ and all adjacent edges. Then,*

$$
\alpha(\mathcal{G}') \geq \alpha(\mathcal{G}) - \frac{(|\mathbb{V}| - 1)^3}{2}. \tag{15}
$$

Proof Let $\mathcal{G}_1$ be the graph by adding a vertex to $\mathcal{G}'$ and all the possible adjacent edges. Then $\mathcal{G}$ is a subgraph of $\mathcal{G}_1$. Denote by $\mathbb{F}(p) := \{\mathbf{x} \in \Re^p \mid \|\mathbf{x}\|_2 = 1, \mathbf{e}^T\mathbf{x} = 0\}$ for $p \geq 4$. Let W be the Laplacian tensor of graph $\mathcal{G}_1$, and A be the Laplacian tensor of graph $\mathcal{G}$. Let $\bar{\mathbb{E}}$ be the set of edges of graph $\mathcal{G}_1$. Then, $\mathbb{E} \subseteq \bar{\mathbb{E}}$ by the construction of $\mathcal{G}_1$. Now, by Theorem 16, $\alpha(\mathcal{G}) = \min\{A\mathbf{x}^4 \mid \mathbf{x} \in \mathbb{F}(|\mathbb{V}(\mathcal{G})|)\}$, and $\alpha(\mathcal{G}_1) = \min\{W\mathbf{x}^4 \mid \mathbf{x} \in \mathbb{F}(|\mathbb{V}(\mathcal{G})|)\}$. While,

$$
\begin{aligned}
A\mathbf{x}^4 &= \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)\mathbf{x}^4, \quad \text{and} \\
W\mathbf{x}^4 &= \sum_{\mathbb{E}_p \in \bar{\mathbb{E}}} L(\mathbb{E}_p)\mathbf{x}^4 = \sum_{\mathbb{E}_p \in \mathbb{E}} L(\mathbb{E}_p)\mathbf{x}^4 + \sum_{\mathbb{E}_p \in \bar{\mathbb{E}} \setminus \mathbb{E}} L(\mathbb{E}_p)\mathbf{x}^4.
\end{aligned}
$$

So, $A\mathbf{x}^4 \leq W\mathbf{x}^4$ for any $\mathbf{x} \in \mathbb{F}(|\mathbb{V}(\mathcal{G})|)$ since every $L(\mathbb{E}_p)$ is positive semidefinite by Lemma 6. Hence,

$$
\alpha(\mathcal{G}_1) \geq \alpha(\mathcal{G}). \tag{16}
$$

Now, let T be the Laplacian tensor of graph $\mathcal{G}'$. Then, by Theorem 16, $\alpha(\mathcal{G}') = T\mathbf{x}^4$ for some $\mathbf{x} \in \mathbb{F}(|\mathbb{V}(\mathcal{G})| - 1)$. Let $l \in \mathbb{V}(\mathcal{G})$ be the removed vertex and $\mathbf{y} \in \Re^{|\mathbb{V}(\mathcal{G})|}$ with $\mathbf{y}_{\mathbb{V}(\mathcal{G})-\{l\}} = \mathbf{x}$ and $y_l = 0$, we have

$$
W\mathbf{y}^4 = T\mathbf{x}^4 + \sum_{\mathbb{E}_p \in \mathbb{M}} L(\mathbb{E}_p)\mathbf{y}^4,
$$

where $\mathbb{M} = \{\mathbb{E}_p \mid \mathbb{E}_p = \{i, j, k, l\},\ i, j, k \in \mathbb{V}(\mathcal{G}')\}$. For any $\{i, j, k, l\} = \mathbb{E}_p \in \mathbb{M}$, we have that

$$\begin{aligned}
L(\mathbb{E}_p)\mathbf{y}^4 &= \frac{1}{84}[(x_i + x_j + x_k)^4 + (x_i + x_j - 3x_k)^4 + (x_i + x_k - 3x_j)^4 \\
&\quad + (x_j + x_k - 3x_i)^4] \\
&= \frac{1}{84}[84(x_i^4 + x_j^4 + x_k^4) - 112(x_i^3 x_j + x_i^3 x_k + x_j^3 x_i + x_j^3 x_k + x_k^3 x_i + x_k^3 x_j) \\
&\quad + 120(x_i^2 x_j^2 + x_j^2 x_k^2 + x_k^2 x_i^2) + 48(x_i^2 x_j x_k + x_j^2 x_i x_k + x_k^2 x_i x_j)].
\end{aligned}$$

Denote by $q := |\mathbb{V}(\mathcal{G}')| = |\mathbb{V}(\mathcal{G})| - 1 \geq 3$ as assumed in Introduction, we have that $|\mathbb{M}| = \binom{q}{3}$, and

$$\begin{aligned}
\frac{1}{3}\sum_{\mathbb{E}_p \in \mathbb{M}} L(\mathbb{E}_p)\mathbf{y}^4 &= \binom{q}{3}\sum_{i=1}^{q} x_i^4 - 2\binom{q-2}{1}\frac{112}{84}\sum_{i=1}^{q} x_i^3\Big(\sum_{j \neq i} x_j\Big) \\
&\quad + \binom{q-2}{1}\frac{120}{84}\sum_{i=1}^{q} x_i^2\Big(\sum_{j \neq i} x_j^2\Big) + \frac{1}{2}\frac{48}{84}\sum_{i \neq j} x_i x_j \sum_{k \neq i,\, k \neq j} x_k^2 \\
&= \binom{q}{3}\sum_{i=1}^{q} x_i^4 + 2\binom{q-2}{1}\frac{112}{84}\sum_{i=1}^{q} x_i^3 x_i \\
&\quad + \binom{q-2}{1}\frac{120}{84}\sum_{i=1}^{q} x_i^2(1 - x_i^2) + \frac{24}{84}\sum_{i \neq j} x_i x_j (1 - x_i^2 - x_j^2) \\
&= \binom{q}{3}\sum_{i=1}^{q} x_i^4 + 2\binom{q-2}{1}\frac{112}{84}\sum_{i=1}^{q} x_i^3 x_i \\
&\quad - \binom{q-2}{1}\frac{120}{84}\sum_{i=1}^{q} x_i^4 + \binom{q-2}{1}\frac{120}{84} \\
&\quad + \frac{24}{84}\sum_{i \neq j}(x_i x_j - x_i^3 x_j - x_i x_j^3) \\
&= \binom{q}{3}\sum_{i=1}^{q} x_i^4 + 2\binom{q-2}{1}\frac{112}{84}\sum_{i=1}^{q} x_i^3 x_i \\
&\quad - \binom{q-2}{1}\frac{120}{84}\sum_{i=1}^{q} x_i^4 + \binom{q-2}{1}\frac{120}{84} \\
&\quad - \frac{48}{84}\sum_{i=1}^{q} x_i^2 + 4\frac{24}{84}\sum_{i=1}^{q} x_i^4
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{q(q-1)(q-2)}{6} + \frac{224(q-2) - 120(q-2) + 96}{84} \right) \sum_{i=1}^{q} x_i^4 \\
&\quad + \frac{120(q-2) - 48}{84} \\
&\le \frac{q(q-1)(q-2)}{6} \\
&\quad + \frac{224(q-2) - 120(q-2) + 96 + 120(q-2) - 48}{84} \\
&= \frac{q(q-1)(q-2)}{6} + \frac{224(q-2) + 48}{84} \\
&= \frac{q^3}{6} + \frac{-42q^2 + 252q - 400}{84} \\
&\le \frac{q^3}{6},
\end{aligned}
$$

where the second and the fourth equalities follow from the fact that $\|\mathbf{x}\|_2 = 1$ and $\mathbf{x}^T\mathbf{e} = 0$, the first inequality from the fact that $\|\mathbf{x}\|_4 \le \|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_2 = 1$, and the last inequality from the fact that $-42q^2 + 252q - 400 < 0$ for $q \ge 3$.

So, by the fact that $\|\mathbf{y}\|_2 = \|\mathbf{x}\|_2$, $\sum_{i\in\mathbb{V}(\mathcal{G})} y_i = \mathbf{e}^T\mathbf{x} = 0$ and Theorem 16,

$$
\alpha(\mathcal{G}_1) \le W\mathbf{y}^4 \le \alpha(\mathcal{G}') + \frac{q^3}{2}. \tag{17}
$$

Hence, (17), together with (16), implies (15). □

The following is a direct corollary from Lemma 19.

Corollary 20 *Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a graph, and $\mathcal{G}'$ be a graph by removing $k \le n := |\mathbb{V}(\mathcal{G})|$ vertices from $\mathcal{G}$ and all adjacent edges. Then,*

$$
\alpha(\mathcal{G}') \ge \alpha(\mathcal{G}) - \frac{k}{2}(n-1)^3.
$$

3 Applications

In this section, we discuss some issues of graphs that relate to its algebraic connectivity. Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a graph. The edge cut means: given any nonempty proper subset $\mathbb{X} \subset \mathbb{V}$, the edge cut of $\mathbb{X}$ is the set of edges

$$
\mathbb{E}_{\mathbb{X}} := \{\mathbb{E}_p \in \mathbb{E} \mid \exists i \in \mathbb{X},\ \exists j \notin \mathbb{X},\ \text{s.t. } \{i, j\} \subset \mathbb{E}_p\}.
$$

The edge connectivity of $\mathcal{G}$, denoted by $e(\mathcal{G})$, is defined as the minimum cardinality of $\mathbb{E}_{\mathbb{X}}$ over all nonempty proper subsets $\mathbb{X}$ of $\mathbb{V}$ such that the resulting graph is disconnected.

Lemma 21 *Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a graph, T be its Laplacian tensor, $\alpha(\mathcal{G})$ be its algebraic connectivity and λ_b be the largest Z-eigenvalue of T. Then, for all $\mathbb{X} \subset \mathbb{V}$*

$$\frac{|\mathbb{X}|^2(n-|\mathbb{X}|)^2}{n^2}\alpha(\mathcal{G}) \leq |\mathbb{E}_{\mathbb{X}}| \leq \frac{21|\mathbb{X}|^2(n-|\mathbb{X}|)^2}{16n^2}\lambda_b. \tag{18}$$

Proof Let $\mathbb{X}$ be a nonempty proper subset of $\mathbb{V}$ and $\mathbb{E}_{\mathbb{X}}$ its associated edge cut. Let $\mathbf{x} := \sum_{i\in\mathbb{X}} \mathbf{e}_i$, we have an orthogonal decomposition of $\mathbf{x}$ as $\mathbf{x} = \beta\mathbf{e} + \mathbf{g}$ such that $\mathbf{e}^T\mathbf{g} = 0$. Actually, $\beta = \frac{|\mathbb{X}|}{n}$ and $\mathbf{g} = (\sum_{i\in\mathbb{X}} \frac{n-|\mathbb{X}|}{n}\mathbf{e}_i - \sum_{i\notin\mathbb{X}} \frac{|\mathbb{X}|}{n}\mathbf{e}_i)$. So,

$$\begin{aligned} T\mathbf{g}^4 &= T\mathbf{x}^4 \\ &= \sum_{\{i,j,k,l\}=\mathbb{E}_p\in\mathbb{E}_{\mathbb{X}}} \frac{1}{84}[(x_i+x_j+x_k-3x_l)^4 + (x_i+x_j+x_l-3x_k)^4 \\ &\quad + (x_i+x_k+x_l-3x_j)^4 + (x_j+x_k+x_l-3x_i)^4]. \end{aligned}$$

For every $\{i, j, k, l\} = \mathbb{E}_p \in \mathbb{E}_{\mathbb{X}}$, there are three situations:

- Three of $\{x_i, x_j, x_k, x_l\}$ are zero and one of them is 1.
- Two of $\{x_i, x_j, x_k, x_l\}$ are zero and two of them are 1.
- One of $\{x_i, x_j, x_k, x_l\}$ is zero and three of them are 1.

So, we have

$$\frac{16}{21} \leq L(\mathbb{E}_p)\mathbf{x}^4 \leq 1$$

by a direct computation for the three cases. Thus,

$$\frac{16|\mathbb{E}_{\mathbb{X}}|}{21} \leq T\mathbf{g}^4 \leq |\mathbb{E}_{\mathbb{X}}|.$$

Hence, by (9), Theorem 16 and the fact that $\|\mathbf{g}\|_2^2 = \frac{|\mathbb{X}|(n-|\mathbb{X}|)}{n}$, we get that

$$\frac{16|\mathbb{E}_{\mathbb{X}}|}{21} \leq \frac{|\mathbb{X}|^2(n-|\mathbb{X}|)^2}{n^2}\lambda_b, \quad \text{and} \quad \frac{|\mathbb{X}|^2(n-|\mathbb{X}|)^2}{n^2}\alpha(\mathcal{G}) \leq |\mathbb{E}_{\mathbb{X}}|. \tag{19}$$

Now, (19) implies (18) directly. □

Here we give an intuitive example for the lower bound in Lemma 21.

Example 22 Consider graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ with vertices set $\mathbb{V} = \{1, 2, 3, 4, 5\}$ and edges set $\mathbb{E} = \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}\}$. Since $|\mathbb{E}_{\mathbb{X}}|$ is easy to compute when $|\mathbb{X}| = 1$ for any graphs. We consider the more nontrivial cases. The lower bound for $|\mathbb{E}_{\mathbb{X}}|$ when $|\mathbb{X}| = 2$ provided by Lemma 21 is $\frac{36}{25}\alpha(\mathcal{G})$. It is easy to see that $|\mathbb{E}_{\mathbb{X}}| = 5$ when $|\mathbb{X}| = 2$. It is difficulty to solve minimization problem (10), so we randomly select 100000 points in the feasible set of (10) to get an approximate $\alpha(\mathcal{G}) = 2.98$. Then, the lower bound computed is 4.29. Since $|\mathbb{E}_{\mathbb{X}}|$ is an integer, we see that the computed lower bound is tight.

The following result is a direct corollary from Lemma 21.

Theorem 23 *Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a graph, T be its Laplacian tensor, $\alpha(\mathcal{G})$ be the algebraic connectivity of $\mathcal{G}$, λ_b be the largest Z-eigenvalue of T, and $e(\mathcal{G})$ be the edge connectivity of $\mathcal{G}$. Then,*

$$\frac{(n-1)^2}{n^2}\alpha(\mathcal{G}) \leq e(\mathcal{G}) \leq \begin{cases} \frac{21n^2}{256}\lambda_b & \text{if } n \text{ is even,} \\ \frac{21(n^2-1)^2}{256n^2}\lambda_b & \text{if } n \text{ is odd.} \end{cases}$$

The vertex connectivity of $\mathcal{G}$, denoted by $v(\mathcal{G})$, is defined as the minimum cardinality of $\mathbb{X} \subset \mathbb{V}$ such that the resulting graph by removing vertices in $\mathbb{X}$ and their associated edges is disconnected.

Theorem 24 *Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a graph, $\alpha(\mathcal{G})$ be the algebraic connectivity of $\mathcal{G}$ and $v(\mathcal{G})$ be the vertex connectivity of $\mathcal{G}$. We have*

$$\alpha(\mathcal{G}) \leq v(\mathcal{G})\frac{(n-1)^3}{2}.$$

Proof Let $\mathbb{X}$ be a subset of vertices such that $\mathbb{X}$ is the vertex cut to disconnect graph $\mathcal{G}$. Then $|\mathbb{X}| = v(\mathcal{G})$, and the resulting graph is disconnected. Hence, its algebraic connectivity is zero by Corollaries 13 and 15. Thus, the result follows from Corollary 20 directly. □

4 Conclusion

We introduced in this paper the Laplacian tensor for an even uniform hypergraph, and the algebraic connectivity through the concept of Z-eigenvalues of tensors. We established several properties of algebraic connectivity for an even hypergraph and its connections with edge connectivity and vertex connectivity.

It is far away from a comprehensive and a complete discussion of spectral theory for hypergraphs like these for graphs (Chung 1997). It is just an initial step towards the investigation of spectral theory for hypergraphs based on eigenvalues of tensors introduced in Qi (2005). Besides the spectral theory for hypergraphs, the class of tensors with structure similar to Laplacian tensors introduced in this paper has its own interest. It has already shown some properties (e.g., Theorem 16) that are not true for general tensors. Furthermore, this class of tensors is related to sums of powers of real linear forms (Reznick 1992) and belongs to the class of Cholesky decomposable tensors introduced in Lim (2011), which has significant applications in medical imaging.

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