Some Spectral Properties of Odd-Bipartite Z-Tensors and Their Absolute Tensors

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Abstract Stimulated by odd-bipartite and even-bipartite hypergraphs, we define odd-bipartite (weakly odd-bipartie) and even-bipartite (weakly even-bipartite) tensors. It is verified that all even order odd-bipartite tensors are irreducible tensors, while all even-bipartite tensors are reducible no matter the parity of the order. Based on properties of odd-bipartite tensors, we study the relationship between the largest H-eigenvalue of a Z-tensor with nonnegative diagonal elements, and the largest H-eigenvalue of absolute tensor of that Z-tensor. When the order is even and the Z-tensor is weakly irreducible, we prove that the largest H-eigenvalue of the Z-tensor and the largest H-eigenvalue of the absolute tensor of that Z-tensor are equal, if and only if the Z-tensor is weakly odd-bipartite. Examples show the authenticity of the conclusions. Then, we prove that a symmetric Z-tensor with nonnegative diagonal entries and the absolute tensor of the Z-tensor are diagonal similar, if and only if the Z-tensor has even order and it is weakly odd-bipartite. After that, it is proved that, when an even order symmetric Z-tensor with nonnegative diagonal entries is weakly irreducible, the equality of the spectrum of the Z-tensor and the spectrum of absolute tensor of that Z-tensor, can be characterized by the equality of their spectral radii.

Keywords H-eigenvalue, Z-tensor, odd-bipartite tensor, absolute tensor
MSC 90C30, 15A06

1 Introduction

Since the early work of [12] and [17], more and more researchers are interested in studying eigenvalue problems of tensors in the past several years [1, 3, 4, 5, 6, 9, 10, 14, 15, 16, 18, 19, 20, 21, 24, 25]. In [17], two kinds of eigenvalues are defined for real symmetric tensors: eigenvalues and E-eigenvalues. An eigenvalue (E-

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eigenvalue) with a real eigenvector (E-eigenvector) is called an H-eigenvalue (Z-eigenvalue). When a symmetric tensor has even order, H-eigenvalues and Z-eigenvalues always exist. An even order symmetric tensor is positive definite (semi-definite) if and only if all of its H-eigenvalues or all of its Z-eigenvalues are positive (nonnegative). Based upon this property, an H-eigenvalue method for the positive definiteness (positive semi-definiteness respectively) identification problem is developed.

The main difficulty in tensor problems is that they are generally nonlinear. Because of the difficulties in studying the properties of a general tensor, researchers focus on some structured tensors. Z-tensors are an important class of structured tensors and have been well studied [7, 13, 26]. They are closely related with spectral graph theory, the stationary distribution of Markov chains and the convergence of iterative methods for linear systems.

Recently, in [10], Hu et al. considered the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue of a k-uniform connected hypergraph. When the order is even and the hypergraph is odd-bipartite, they proved that the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue are equal. For the odd order case, it is proved that the largest Laplacian H-eigenvalue is strictly less than the largest signless Laplacian H-eigenvalue [10]. Later, Shao et al. [23] gave several spectral characterizations of the connected odd-bipartite hypergraphs. They proved that the spectrum of the Laplacian tensor and the spectrun of the signless Laplacian tensor of an uniform hypergraph are equal if and only if the hypergraph is an even order connected odd-bipartite hypergraph. Since the Laplacian tensor is a special case of Z-tensors and the signless Laplacian tensor is a special case of the absolute tensors of Z-tensors, questions comes naturally: what is the relation between the largest H-eigenvalue of a general Z-tensor, and the largest H-eigenvalue of the Z-tensor's absolute tensor? What is the relation between spectrums of a general Z-tensor and its absolute tensor? These constitute main motivations of the paper.

In this article, some spectral properties of Z-tensors with nonnegative diagonal entries, and absolute tensors of Z-tensors are studied. The rest of this paper is organized as follows. In Section 2, some basic notions and preliminaries of tensors are presented. In Section 3, stimulated by odd-bipartite and even-bipartite hypergraphs [9], odd-bipartite (weakly odd-bipartite) and even-bipartite (weakly even-bipartite) tensors are defined. Odd-bipartite (even-bipartite) tensors are weakly odd-bipartite (weakly even-bipartite) tensors. Examples show that the converse, generally, may not hold. A square odd-bipartite matrix is irreducible. For high order tensors, we prove that an even order odd-bipartite tensor is irreducible, while a tensor is reducible if it is even-bipartite no matter the parity of the order.

In Section 4, we study the relation between the largest H-eigenvalue of a Z-tensor with nonnegative diagonal entries, and the largest H-eigenvalue of the Z-tensor's absolute tensor. For an even order Z-tensor with nonnegative diagonal entries, if it is weakly irreducible, we show that the largest H-eigenvalues of the

Z-tensor and its absolute tensor are equal if and only if the Z-tensor is weakly odd-bipartite. For the odd order case, sufficient conditions for the equality of these largest H-eigenvalues are given. Examples show the authenticity of the conclusions. In Section 5, we prove that, when an even order symmetric Z-tensor with nonnegative diagonal entries is weakly irreducible, its spectrum and the spectrum of its absolute tensor are equal if and only if the Z-tensor is odd-bipartite. Furthermore, it is shown that the equality of the spectrum of a symmetric Z-tensor with nonnegative diagonal entries, and the spectrum of the absolute tensor of that Z-tensor, can be characterized by the equality of their spectral radii. We conclude this paper with some final remarks in Section 6.

By the end of the introduction, we add some comments on notation that will be used in the sequel. Let $\mathbb{R}^n$ be the n dimensional real Euclidean space and the set consisting of all natural numbers is denoted by $\mathbb{N}$. Suppose $m, n \in \mathbb{N}$ are two natural numbers. Denote $[n] = \{1, 2, \cdots, n\}$. Vectors are denoted by italic lowercase letters i.e. $x,\ y, \cdots$, and tensors are written as calligraphic capitals such as $\mathscr{A}, \mathscr{T}, \cdots$. The i-th unit coordinate vector in $\mathbb{R}^n$ is denoted by e_i. Let $|V|$ denote the number of elements when the symbol $|\cdot|$ be used on a subset $V \subseteq \mathbb{N}$. If the symbol $|\cdot|$ is used on a tensor $\mathscr{A} = (a_{i_1\cdots i_m})_{1\leqslant i_j\leqslant n}$, $j = 1, \cdots, m$, we get another tensor $|\mathscr{A}| = (|a_{i_1\cdots i_m}|)_{1\leqslant i_j\leqslant n},\ j = 1, \cdots, m$. If both $\mathscr{A} = (a_{i_1\cdots i_m})_{1\leqslant i_j\leqslant n}$ and $\mathscr{B} = (b_{i_1\cdots i_m})_{1\leqslant i_j\leqslant n},\ j = 1, \cdots, m$, are real mth order n dimensional tensors, then $\mathscr{A} \leqslant \mathscr{B}$ means $a_{i_1\cdots i_m} \leqslant b_{i_1\cdots i_m}$ for all $i_1, \cdots, i_m \in [n]$.

2 Preliminaries

In this section, we will review some basic notions of tensors. For more details, see [17] and the references therein.

A real mth order n-dimensional tensor $\mathscr{A} = (a_{i_1 i_2\cdots i_m})$ is a multi-array of real entries $a_{i_1 i_2\cdots i_m}$, where $i_j \in [n]$ for $j \in [m]$. If the entries $a_{i_1 i_2\cdots i_m}$ are invariant under any permutation of their indices, then tensor $\mathscr{A}$ is called a symmetric tensor.

The following definition on eigenvalue-eigenvector comes from [17].

Definition 1. *Let $\mathbb{C}$ be the complex field. A pair $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ is called an eigenvalue-eigenvector pair of $\mathscr{T}$, if they satisfy*

$$\mathscr{T}x^{m-1} = \lambda x^{[m-1]}, \tag{1}$$

where $\mathscr{T}x^{m-1}$ and $x^{[m-1]}$ are all n dimensional column vectors such as

$$\mathscr{T}x^{m-1} = \left(\sum_{i_2,\cdots,i_m=1}^{n} t_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1\leqslant i\leqslant n}$$

and $x^{[m-1]} = (x_i^{m-1})_{1\leqslant i\leqslant n}$.

For real tensor $\mathscr{T}$ and $x \in \mathbb{R}^n$ in (1), λ is a real number since $\lambda = \frac{(\mathscr{T}x^{m-1})_j}{x_j^{m-1}}$ for some j with $x_j \neq 0$. In this case, λ is called an H-eigenvalue of $\mathscr{T}$ and x is its corresponding H-eigenvector [17].

Next, we present a fundamental result which will be much used in the sequel.

Proposition 1. [17] *Suppose that $\mathscr{T} = a(\mathscr{B} + b\mathscr{I})$, where a and b are two real numbers. Then μ is an eigenvalue (H-eigenvalue) of tensor $\mathscr{T}$ if and only if $\mu = a(\lambda + b)$, where λ is an eigenvalue (H-eigenvalue) of tensor $\mathscr{B}$. In this case, they have the same eigenvectors (H-eigenvectors).*

The spectral radius of tensor $\mathscr{T}$ is denoted by

$$\rho(\mathscr{T}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathscr{T}\}.$$

All eigenvalues of tensor $\mathscr{T}$ construct the spectrum denoted by $\sigma(\mathscr{T})$.

3 Odd-bipartite and even-bipartite tensors

In this section, we first define odd-bipartite tensors and even-bipartite tensors. Then, some special characteristics of this kinds of tensors are shown.

Definition 2. *Assume $\mathscr{A} = (a_{i_1\cdots i_m})$ is a tensor with order m and dimension n. If there is a nonempty proper index subset $V \subset [n]$ such that*

$$a_{i_1\cdots i_m} \neq 0, \text{ when } |V \cap \{i_1, \cdots, i_m\}| \text{ is odd}$$

and $a_{i_1\cdots i_m} = 0$ for the others, then $\mathscr{A}$ is called an odd-bipartite tensor corresponding to set V or $\mathscr{A}$ is odd-bipartite for simple.

Here, we should note that indices of an edge $\{i_1, \cdots, i_m\}$ in hypergraph [9] are different from each other, which is a notable distinction to general tensors. So, in this paper, we define that $|V \cap \{i_1, \cdots, i_m\}|$ is the number of indices $V \cap \{i_1, \cdots, i_m\}$, and duplicate indices should be calculated. For example, suppose $V = \{1, 2, 3\}$ and $\mathscr{A}$ is a 4th order 6 dimensional tensor, then

$$|V \cap \{1, 1, 3, 3\}| = 4, \ |V \cap \{1, 2, 3, 5\}| = 3, \ |V \cap \{4, 6, 4, 5\}| = 0.$$

Definition 3. *Assume $\mathscr{A} = (a_{i_1\cdots i_m})$ is a tensor with order m and dimension n. $\mathscr{A}$ is called weakly odd-bipartite if there is a nonempty proper index subset $V \subset [n]$ such that*

$$a_{i_1\cdots i_m} = 0, \text{ when } |V \cap \{i_1, \cdots, i_m\}| \text{ is even.}$$

From Definitions 2 and 3, even-bipartite and weakly even-bipartite tensors can be defined similarly. Furthermore, we can easily prove that, if $\mathscr{A}$ is odd-bipartite (even-bipartite, respectively), then $\mathscr{A}$ is weakly odd-bipartite (weakly

even-bipartite respectively), but not vice versa. For example, suppose $\mathscr{A}$ is a 3rd order 2 dimensional tensor with entries such that

$$a_{222} = 1 \text{ and } a_{i_1i_2i_3} = 0$$

for the others. It is easy to check that $\mathscr{A}$ is weakly odd-bipartite corresponding to the index set $V = \{2\}$ but not odd-bipartite corresponding to $\{1\}$ or $\{2\}$.

When m is odd, for all $i_1, i_2, \cdots, i_m \in [n]$ and a nonempty proper index subset $V \subset [n]$, it holds that $|\{i_1, i_2, \cdots, i_m\} \cap V|$ is odd if and only if $|\{i_1, i_2, \cdots, i_m\} \cap \bar{V}|$ is even, where $\bar{V} = [n]\backslash V$. So, by Definitions 3.1 and 3.2, we can readily obtain the following conclusion.

Lemma 1. *Let $\mathscr{A}$ be a tensor with order m and dimension n. Assume m is odd. Then, $\mathscr{A}$ is odd-bipartite (or weakly odd-bipartite respectively) corresponding to nonempty proper index subset $V \subset [n]$ if and only if $\mathscr{A}$ is even-bipartite (or weakly even-bipartite respectively) corresponding to the nonempty proper index subset $\bar{V} = [n]\backslash V$.*

Irreducible tensors are a class of important and useful tensors, which have been repeatedly used in Perron Frobenius Theorem for nonnegative tensors [2, 24, 25]. Next, we will study the relation between irreducible tensors and odd-bipartite tensors. To do this, we first list the corresponding definition below.

Definition 4. [2] *For a tensor $\mathscr{T}$ with order m and dimension n. We call $\mathscr{T}$ is reducible if there is a nonempty proper index subset $V \subset [n]$ such that*

$$t_{i_1i_2\cdots i_m} = 0, \ \forall\ i_1 \in V, \ \forall\ i_2, i_3, \cdots, i_m \notin V.$$

Otherwise we call $\mathscr{T}$ is irreducible.

Theorem 1. *Let m be even. Assume tensor $\mathscr{A} = (a_{i_1\cdots i_m})$ with order m and dimension n is odd-bipartite. Then $\mathscr{A}$ is irreducible.*

Proof. Since $\mathscr{A}$ is odd-bipartite, there exists a nonempty proper index subset $V \subset [n]$ satisfying

$$a_{i_1\cdots i_m} \neq 0, \text{ when the number } |V \cap \{i_1, \cdots, i_m\}| \text{ is odd.} \tag{2}$$

By contradiction, suppose $\mathscr{A} = (a_{i_1\cdots i_m})$ is reducible, then there is a nonempty proper index subset $V_1 \subset [n]$ such that

$$a_{i_1\cdots i_m} = 0, \ \forall\ i_1 \in V_1, \ \forall\ i_2, \cdots, i_m \notin V_1. \tag{3}$$

We will break the proof into four cases.

(i) If $V_1 \subseteq V$, let $i_1 \in V_1, i_2, \cdots, i_m \notin V$. Here, several indices in $i_2, \cdots, i_m$ may equal to each other when the number of elements in $[n]\backslash V$ is strictly less than $m - 1$. Then, by (3) we have

$$a_{i_1\cdots i_m} = 0,$$

which contradicts (2) since $|V \cap \{i_1, \cdots, i_m\}| = 1$ is odd.

(ii) If $V \subseteq V_1$, let $i_1 \in V, i_2, \cdots, i_m \notin V_1$. Then, by (3) one has

$$a_{i_1 \cdots i_m} = 0,$$

which is a contradiction with (2).

(iii) If $V \cap V_1 \neq \emptyset$ and neither $V \subseteq V_1$ nor $V_1 \subseteq V$, let $i_1 \in V_1 \setminus V$, $i_2, \cdots, i_m \in V \setminus V_1$. Then it follows that

$$a_{i_1 \cdots i_m} = 0,$$

which also contradicts (2), since $|V \cap \{i_1, \cdots, i_m\}| = m - 1$ is a odd number.

(iv) If $V \cap V_1 = \emptyset$, let $i_1 \in V_1, i_2, \cdots, i_m \in V$. By Definition 4, we have

$$a_{i_1 \cdots i_m} = 0.$$

Since $|V \cap \{i_1, \cdots, i_m\}| = m - 1$ be odd, by (2), one has

$$a_{i_1 \cdots i_m} \neq 0,$$

which is a contradiction.

From (i) to (iv), we conclude that $\mathscr{A}$ is not reducible and the desired results follows. □

If a tensor $\mathscr{A}$ is even-bipartite, no matter the order of $\mathscr{A}$ is odd or even, we have the following result.

Theorem 2. *Assume tensor $\mathscr{A} = (a_{i_1 \cdots i_m})$ with order m and dimension n is even-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. Then $\mathscr{A}$ is reducible corresponding to V.*

Proof. By definitions of reducible tensors and even-bipartite tensors, the conclusion obviously holds. □

Suppose an even order Z-tensor and its absolute tensor are defined such that,

$$\mathscr{A} = \mathscr{D} - \mathscr{C}, \quad |\mathscr{A}| = \mathscr{D} + \mathscr{C}, \tag{4}$$

where $\mathscr{D}$ is a nonnegative diagonal tensor and $\mathscr{C}$ is a nonnegative tensor with zero diagonal entries. From Theorem 2, if $\mathscr{C}$ is odd-bipartite, then tensors $\mathscr{A}$ and $|\mathscr{A}|$ are irreducible. Combining this with Theorem 3.1 of [8] we have the following result.

Corollary 1. *Let m be even. Suppose tensor $\mathscr{A} = \mathscr{D} - \mathscr{C}$ with order m and dimension n is defined as in (4). Then, $\mathscr{A}$ and its absolute tensor $|\mathscr{A}|$ are both weakly irreducible if nonnegative tensor $\mathscr{C}$ is odd-bipartite.*

By the Perron-Frobenius theorem on nonnegative tensors [2] and by Theorem 4.1 of [8], the following result follows.

Corollary 2. *Let m be even. Assume tensor $\mathscr{A}$ is defined as in Corollary 1. If $\mathscr{C}$ is odd-bipartite, the largest H-eigenvalue of $|\mathscr{A}|$ is $\rho(|\mathscr{A}|)$. Furthermore, there exists a positive n dimensional eigenvector $x \in \mathbb{R}^n$ such that*

$$|\mathscr{A}|x^{m-1} = \rho(|\mathscr{A}|)x^{[m-1]}.$$

4 The relation between the largest H-eigenvalues of a Z-tensor and its absolute tensor

In this section, suppose an order m dimension n Z-tensor $\mathscr{A}$ with nonnegative diagonal elements has format

$$\mathscr{A} = \mathscr{D} - \mathscr{C}, \tag{5}$$

where $\mathscr{D}$ is a nonnegative diagonal tensor and $\mathscr{C}$ is a nonnegative tensor with zero diagonal elements. So the absolute format of $\mathscr{A}$ is $|\mathscr{A}| = \mathscr{D} + \mathscr{C}$. In the following analysis, entries of $\mathscr{A}$, $\mathscr{C}$ and $\mathscr{D}$ are always defined as below

$$\mathscr{A} = (a_{i_1\cdots i_m}),\ \mathscr{C} = (c_{i_1\cdots i_m}),\ \mathscr{D} = (d_{i_1\cdots i_m}),\ i_1, i_2, \cdots, i_m \in [n].$$

For the sake of simple, let $d_{ii\cdots i} = d_i$, $i \in [n]$.

During this part, we mainly study the relationship between the largest H-eigenvalue of a Z-tensor $\mathscr{A}$ in (5), and the largest H-eigenvalue of the absolute tensor of $\mathscr{A}$. Sufficient and necessary conditions or sufficient conditions to guarantee the equality of these largest H-eigenvalues are shown. It should be noted that all even order nonnegative tensors always have H-eigenvalues [24]. To proceed, we make an assumption in advance, all tensors considered in this part always have H-eigenvalues.

The largest H-eigenvalues of $\mathscr{A}$ and $|\mathscr{A}|$ are denoted by $\lambda(\mathscr{A})$ and $\lambda(|A|)$ respectively. From Corollary 2, we know that $\lambda(|A|) = \rho(|\mathscr{A}|)$.

Theorem 3. *Let m be even. Suppose $\mathscr{A} = \mathscr{D} - \mathscr{C}$ is defined as (5). Then,*

$$\lambda(\mathscr{A}) = \lambda(|A|)$$

if $\mathscr{C}$ is odd-bipartite.

Proof. By Lemma 13 of [19], we have

$$\lambda(\mathscr{A}) \leqslant \rho(\mathscr{A}) \leqslant \rho(|\mathscr{A}|) = \lambda(|\mathscr{A}|).$$

Thus, in order to prove the conclusion, we only need to prove

$$\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A}).$$

Since $\mathscr{C}$ is odd-bipartite, there exists a nonempty proper index subset $V \subset [n]$ satisfying

$$c_{i_1\cdots i_m} \neq 0, \text{ if } |V \cap \{i_1, \cdots, i_m\}| \text{ is odd},$$

and $c_{i_1\cdots i_m} = 0$ for the others. So, for all entries of $\mathscr{A}$, it follows that

$$a_{i_1\cdots i_m} \neq 0, \text{ if } |V \cap \{i_1, \cdots, i_m\}| \text{ is odd},$$

and $a_{i_1\cdots i_m} = 0$ for the others except the diagonal entries $a_{ii\cdots i}$, $i \in [n]$. By Theorem 2, we know that $\mathscr{C}$, $\mathscr{A}$ and $|\mathscr{A}|$ are all irreducible tensors. From Theorem 4.1 of [8] and Definition 1, there is a vector $x \in \mathbb{R}^n$, $x > 0$ satisfying

$$|\mathscr{A}|x^{m-1} = \lambda(|\mathscr{A}|)x^{[m-1]}.$$

Let $y \in \mathbb{R}^n$ be defined with $y_i = x_i$ whenever $i \in V$ and $y_i = -x_i$ for the others. When $i \in V$, we have

$$\begin{aligned}
(\mathscr{A}y^{m-1})_i &= \left[(\mathscr{D} - \mathscr{C})y^{m-1}\right]_i \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= d_i x_i^{m-1} + \textstyle\sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&= \left[(\mathscr{D} + \mathscr{C})x^{m-1}\right]_i \\
&= \lambda(|\mathscr{A}|)x_i^{m-1} \\
&= \lambda(|\mathscr{A}|)y_i^{m-1}.
\end{aligned} \tag{6}$$

Here the fourth equality follows the fact that m is even and exactly odd number indices take negative values for each $\{i_2, \cdots, i_m\} \subseteq [n]$. When $i \notin V$, we have

$$\begin{aligned}
(\mathscr{A}y^{m-1})_i &= \left[(\mathscr{D} - \mathscr{C})y^{m-1}\right]_i \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= -d_i x_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&= -\left[(\mathscr{D} + \mathscr{C})x^{m-1}\right]_i \\
&= -\lambda(|\mathscr{A}|)x_i^{m-1} \\
&= \lambda(|\mathscr{A}|)y_i^{m-1}.
\end{aligned} \tag{7}$$

Here the fourth equality follows the fact that m is even and exactly even number indices take negative values for each $\{i_2, \cdots, i_m\} \subseteq [n]$. The last equality of (7) follows from the definition of $y_i = -x_i$ when $i \notin V$. Thus, by (6), (7) and Definition 1, $\lambda(|\mathscr{A}|)$ is an H-eigenvalue of $\mathscr{A}$ with H-eigenvector y. So, we have

$$\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A}),$$

and the desired result follows. □

Here, in the proof of Theorem 3, odd-bipartite property of $\mathscr{C}$ guarantees that $|\mathscr{A}|$ has a positive H-eigenvector. Actually, if the H-eigenvector is nonnegative, one can obtain the same result. Before proving this, we first cite an useful conclusion.

Lemma 2. [24] *If $\mathscr{A}$ is a nonnegative tensor with order m and dimension n, then $\rho(\mathscr{A})$ is an eigenvalue of $\mathscr{A}$ with a nonnegative eigenvector $y \neq 0$.*

Theorem 4. *Let m be even. Suppose $\mathscr{A}$ is defined as in Theorem 3. If $\mathscr{C}$ is weakly odd-bipartite, then it holds that*

$$\lambda(\mathscr{A}) = \lambda(|A|).$$

Proof. Since tensor $\mathscr{C}$ is weakly odd-bipartite, so there is a nonempty proper index subset $V \subseteq [n]$ such that

$$c_{i_1\cdots i_m} = 0, \text{ when } |\{i_1, \cdots, i_m\} \cap V| \text{ is even},$$

and $|\{i_1, \cdots, i_m\} \cap V|$ must be an odd number for nonzero entries $c_{i_1\cdots i_m} \neq 0$, $i_1, \cdots, i_m \in [n]$.

On the other hand, by Lemma 2, there is a nonnegative H-eigenvector $x \geqslant 0$ of $|\mathscr{A}|$ corresponding to $\lambda(|\mathscr{A}|)$. Suppose vector $y \in \mathbb{R}^n$ be defined such that $y_i = x_i$ whenever $i \in V$ and $y_i = -x_i$ for the others. Then, the remaining process is similar with the proof of Theorem 3. □

Now, we will give an example to show that the conditions in Theorem 4 is not necessary. For example, suppose 4th order 2 dimensional tensor $\mathscr{A}$ with entries such that

$$a_{1111} = a_{2222} = 1, \ a_{1122} = -1,$$

and $a_{i_1i_2i_3i_4} = 0$ for the others. After calculating the largest H-eigenvalues of $\mathscr{A}$ and $|\mathscr{A}|$, we obtain

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 1.$$

But, the nonnegative tensor $\mathscr{C}$ is not weakly odd-bipartite corresponding to any nonempty proper index subset of $\{1, 2\}$. In the following, sufficient and necessary conditions for the equality of the two largest H-eigenvalues are presented, and it is proved that the necessity of the Theorem 4 holds when the nonnegative tensor $\mathscr{C}$ is weakly irreducible. Before doing this, we cite a definition.

Definition 5. [19] *Assume that $\mathscr{T}$ is a tensor with order m and dimension n. Construct a graph $\hat{G} = (\hat{V}, \hat{E})$, where $\hat{V} = \cup_{j=1}^{d} V_j$ and V_j are subsets of $\{1, 2, \cdots, n\}$ for $j = 1, \cdots, d$. Suppose that $i_j \in V_j, i_l \in V_l, j \neq l$. $(i_j, i_l) \in \hat{E}$ if and only if $t_{i_1i_2\cdots i_m} \neq 0$ for some $m-2$ indices $\{i_1, \cdots, i_m\}\backslash\{i_j, i_l\}$. Then, tensor $\mathscr{T}$ is called weakly irreducible if $\hat{G}$ is connected.*

As observed in [8], an irreducible tensor must be always weakly irreducible.

Theorem 5. *Let $\mathscr{A}$ be defined as in Theorem 4. Assume $\mathscr{C}$ is weakly irreducible. Then,*

$$\lambda(\mathscr{A}) = \lambda(|A|),$$

if and only if $\mathscr{C}$ is weakly odd-bipartite.

Proof. The sufficient condition has been proved in Theorem 4, and we only need to prove the necessary part.

Suppose $x \in \mathbb{R}^n$ is an H-eigenvector of $\mathscr{A}$ corresponding to $\lambda(\mathscr{A})$ such that $\sum_{i=1}^n x_i^m = 1$. Assume $y \in \mathbb{R}^n$ be defined by $y_i = |x_i|$, for $i \in [n]$. Since m is even, one has $\sum_{i=1}^n y_i^m = 1$. By Lemma 3.1 of [11], we have

$$\begin{array}{rl} \lambda(\mathscr{A}) = \mathscr{A}x^m = (\mathscr{D} - \mathscr{C})x^m \\ = \sum_{i=1}^n d_i x_i^m - \sum_{i_1,\cdots,i_m\in[n]} c_{i_1 i_2\cdots i_m} x_{i_1}\cdots x_{i_m} \\ \leqslant \sum_{i=1}^n d_i y_i^m + \sum_{i_1,\cdots,i_m\in[n]} c_{i_1 i_2\cdots i_m} y_{i_1}\cdots y_{i_m} \\ = (\mathscr{D} + \mathscr{C})y^m \leqslant \lambda(|\mathscr{A}|). \end{array} \tag{8}$$

Hence, by the fact that $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$, all inequalities in equation (8) should be equalities, which implies that y is an H-eigenvector of $|\mathscr{A}|$ corresponding to $\lambda(|\mathscr{A}|)$. Since $\mathscr{C}$ is weakly irreducible, $|\mathscr{A}|$ is also weakly irreducible. According to Theorem 4.1 of [8], it holds that $y > 0$ i.e., all elements in y are positive. Let $V = \{i \in [n] |\ x_i > 0\}$ and $\bar{V} = \{i \in [n] |\ x_i < 0\}$. Then $V \cup \bar{V} = [n]$. By (8), we obtain

$$\sum_{i_1,\cdots,i_m\in[n]} c_{i_1 i_2\cdots i_m}(|x_{i_1}|\cdots|x_{i_m}| + x_{i_1}\cdots x_{i_m}) = 0,$$

which implies that

$$c_{i_1 i_2\cdots i_m}(|x_{i_1}|\cdots|x_{i_m}| + x_{i_1}\cdots x_{i_m}) = 0,$$

for all $i_1, i_2, \cdots, i_m \in [n]$ since $\mathscr{C}$ is nonnegative. When $|\{i_1, i_2, \cdots, i_m\} \cap V|$ is even, we have

$$|x_{i_1}|\cdots|x_{i_m}| + x_{i_1}\cdots x_{i_m} > 0,$$

which implies $c_{i_1 i_2\cdots i_m} = 0$. When $|\{i_1, i_2, \cdots, i_m\} \cap V|$ is odd, we have

$$|x_{i_1}|\cdots|x_{i_m}| + x_{i_1}\cdots x_{i_m} = 0.$$

In this case, the value $c_{i_1 i_2\cdots i_m}$ may be zero or may not be zero . Thus, from Definition 3, it follows that $\mathscr{C}$ is weakly odd-bipartite corresponding to set V and the desired conclusion holds. □

Next, we study the relationship between a Z-tensor and its absolute tensor in the odd order case. In [10], Hu et al. proved that the largest H-eigenvalue of an odd order Laplacian tensor is always strictly less than the largest H-eigenvalue of the signless Laplacian tensor corresponding to the Laplacian tensor. By definitions of Laplacian tensor and signless Laplacian tensor in connected hypergraphs, we know that their diagonal entries are positive, and subscripts of

each nonzero element are mutually distinct. However, general Z-tensors (5) may not possess those advantages. Hence, for a general odd order Z-tensor (5), the largest H-eigenvalue of $\mathscr{A}$ may not be strictly less than the largest H-eigenvalue of $|\mathscr{A}|$ when the order is odd.

The following example shows that the largest H-eigenvalues of a Z-tensor (5) and its absolute tensor are equal.

Example 1. *Let $\mathscr{A}$ be a 5th order 3 dimensional tensor. Its entries are given by*

$$a_{11111} = a_{22222} = a_{33333} = 1,\ a_{11122} = a_{22233} = -1$$

and $a_{i_1i_2i_3i_4i_5} = 0$ for the others. Then the H-eigenvalue problems for $\mathscr{A}$ and $|\mathscr{A}|$ are

$$\begin{cases} x_1^4 - x_1^2x_2^2 = \lambda x_1^4 \\ x_2^4 - x_2^2x_3^2 = \lambda x_2^4 \\ x_3^4 = \lambda x_3^4 \end{cases}$$

and

$$\begin{cases} x_1^4 + x_1^2x_2^2 = \lambda x_1^4 \\ x_2^4 + x_2^2x_3^2 = \lambda x_2^4 \\ x_3^4 = \lambda x_3^4 \end{cases}.$$

After calculating the equation system, we know that $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 1$.

Theorem 6. *Let A be defined as (5). Assume m is odd. Suppose $\mathscr{C}$ is weakly odd-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. If for all $i \in V$, it satisfies*

$$c_{ii_2i_3\cdots i_m} = 0,\ \forall\ i_2, i_3, \cdots, i_m \in [n],$$

then $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$.

Proof. By the analysis in Theorems 3-5, from Lemma 13 of [19] and Corollary 2, it follows that

$$\lambda(\mathscr{A}) \leqslant \rho(\mathscr{A}) \leqslant \rho(|\mathscr{A}|) = \lambda(|\mathscr{A}|).$$

Thus, we only need to prove

$$\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A}).$$

Let $x \in \mathbb{R}^n$ be a nonnegative H-eigenvector of $|\mathscr{A}|$ corresponding to $\lambda(|\mathscr{A}|)$. So, for all $i \in [n]$, we have

$$(|\mathscr{A}|x^{m-1})_i = [(\mathscr{D} + \mathscr{C})x^{m-1}]_i = \lambda(|\mathscr{A}|)x_i^{m-1}. \tag{9}$$

Suppose $y \in \mathbb{R}^n$ be defined as $y_i = -x_i$, $i \in V$ and $y_i = x_i$, $i \notin V$. By conditions, $\mathscr{C}$ is weakly odd-bipartite corresponding to subset V, which means

$$c_{i_1i_2i_3\cdots i_m} = 0,\ i_1, i_2, \cdots, i_m \in [n]$$

when $|\{i_1, i_2, i_3, \cdots, i_m\} \cap V|$ is even. Then, for all $i \in [n]$, one has

$$\begin{aligned}(\mathscr{A}y^{m-1})_i &= [(\mathscr{D}-\mathscr{C})y^{m-1}]_i \\ &= d_i y_i^{m-1} - \sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\ &= d_i x_i^{m-1} - \sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m},\end{aligned} \tag{10}$$

where the third equality follows $m-1$ is even and $y_i^{m-1} = x_i^{m-1}$. When $i \in V$, by the fact that $c_{ii_2i_3\cdots i_m} = 0,\ i_2, i_3, \cdots, i_m \in [n]$, and by (9), (10), we have

$$\begin{aligned}(\mathscr{A}y^{m-1})_i &= [(\mathscr{D}-\mathscr{C})y^{m-1}]_i \\ &= d_i y_i^{m-1} - \sum_{i_2,\cdots,i_m\in[n]} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\ &= d_i x_i^{m-1} = \lambda(|\mathscr{A}|)x_i^{m-1} \\ &= \lambda(|\mathscr{A}|)y_i^{m-1}.\end{aligned} \tag{11}$$

Similarly, when $i \notin V$, it holds that

$$\begin{aligned}(\mathscr{A}y^{m-1})_i &= [(\mathscr{D}-\mathscr{C})y^{m-1}]_i \\ &= d_i y_i^{m-1} - \sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\ &= d_i x_i^{m-1} + \sum_{i_2,\cdots,i_m\in[n] \ \ |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\ &= d_i x_i^{m-1} + (\mathscr{C}x^{m-1})_i \\ &= [(\mathscr{D}+\mathscr{C})x^{m-1}]_i \\ &= (|\mathscr{A}|x^{m-1})_i = \lambda(|\mathscr{A}|)x_i^{m-1} \\ &= \lambda(|\mathscr{A}|)y_i^{m-1},\end{aligned} \tag{12}$$

where the third equality follows the fact that m is odd and exactly odd indices take negative values. By (11) and (12), we know that $\lambda(|\mathscr{A}|)$ is an H-eigenvalue of $\mathscr{A}$. Hence, we have $\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A})$ and the desired result follows. □

Now, we present a example to verify the authenticity of Theorem 6.

Example 2. *Set a 5th order 3 dimensional tensor $\mathscr{A}$ such that*

$$a_{11111} = 1,\ a_{22222} = 1,\ a_{33333} = 3,\ a_{11333} = -1,\ a_{22333} = -2$$

and $a_{i_1i_2i_3i_4} = 0$ for the others. Let $V = \{3\}$. Then $\mathscr{C}$ is weakly odd-bipartite corresponding to the set V and $c_{3i_2i_3i_4i_5} = 0,\ \forall\ i_2, i_3, i_4, i_5 \in [3]$.

The H-eigenvalue problems for $\mathscr{A}$ and $|\mathscr{A}|$ are to solve

$$\begin{cases} x_1^4 - x_1x_3^3 = \lambda x_1^4 \\ x_2^4 - 2x_2x_3^3 = \lambda x_2^4 \\ 3x_3^4 = \lambda x_3^4 \end{cases}$$

and

$$\begin{cases} x_1^4 + x_1x_3^3 = \lambda x_1^4 \\ x_2^4 + 2x_2x_3^3 = \lambda x_2^4 \\ 3x_3^4 = \lambda x_3^4 \end{cases}.$$

After calculating the largest H-eigenvalues of $\mathscr{A}$ and $|\mathscr{A}|$, we obtain

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 3.$$

The next example shows that the conditions in Theorem 6 are not necessary.

Example 3. *Let $\mathscr{A}$ be a 5th order 3 dimensional tensor. Its entries are given by*

$$a_{11111} = 1,\ a_{22222} = 2,\ a_{33333} = 4,\ a_{11122} = a_{11333} = -1,\ a_{22233} = -2$$

and $a_{i_1i_2i_3i_4i_5} = 0$ for the others. Then the H-eigenvalue problems for $\mathscr{A}$ and $|\mathscr{A}|$ are

$$\begin{cases} x_1^4 - x_1^2x_2^2 - x_1x_3^3 = \lambda x_1^4 \\ 2x_2^4 - 2x_2^2x_3^2 = \lambda x_2^4 \\ 4x_3^4 = \lambda x_3^4 \end{cases}$$

and

$$\begin{cases} x_1^4 + x_1^2x_2^2 + x_1x_3^3 = \lambda x_1^4 \\ 2x_2^4 + 2x_2^2x_3^2 = \lambda x_2^4 \\ 4x_3^4 = \lambda x_3^4 \end{cases}$$

After calculating these equation sets, we know that $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 4$, but the nonnegative tensor $\mathscr{C}$ is not weakly odd-bipartite corresponding to any nonempty proper index subset of $\{1, 2, 3\}$.

By Lemma 1 and Theorem 6, we have the following conclusion.

Corollary 3. *Let A be defined as in (5). Assume m is odd. Suppose $\mathscr{C}$ is weakly even-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. If for all $i \notin V$, it satisfies*

$$c_{ii_2i_3\cdots i_m} = 0,\ \forall\ i_2, i_3, \cdots, i_m \in [n],$$

then $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$.

5 The relation between spectrums of a symmetric Z-tensor and its absolute tensor

In this section, we will study the relation between the spectrum of an even order symmetric Z-tensor with nonnegative diagonal entries, and the spectrum of the absolute tensor of the Z-tensor. It is proved that, if the symmetric Z-tensor is weakly irreducible and odd-bipartite, then the two spectral sets equal. Furthermore, for a weakly irreducible symmetric Z-tensor with nonnegative diagonal entries, we show that the spectral sets of the Z-tensor and its absolute tensor equal if and only if their spectral radii equal. Before proving the conclusion, we firstly cite the definition of diagonal similar tensors [22], which is useful in the following analysis.

Definition 6. *Let $\mathscr{A}$ and $\mathscr{B}$ be two order $m \geqslant 2$ dimension n tensors. If there exists a nonsingular diagonal matrix P of dimension n such that $\mathscr{B} = P^{-(m-1)}\mathscr{A}P$, then $\mathscr{A}$ and $\mathscr{B}$ are called diagonal similar.*

Here, tensor $\mathscr{B} = P^{-(m-1)}\mathscr{A}P$ is defined by

$$b_{i_1 i_2 \cdots i_m} = \sum_{j_1, j_2, \cdots, j_m \in [n]} a_{j_1 j_2 \cdots j_m} p_{i_1 j_1}^{m-1} p_{j_2 i_2} \cdots p_{j_m i_m}, \ i_1, i_2, \cdots, i_m \in [n].$$

Theorem 7. *Assume order m dimension n symmetric Z-tensor $\mathscr{A}$ is defined as in (5). Suppose $\mathscr{C}$ is weakly irreducible. Then, $\mathscr{A}$ and $|\mathscr{A}|$ are diagonal similar if and only if m is even and $\mathscr{C}$ is weakly odd-bipartite.*

Proof. For necessary, from Definition 6, we know that there is a nonsingular diagonal matrix P satisfying

$$\mathscr{A} = P^{-(m-1)}|\mathscr{A}|P,$$

i.e.,

$$\mathscr{D} - \mathscr{C} = P^{-(m-1)}(\mathscr{D} + \mathscr{C})P.$$

Since $\mathscr{D} = P^{-(m-1)}\mathscr{D}P$, we have

$$-\mathscr{C} = P^{-(m-1)}\mathscr{C}P,$$

which implies that

$$-c_{i_1 i_2 \cdots i_m} = c_{i_1 i_2 \cdots i_m} p_{i_1 i_1}^{-(m-1)} p_{i_2 i_2} \cdots p_{i_m i_m}. \tag{13}$$

If $p_{11} = p_{22} = \cdots = p_{nn}$, by (13), we get $\mathscr{C} = 0$, which is a contradiction to the fact that $\mathscr{C}$ is weakly irreducible. So there are at least two distinct diagonal entries in P.

When $c_{i_1 i_2 \cdots i_m} \neq 0$, by (13), one has

$$-p_{i_1 i_1}^m = p_{i_1 i_1} p_{i_2 i_2} \cdots p_{i_m i_m}. \tag{14}$$

By (14), and by the fact that $\mathscr{C}$ is weakly irreducible, we obtain

$$p_{ii}^m = p_{jj}^m, \quad i, j \in [n],$$

which implies that m is even and

$$V = \{i \in [n] \mid p_{ii} < 0\} \neq \emptyset, \quad \tilde{V} = \{i \in [n] \mid p_{ii} > 0\} \neq \emptyset.$$

Combining this with (13)-(14), we know that

$$c_{i_1 i_2 \cdots i_m} = 0, \ \ \textit{when } |\{i_1, i_2, \cdots, i_m\} \cap V| \textit{ is even}.$$

Thus, tensor $\mathscr{C}$ is weakly odd-bipartite corresponding to V and the only if part holds.

For the if part, without loss of generality, suppose $\mathscr{C}$ is weakly odd-bipartite corresponding to $\Omega \subset [n]$. Let P be a diagonal matrix with i-th diagonal entries being -1 when $i \in \Omega$ and 1 when $i \notin \Omega$. By a direct computation, one has

$$\mathscr{A} = P^{-(m-1)}|\mathscr{A}|P.$$

Apparently, P is a nonsingular diagonal matrix. From Definition 6, it follows that $\mathscr{A}$ and $|\mathscr{A}|$ are diagonal similar. □

It should be noted that diagonal similar tensors have the same characteristic polynomials, and thus they have the same spectrum (see Theorem 2.1 of [22]), which is similar to the matrix case.

Corollary 4. *Assume tensor $\mathscr{A}$ is defined as in Theorem 7. Let m be even. Suppose $\mathscr{C}$ is odd-bipartite. Then $\sigma(\mathscr{A}) = \sigma(|\mathscr{A}|)$.*

Lemma 3. [25] *Let $\mathscr{A}$ and $\mathscr{B}$ be two order m dimension n tensors with $|\mathscr{B}| \leqslant \mathscr{A}$. Then*

(1) $\rho(\mathscr{B}) \leqslant \rho(\mathscr{A})$.

(2) If $\mathscr{A}$ is weakly irreducible and $\rho(\mathscr{B}) = \rho(\mathscr{A})$, where $\lambda = \rho(\mathscr{A})e^{i\psi}$ is an eigenvalue of $\mathscr{B}$ with an eigenvector y, then,

(i) all the components of y are nonzero;

(ii) let $U = diag(y_1/|y_1|, \cdots, y_n/|y_n|)$ be a nonsingular diagonal matrix, we have $\mathscr{B} = e^{i\psi}U^{-(m-1)}\mathscr{A}U$.

Theorem 8. *Assume order m dimension n symmetric Z-tensor $\mathscr{A}$ is defined as in (5). If $\mathscr{C}$ is weakly irreducible, then $\rho(\mathscr{A}) = \rho(|\mathscr{A}|)$ if and only if $\sigma(\mathscr{A}) = \sigma(|\mathscr{A}|)$.*

Proof. The sufficient condition is obvious. Now, we prove the only if part. Suppose $\lambda = \rho(|\mathscr{A}|)e^{i\psi}$ is an eigenvalue of $\mathscr{A}$. Since $\mathscr{C}$ is weakly irreducible, from Lemma 3, we know that there exists a nonsingular diagonal matrix P such that

$$\mathscr{A} = e^{i\psi}P^{-(m-1)}|\mathscr{A}|P, \tag{15}$$

which means

$$\mathscr{D} - \mathscr{C} = e^{i\psi}P^{-(m-1)}(\mathscr{D} + \mathscr{C})P. \tag{16}$$

By the fact that all diagonal elements of $\mathscr{C}$ equal zero, by (16), one has

$$\mathscr{D} = e^{i\psi}P^{-(m-1)}\mathscr{D}P = e^{i\psi}\mathscr{D},$$

which implies $e^{i\psi} = 1$. So, by Definition 6 and (15), we know that $\mathscr{A}$ and $|\mathscr{A}|$ are diagonal similar tensors. Thus, from Theorem 2.3 of [22], it holds that $\sigma(\mathscr{A}) = \sigma(|\mathscr{A}|)$. □

6 Final remarks

Odd-bipartite and even-bipartite tensors are defined in this paper. Using this, we studied the relation between the largest H-eigenvalue of a Z-tensor with nonnegative diagonal elements, and the largest H-eigenvalue of the Z-tensor's absolute tensor. Sufficient and necessary conditions for the equality of these largest H-eigenvalues are given when the Z-tensor has even order. For the odd order case, sufficient conditions are presented. Examples are given to verify the authenticity of the conclusions. On the other side, relation between spectral sets of an even order symmetric Z-tensor with nonnegative diagonal entries and its absolute tensor are studied.

In this paper, we only study the case of H-eigenvalues of Z-tensors. Do Z-eigenvalues of Z-tensors also hold in such case? This may be an interesting work in the future.

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