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Spectral directed hypergraph theory via tensors

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In this paper, we show that each of the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of a uniform directed hypergraph has n linearly independent H-eigenvectors. Some lower and upper bounds for the largest and smallest adjacency, Laplacian and signless Laplacian H-eigenvalues of a uniform directed hypergraph are given. For a uniform directed hypergraph, the smallest Laplacian H-eigenvalue is 0. On the other hand, the upper bound of the largest adjacency and signless Laplacian H-eigenvalues are achieved if and only if it is a complete directed hypergraph. For a uniform directed hyperstar, all adjacency H-eigenvalues are 0. At the same time, we make some conjectures about the nonnegativity of one H-eigenvector corresponding to the largest H-eigenvalue, and raise some questions about whether the Laplacian and signless Laplacian tensors are positive semi-definite for a uniform directed hypergraph.

Keywords: Spectrum; directed hypergraph; *H*-eigenvalue; adjacency tensor; Laplacian tensor; signless Laplacian tensor

AMS Subject Classifications: 05C65; 15A18

1. Introduction

In 2005, Lim [1] and Qi [2] independently defined eigenvalues and eigenvectors of a real tensor. Qi [2] explored the application of eigenvalues of tensors in determining positive definiteness of an even degree multivariate form. Lim [1,3] pointed out that a potential application of eigenvalues of tensors is on the spectral (undirected) hypergraph theory. Qi [2,4] proposed several kinds of eigenvalues for a tensor, such as *H*-eigenvalues, *Z*-eigenvalues, *N*-eigenvalues, *E*-eigenvalues, *H*⁺-eigenvalues and *H*⁺⁺-eigenvalues. Recently, a number of papers appeared on various kinds of structured tensors [5–14] and spectral hypergraph theory via the adjacency tensors, Laplacian tensors and signless Laplacian tensors of undirected uniform hypergraphs.[15–31,37] Recently, Chen and Qi [32] studied spectral properties of circulant tensors and found their applications in spectral directed hypergraph theory. However, unlike spectral theory of undirected hypergraphs, there is almost blank for spectral directed graphs, and have found applications in imaging

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processing,[33] optical network communications,[34] computer science and combinatorial optimization.[35] Thus, in this paper, we studied spectral directed hypergraph theory via tensors.

In the next section, definitions of *H*-eigenvalues of tensors and directed hypergraphs with their adjacency tensors, Laplacian tensors and signless Laplacian tensors are given. We, respectively, present in Sections 3, 4 and 5 some basic results about *H*-eigenvalues of the adjacency tensor, Laplacian tensor and signless Laplacian tensor of a uniform directed hypergraph. In the last section, some results about *H*-eigenvalues of the adjacency tensor, Laplacian tensor of a uniform directed hypergraph. In the last section, some results about *H*-eigenvalues of the adjacency tensor, Laplacian tensor of a uniform directed hypergraph.

2. Preliminaries

Some definitions of *H*-eigenvalues of tensors and directed hypergraphs with their adjacency tensors, Laplacian tensors and signless Laplacian tensors are presented in this section.

2.1. H-Eigenvalues of tensors

In this subsection, some basic definitions on H-eigenvalues of tensors are reviewed. For comprehensive references, see [2,10] and references therein. Especially, for spectral hyper-graph theory oriented facts on H-eigenvalues of tensors, please see [4,17].

Let \mathbb{R} be the field of real numbers and \mathbb{R}^n be the *n*-dimensional real space. \mathbb{R}^n_+ denotes the nonnegative orthant of \mathbb{R}^n . For integers $k \ge 3$ and $n \ge 2$, a real tensor $\mathcal{T} = (t_{i_1...i_k})$ of order *k* and dimension *n* refers to a multiway array (also called hypermatrix) with entries $t_{i_1...i_k}$ such that $t_{i_1...i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1, ..., n\}$ and $j \in [k]$. Tensors are always referred to *k*-order real tensors in this paper, and the dimensions will be clear from the context. Given a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{T}\mathbf{x}^{k-1}$ is defined as an *n*-dimensional vector such that its *i*-th element being $\sum_{i_2,...,i_k \in [n]} t_{ii_2...i_k} x_{i_2} \dots x_{i_k}$ for all $i \in [n]$. Let \mathcal{I} be the identity tensor of appropriate dimension, e.g. $i_{i_1...i_k} = 1$ if and only if $i_1 = \dots = i_k \in [n]$, and zero otherwise when the dimension is *n*. The following definition was introduced by Qi [2].

Definition 2.1 Let \mathcal{T} be a *k*-order *n*-dimensional real tensor. For some $\lambda \in \mathbb{R}$, if polynomial system $(\lambda \mathcal{I} - \mathcal{T}) \mathbf{x}^{k-1} = 0$ has a solution $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, then λ is called an *H*-eigenvalue and \mathbf{x} an *H*-eigenvector corresponding to λ .

For a subset $S \subseteq [n]$, we denoted by |S| its cardinality, and $\sup(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$ its *support*. A tensor \mathcal{T} is called *symmetric* if its entries $t_{i_1i_2...i_k} = t_{i'_1i'_2...i'_k}$ for arbitrary permutation $(i'_1, i'_2, ..., i'_k)$ of $(i_1, i_2, ..., i_k)$. A tensor \mathcal{T} is called *positive semi-definite* if for any vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{T}\mathbf{x}^k \ge 0$, and is called *positive definite* if for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{T}\mathbf{x}^k > 0$. Let tensor $\mathcal{T} = (t_{i_1...i_k})$. If for $i \in [n]$, $2t_{i...i} \ge \sum_{i_2,...,i_k \in [n]} |t_{ii_2...i_k}|$, then \mathcal{T} is called a *diagonally dominated tensor*. Denote $\mathbf{1}_j \in \mathbb{R}^n$ as the *j*th unit vector (i.e. the *j*th element is 1, otherwise 0) for $j \in [n]$, **0** the zero vector in \mathbb{R}^n , **1** the all 1 vector in \mathbb{R}^n .

2.2. Directed hypergraphs

In this subsection, we present some essential concepts of directed hypergraphs with their adjacency tensors, Laplacian tensors and signless Laplacian tensors, which will be used in the sequel. Our definition for a directed hypergraph is the same as in [34], which is a special case of the definition in [33], i.e. we discuss the case that each arc has only one tail.

In this paper, unless stated otherwise, a directed hypergraph means a simple k-uniform directed hypergraph G with vertex set V, which is labelled as $[n] = \{1, ..., n\}$, and arc set E. E is a set of ordered subsets of V. The elements of E are called arcs. An arc $e \in E$ has the form $e = (j_1, ..., j_k)$, where $j_l \in V$ for $l \in [k]$ and $j_l \neq j_m$ if $l \neq m$. The order of $j_2, ..., j_k$ is irrelevant. But the order of j_1 is special. The vertex j_1 is called the tail (or out-vertex) of the arc e. It must be in the first position of the arc. Each other vertex $j_2, ..., j_k$ is called a head (or in-vertex) of the arc e. The out-degree of a vertex $j \in V$ is defined as $d_j^+ = |E_j^+|$, where $E_j^+ = \{e \in E : j \text{ is the tail of } e\}$. The in-degree of a vertex $j \in V$ is defined as $d_j^- = \frac{1}{k-1}|E_j^-|$, where $E_j^+ = \{e \in E : j \text{ is a head of } e\}$. The degree (or all-degree) of a vertex $j \in V$ is defined as $d_j = d_j^+ + d_j^-$. If for each $j \in V$, the degree d_j^+ (or d_j^+ or d_j , respectively) has the same value d, then G is called a directed d-out-regular (or d-in-regular, respectively) hypergraph.

By *k*-uniformity, we mean that for every arc $e \in E$, the cardinality |e| of *e* is equal to *k*. Throughout this paper, $k \ge 3$ and $n \ge k$. Moreover, since the trivial hypergraph (i.e. $E = \emptyset$) is of less interest, we consider only hypergraphs having at least one arc (i.e. nontrivial) in this paper.

For a subset $S \subset [n]$, we denoted by E_S the set of arcs $\{e \in E \mid S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i . It is the set of arcs containing the vertex i, i.e. $E_i := \{e \in E \mid i \in e\}$. Two different vertices i and j are weak-connected, if there is a sequence of arcs (e_1, \ldots, e_m) such that $i \in e_1, j \in e_m$ and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [m-1]$. Two different vertices i to j is strong-connected, denoted by $i \rightarrow j$, if there is a sequence of arcs (e_1, \ldots, e_m) such that i is the tail of e_1 , j is a head of e_m and a head of e_r is the tail of e_{r+1} for all $r \in [m-1]$. A directed hypergraph is called *weak-connected*, if every pair of different vertices of G is weak-connected. A directed hypergraph is called *strongconnected*, if every pair of different vertices i and j of G satisfying $i \rightarrow j$ and $j \rightarrow i$. Let $S \subseteq V$, the directed hypergraph with vertex set S and arc set $\{e \in E \mid e \subseteq S\}$ is called the directed sub-hypergraph of G induced by S. We will denote it by G_S . A weak-connected component G_S is a sub-hypergraph of G such that any two vertices in S are weak-connected and no other vertex in $V \setminus S$ is weak-connected to any vertex in S. A directed hypergraph G = (V, E) is complete if E consists of all the possible arcs. In the sequel, unless stated otherwise, all the notations introduced above are reserved for the specific meanings. For others definitions and notations not mentioned here, please see [36].

The following definition for the adjacency tensor, Laplacian tensor and signless Laplacian tensor of a directed hypergraph was proposed by Chen and Qi [32].

Definition 2.2 Let G = (V, E) be a k-uniform directed hypergraph. The *adjacency tensor* of the directed hypergraph G is defined as the k-order n-dimensional tensor \mathcal{A} whose $(i_1 \dots i_k)$ -entry is:

$$a_{i_1\dots i_k} := \begin{cases} \frac{1}{(k-1)!} \text{ if } (i_1,\dots,i_k) = e \in E \text{ and } i_1 \text{ is the tail of } e, \\ 0 \text{ otherwise.} \end{cases}$$

Let \mathcal{D} be a *k*-order *n*-dimensional diagonal tensor with its diagonal element $d_{i...i}$ being d_i^+ , the out-degree of vertex *i*, for all $i \in [n]$. Then $\mathcal{L} := \mathcal{D} - \mathcal{A}$ is the *Laplacian tensor* of the directed hypergraph *G*, and $\mathcal{Q} := \mathcal{D} + \mathcal{A}$ is the *signless Laplacian tensor* of the directed hypergraph *G*.

By the definition above, the adjacency tensor, the Laplacian tensor and the signless Laplacian tensors of a uniform directed hypergraph are not symmetric in general. The adjacency tensor and the signless Laplacian tensor are still nonnegative. In general, we do not know if the Laplacian tensor and the signless Laplacian tensor of an even-uniform directed hypergraph are positive semi-definite or not. We may still show that the smallest H-eigenvalue of the Laplacian tensor of an k-uniform directed hypergraph is zero with an H-eigenvector 1, and the largest H-eigenvalues of the adjacency tensor and the signless Laplacian tensor of a directed d-out-regular hypergraph are d and 2d, respectively.

3. *H*-Eigenvalues of the adjacency tensor for a uniform directed hypergraph

This section presents some basic results about the H-eigenvalues of the adjacency tensor of a uniform directed hypergraph. We start the discussion on the H-eigenvalue **0** of the adjacency tensor of a uniform directed hypergraph.

The next proposition is a direct consequence of Definition 2.1.

PROPOSITION 3.1 Given an n-vertex k-uniform directed hypergraph G, let A be its adjacency tensor and $\mathbf{1}_j$ $(1 \le j \le n)$ be the jth unit vector. Then any vector \mathbf{x} , which is linear combination of at most k - 2 different $\mathbf{1}_j$, is an H-eigenvector of A corresponding to H-eigenvalue 0.

Proof For any vector **x** which is linear combination of at most k - 2 different $\mathbf{1}_j$, one can get that the multiplication of any k - 1 entries in vector **x** is equal to 0. Then

$$A\mathbf{x}^{k-1} = \mathbf{0} = 0\mathbf{x}.$$

By Definition 2.1, the proposition follows.

COROLLARY 3.1 Given an n-vertex k-uniform directed hypergraph G, let A be its adjacency tensor and $\mathbf{1}_j$ $(1 \le j \le n)$ be the jth unit vector. Then A has n linearly independent H-eigenvectors.

Proof Note that $k \ge 3$ throughout this paper. Hence, from Proposition 3.1, there exist *n* linearly independent unit vectors $\mathbf{1}_j$ (j = 1, ..., n) as *H*-eigenvectors of \mathcal{A} all corresponding to *H*-eigenvalue 0.

By Proposition 3.1, the adjacency tensor \mathcal{A} of uniform directed hypergraph G has at least one H-eigenvalue. Now, we study the largest and smallest H-eigenvalues of uniform directed hypergraph G via its adjacency tensor \mathcal{A} . Denote by $\lambda(\mathcal{A})$ (respectively, $\mu(\mathcal{A})$) as the largest (respectively, smallest) H-eigenvalue of uniform directed hypergraph G via its adjacency tensor \mathcal{A} .

THEOREM 3.1 Let A be the adjacency tensor of an n-vertex k-uniform directed hypergraph G = (V, E). Then we have

$$-\Delta^+ \le \mu(\mathcal{A}) \le 0 \le \lambda(\mathcal{A}) \le \Delta^+,$$

where the maximum out-degree $\Delta^+ := \max_{1 \le i \le n} d_i^+$.

Proof By Proposition 3.1, $\mu(\mathcal{A}) \leq 0 \leq \lambda(\mathcal{A})$. Hence, we only have to prove $-\Delta^+ \leq \mu(\mathcal{A})$ and $\lambda(\mathcal{A}) \leq \Delta^+$.

For any *H*-eigenvalue λ of A, let **y** be its corresponding *H*-eigenvector of A and y_i ($0 < |y_i|$) be an entry of **y** with maximum absolute value. Then the *i*th eigenvalue equation gives that

$$\lambda y_i^{k-1} = \sum_{i_2, \dots, i_k \in [n]} a_{ii_2 \dots i_k} y_{i_2} \dots y_{i_k} = \sum_{(i, i_2, \dots, i_k) \in E} y_{i_2} \dots y_{i_k}.$$

Then

$$\begin{aligned} |\lambda||y_i|^{k-1} &= \left| \sum_{(i,i_2,\dots,i_k)\in E} y_{i_2}\dots y_{i_k} \right| \le \sum_{(i,i_2,\dots,i_k)\in E} |y_{i_2}|\dots |y_{i_k}| \\ &\le \sum_{(i,i_2,\dots,i_k)\in E} |y_i|^{k-1} = d_i^+ |y_i|^{k-1} \le \Delta^+ |y_i|^{k-1}. \end{aligned}$$

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$$-\Delta^+ \leq \lambda \leq \Delta^+.$$

Then we have $-\Delta^+ \leq \mu(\mathcal{A})$ and $\lambda(\mathcal{A}) \leq \Delta^+$, since $\mu(\mathcal{A}) \leq \lambda \leq \lambda(\mathcal{A})$.

By Theorem 3.1 together with its proof, we have the following two corollaries.

COROLLARY 3.2 The H-eigenvalues of adjacency tensor A for n-vertex k-uniform directed hypergraph G lie in the union of n intervals in \mathbb{R} . These n intervals have the diagonal elements 0 as their centres, and $d_i^+(i = 1, ..., n)$ as their radii. That is, all the H-eigenvalues of adjacency tensor A for n-vertex k-uniform directed hypergraph G lie in the interval with centre 0 and radius Δ^+ .

COROLLARY 3.3 The largest H-eigenvalue $\lambda(A)$ of adjacency tensor A for directed d-out-regular hypergraph G is d, with a corresponding H-eigenvector 1.

THEOREM 3.2 Let A be the adjacency tensor of an *n*-vertex *k*-uniform directed hypergraph G = (V, E), *r* be a real number nonzero. Then we have

$$\lambda(\mathcal{A}) \leq \binom{n-1}{k-1},$$

and equality holds if and only if G is an n-vertex k-uniform complete directed hypergraph with H-eigenvector r1 corresponding to H-eigenvalue $\lambda(A)$.

Proof For the largest *H*-eigenvalue $\lambda(A)$ of A, let **y** be its corresponding *H*-eigenvector of A and y_i ($0 < |y_i|$) be an entry of **y** with maximum absolute value. Then the *i*th eigenvalue equation gives that

$$\lambda(\mathcal{A})y_i^{k-1} = \sum_{i_2,...,i_k \in [n]} a_{ii_2...i_k} y_{i_2} \dots y_{i_k} = \sum_{(i,i_2,...,i_k) \in E} y_{i_2} \dots y_{i_k}.$$

Then

$$|\lambda(\mathcal{A})||y_i|^{k-1} = \left|\sum_{(i,i_2,\dots,i_k)\in E} y_{i_2}\dots y_{i_k}\right| \le \sum_{(i,i_2,\dots,i_k)\in E} |y_{i_2}|\dots |y_{i_k}|.$$

That is

$$\lambda(\mathcal{A})| \leq \sum_{(i,i_2,...,i_k)\in E} \frac{|y_{i_2}|}{|y_i|} \dots \frac{|y_{i_k}|}{|y_i|} \leq \sum_{i_2,...,i_k\in [n]\setminus\{i\}} 1 = \binom{n-1}{k-1},$$

where the last inequality follows from the fact that $|E| \leq {\binom{n-1}{k-1}}$ and equality holds if and only if \mathcal{G} is an *n*-vertex *k*-uniform complete directed hypergraph, the first two inequalities follow from the absolute inequalities and two equalities both holding if and only if $y_1 = y_2 = \cdots = y_n$, i.e. $\mathbf{y} = r\mathbf{1}$. This completes the proof.

In spectral theory of undirected hypergraphs, it is obvious that there exists a nonnegative H-eigenvector corresponding to the largest H-eigenvalue $\lambda(\mathcal{A})$. But, unlike spectral theory of undirected hypergraphs, until now we still cannot prove the following conjecture.

Conjecture 3.1 Given an *n*-vertex *k*-uniform directed hypergraph *G*, let \mathcal{A} be its adjacency tensor and *k* be even. Then there exists a nonnegative *H*-eigenvector **x** corresponding to the largest *H*-eigenvalue $\lambda(\mathcal{A})$ such that $\lambda(\mathcal{A}) = \frac{\mathcal{A}\mathbf{x}^k}{||\mathbf{x}||_k^k}$.

4. H-Eigenvalues of the Laplacian tensor for a uniform directed hypergraph

This section presents some basic results about the H-eigenvalue of the Laplacian tensor for a uniform directed hypergraph. We start the discussion on the H-eigenvalue **0** of the Laplacian tensor of a uniform directed hypergraph.

The next proposition is a direct consequence of Definition 2.1.

PROPOSITION 4.1 Given an *n*-vertex *k*-uniform directed hypergraph G, let \mathcal{L} be its Laplacian tensor. Then vector **1**, is an *H*-eigenvector of \mathcal{L} corresponding to *H*-eigenvalue 0.

Proof For vector **1**, one can get that for each i = 1, ..., n

$$[\mathcal{L}\mathbf{1}^{k-1}]_i = \sum_{i_2,\dots,i_k \in [n]} l_{ii_2\dots i_k} \mathbf{1} = d_{ii\dots i} - \sum_{(i,i_2,\dots,i_k) \in E} \mathbf{1} = d_i^+ - d_i^+ = \mathbf{0}[\mathbf{1}]_i^{k-1}.$$

By Definition 2.1, the proposition follows.

PROPOSITION 4.2 Given an n-vertex k-uniform directed hypergraph G, let \mathcal{L} be its Laplacian tensor. Then each vector $\mathbf{1}_j$ (j = 1, ..., n) is an H-eigenvector of \mathcal{L} corresponding to H-eigenvalue d_j^+ .

Proof For vector $\mathbf{1}_i$, one can get that

$$\left[\mathcal{L}\mathbf{1}_{j}^{k-1}\right]_{j} = d_{jj\dots j}\mathbf{1} - \sum_{(j,j_{2},\dots,j_{k})\in E} \mathbf{0} = d_{i}^{+}$$

and for each $i \in [n] \setminus \{j\}$

$$\left[\mathcal{L}\mathbf{1}_{j}^{k-1}\right]_{i} = d_{ii\dots i}\mathbf{0} - \sum_{(i,i_{2},\dots,i_{k})\in E}\mathbf{0} = \mathbf{0}.$$

By Definition 2.1, the proposition follows.

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The next corollary is a direct consequence of Propositions 4.1 and 4.2.

COROLLARY 4.1 Given an n-vertex k-uniform directed hypergraph G, let L be its Laplacian tensor. Then \mathcal{L} has n linearly independent H-eigenvectors and at least n + 1 Heigenvalues (Repeatable).

By Corollary 4.1, the Laplacian tensor \mathcal{L} of uniform directed hypergraph G has at least n + 1 H-eigenvalues. Now, we study the largest and smallest H-eigenvalues of uniform directed hypergraph G via its Laplacian tensor \mathcal{L} . Denote by $\lambda(\mathcal{L})$ (respectively, $\mu(\mathcal{L})$) as the largest (respectively, smallest) H-eigenvalue of uniform directed hypergraph G via its Laplacian tensor \mathcal{L} .

THEOREM 4.1 Let \mathcal{L} be the Laplacian tensor of an *n*-vertex *k*-uniform directed hypergraph G = (V, E). Then we have

$$0 = \mu(\mathcal{L}) \leq \delta^+ \leq \Delta^+ \leq \lambda(\mathcal{L}) \leq 2\Delta^+,$$

where the minimum out-degree $\delta^+ := \min_{1 \le i \le n} d_i^+$.

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Proof By Propositions 4.1 and 4.2, $\mu(\mathcal{L}) \leq 0 \leq \delta^+ \leq \Delta^+ \leq \lambda(\mathcal{L})$. Hence, we only have to prove $0 \le \mu(\mathcal{L})$ and $\lambda(\mathcal{L}) \le 2\Delta^+$.

For any *H*-eigenvalue λ of \mathcal{L} , let **y** be its corresponding *H*-eigenvector of \mathcal{L} and y_i $(0 < |y_i|)$ be an entry of y with maximum absolute value. Then the *i*-th eigenvalue equation gives that

$$\lambda y_i^{k-1} = \sum_{i_2, \dots, i_k \in [n]} l_{ii_2 \dots i_k} y_{i_2} \dots y_{i_k} = d_i^+ y_i^{k-1} - \sum_{(i, i_2, \dots, i_k) \in E} y_{i_2} \dots y_{i_k}.$$

Then

$$\begin{aligned} |\lambda - d_i^+| |y_i|^{k-1} &= \left| \sum_{(i,i_2,\dots,i_k) \in E} y_{i_2} \dots y_{i_k} \right| \le \sum_{(i,i_2,\dots,i_k) \in E} |y_{i_2}| \dots |y_{i_k}| \\ &\le \sum_{(i,i_2,\dots,i_k) \in E} |y_i|^{k-1} = d_i^+ |y_i|^{k-1}. \end{aligned}$$

That is

$$|\lambda - d_i^+| \le d_i^+.$$

Hence

$$0 \le \lambda \le 2d_i^+ \le 2\Delta^+.$$

Then we have $0 \le \mu(\mathcal{L})$ and $\lambda(\mathcal{L}) \le 2\Delta^+$.

By Theorem 4.1 together with its proof, we have the following two corollaries.

COROLLARY 4.2 The H-eigenvalues of Laplacian tensor \mathcal{L} for n-vertex k-uniform directed hypergraph G lie in the union of n intervals in \mathbb{R} . These n intervals have the diagonal elements d_i^+ (i = 1, ..., n) as their centres, and d_i^+ (respectively) as their radii. That is, all the H-eigenvalues of Laplacian tensor $\mathcal L$ for n-vertex k-uniform directed hypergraph G *lie in the interval with centre* Δ^+ *and radius* Δ^+ *.*

COROLLARY 4.3 Let \mathcal{L} be the Laplacian tensor of an *n*-vertex *k*-uniform directed hypergraph G = (V, E). Then we have

$$\lambda(\mathcal{L}) \leq 2\binom{n-1}{k-1}.$$

In spectral theory of undirected hypergraphs, since the Laplacian tensor \mathcal{L} of an evenuniform undirected hypergraph G is symmetric and diagonally dominated, then \mathcal{L} is positive semi-definite. But, unlike spectral theory of undirected hypergraphs, until now we still cannot answer the following question.

Question 4.1 Given an even-uniform directed hypergraph G, let \mathcal{L} be its Laplacian tensor. Is \mathcal{L} a positive semi-definite tensor?

5. H-Eigenvalues of the signless Laplacian tensor for a uniform directed hypergraph

This section presents some basic results about the *H*-eigenvalue of the signless Laplacian tensor of a uniform directed hypergraph.

The next proposition is a direct consequence of Definition 2.1.

PROPOSITION 5.1 Given an n-vertex k-uniform directed hypergraph G, let Q be its signless Laplacian tensor. Then each vector $\mathbf{1}_j$ (j = 1, ..., n) is an H-eigenvector of Q corresponding to H-eigenvalue d_j^+ .

Proof For vector $\mathbf{1}_i$, one can get that

$$\left[\mathcal{Q}\mathbf{1}_{j}^{k-1}\right]_{j} = d_{jj\dots j}\mathbf{1} + \sum_{(j,j_{2},\dots,j_{k})\in E} \mathbf{0} = d_{i}^{+}$$

and for each $i \in [n] \setminus \{j\}$

$$\left[\mathcal{Q}\mathbf{1}_{j}^{k-1}\right]_{i} = d_{ii\dots i}\mathbf{0} + \sum_{(i,i_{2},\dots,i_{k})\in E}\mathbf{0} = \mathbf{0}.$$

By Definition 2.1, the proposition follows.

The next corollary is a direct consequence of Proposition 5.1.

COROLLARY 5.1 Given an n-vertex k-uniform directed hypergraph G, let Q be its signless Laplacian tensor. Then Q has n linearly independent H-eigenvectors and at least n H-eigenvalues (Repeatable).

By Corollary 5.1, the signless Laplacian tensor Q of uniform directed hypergraph *G* has at least *n H*-eigenvalues. Now, we study the largest and smallest *H*-eigenvalues of uniform directed hypergraph *G* via its signless Laplacian tensor Q. Denote by $\lambda(Q)$ (respectively, $\mu(Q)$) as the largest (respectively, smallest) *H*-eigenvalue of uniform directed hypergraph *G* via its signless Laplacian tensor Q.

THEOREM 5.1 Let Q be the signless Laplacian tensor of an n-vertex k-uniform directed hypergraph G = (V, E). Then we have

$$0 \le \mu(\mathcal{Q}) \le \delta^+ \le \Delta^+ \le \lambda(\mathcal{Q}) \le 2\Delta^+$$

Proof By Proposition 5.1, $\mu(Q) \leq \delta^+ \leq \Delta^+ \leq \lambda(Q)$. Hence, we only have to prove $0 \leq \mu(Q)$ and $\lambda(Q) \leq 2\Delta^+$.

For any *H*-eigenvalue λ of Q, let **y** be its corresponding *H*-eigenvector of Q and y_i ($0 < |y_i|$) be an entry of **y** with maximum absolute value. Then the *i*-th eigenvalue equation gives that

$$\lambda y_i^{k-1} = \sum_{i_2, \dots, i_k \in [n]} l_{ii_2 \dots i_k} y_{i_2} \dots y_{i_k} = d_i^+ y_i^{k-1} + \sum_{(i, i_2, \dots, i_k) \in E} y_{i_2} \dots y_{i_k}.$$

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Then

$$\begin{aligned} |\lambda - d_i^+| |y_i|^{k-1} &= \left| \sum_{(i,i_2,\dots,i_k)\in E} y_{i_2}\dots y_{i_k} \right| \le \sum_{(i,i_2,\dots,i_k)\in E} |y_{i_2}|\dots |y_{i_k}| \\ &\le \sum_{(i,i_2,\dots,i_k)\in E} |y_i|^{k-1} = d_i^+ |y_i|^{k-1}. \end{aligned}$$

That is

$$|\lambda - d_i^+| \le d_i^+.$$

Hence

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$$0 \le \lambda \le 2d_i^+ \le 2\Delta^+.$$

Then we have $0 \le \mu(Q)$ and $\lambda(Q) \le 2\Delta^+$.

By the proof of Theorem 5.1, we have the following two corollaries.

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COROLLARY 5.2 The H-eigenvalues of signless Laplacian tensor \mathcal{Q} for n-vertex k-uniform directed hypergraph G lie in the union of n intervals in \mathbb{R} . These n intervals have the diagonal elements d_i^+ (i = 1, ..., n) as their centres, and d_i^+ (respectively) as their radii. That is, all the H-eigenvalues of signless Laplacian tensor \mathcal{Q} for n-vertex k-uniform directed hypergraph G lie in the interval with centre Δ^+ and radius Δ^+ .

COROLLARY 5.3 The largest H-eigenvalue $\lambda(Q)$ of signless Laplacian tensor Q for directed d-out-regular hypergraph G is 2d, with a corresponding H-eigenvector **1**.

By Theorem 5.1 and the proof of Theorem 3.2, we have the following corollary.

COROLLARY 5.4 Let Q be the signless Laplacian tensor of an *n*-vertex *k*-uniform directed hypergraph G = (V, E). Then we have

$$\lambda(\mathcal{Q}) \leq 2\binom{n-1}{k-1},$$

and equality holds if and only if G is an n-vertex k-uniform complete directed hypergraph with H-eigenvector $r\mathbf{1}$ corresponding to H-eigenvalue $\lambda(Q)$.

In spectral theory of undirected hypergraphs, it is obvious that there exists a nonnegative H-eigenvector corresponding to the largest H-eigenvalue $\lambda(Q)$. But, unlike spectral theory of undirected hypergraphs, until now we still cannot prove the following conjecture is true.

Conjecture 5.1 Given an *n*-vertex *k*-uniform directed hypergraph *G*, let Q be its signless Laplacian tensor and *k* be even. Then there exists a nonnegative *H*-eigenvector **x** corresponding to the largest *H*-eigenvalue $\lambda(Q)$ such that $\lambda(Q) = \frac{Q\mathbf{x}^k}{||\mathbf{x}||_k^k}$.

In spectral theory of undirected hypergraphs, since the signless Laplacian tensor Q of an even-uniform undirected hypergraph G is symmetric and diagonally dominated, then Q is positive semi-definite. But, unlike spectral theory of undirected hypergraphs, until now we still can't answer the following question.

Question 5.1 Given an even-uniform directed hypergraph G, let Q be its signless Laplacian tensor. Is Q a positive semi-definite tensor?

6. H-Eigenvalues of the uniform directed hyperstar

In the following, we introduce the special class of directed hyperstars.

Definition 6.1 Let G = (V, E) be a k-uniform directed hypergraph. If there is a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \ldots \cup V_d$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_d| = k - 1$, $E = \{V_0 \cup V_i \mid i \in [d]\}$ and only V_0 is tail, then G is called a *directed hyperstar*. The degree d of the vertex in V_0 , which is called the *heart*, is the *size* of the directed hyperstar. The arcs of G are *leaves*, and the vertices other than the heart are vertices of leaves.

It is an obvious fact that, with a possible renumbering of the vertices, all the directed hyperstars with the same size are identical. Moreover, by Definition 2.1, we see that the process of renumbering does not change the *H*-eigenvalues of the adjacency tensor, Laplacian tensor and signless Laplacian tensor of the hyperstar.

An example of a directed hyperstar is given in Figure 1.

The next proposition is a direct consequence of Definition 6.1.

PROPOSITION 6.1 Let G = (V, E) be a k-uniform directed hyperstar of size d > 0. Then except for one vertex $i \in [n]$ with $d_i = d_i^+ = d$, we have $d_j = d_j^- = \frac{1}{k-1}$ for the others.

By Theorems 3.1, 4.1 and 5.1, we have the following lemma.

COROLLARY 6.1 Let G = (V, E) be a k-uniform directed hyperstar with its maximum degree d > 0 and A, L, Q, respectively, be its adjacency tensor, Laplacian tensor, signless Laplacian tensor. Then $\lambda(A) \leq d \leq \lambda(L)$ and also $d \leq \lambda(Q)$.

When G is a k-uniform directed hyperstar, Theorem 3.1 and part of Corollary 6.1 can be strengthened as in the next theorem.



Figure 1. An example of a 3-uniform directed hyperstar of size 3. An arc is pictured as a closed curve with the containing solid discs the vertices in that arc. Different arcs are in different curves with different colours. The red (also in dashed margin) disc represents the heart.

THEOREM 6.1 Let G = (V, E) be a k-uniform directed hyperstar of size d > 0 and A be its adjacency tensor. Then $\mu(A) = 0 = \lambda(A)$, that is, all H-eigenvalues of the adjacency tensor for G are 0.

Proof Let $V(G) = \{1, 2, ..., n\}$. Without loss of generality, suppose that 1 is the heart vertex of *G* satisfying $d_1 = d$, and the arcs of *G* can be supposed as

$$E(G) = \left\{ e_j := (1, j_2, j_3, \dots, j_k) \mid j_i \in \{2, \dots, n\}, 1 \le j \le d, 2 \le i \le k \right\}$$

Let $\mathbf{x} \in \mathbb{R}^n$ be a nonzero *H*-eigenvector corresponding to $\lambda(\mathcal{A})$. Then we have that

$$\left(\mathcal{A}\mathbf{x}^{k-1}\right)_1 = \sum_{j=1}^d x_{j_2} x_{j_3} \dots x_{j_k} = \lambda(\mathcal{A}) x_1^{k-1},$$

and for each $i \in \{2, \ldots, n\}$

$$\left(\mathcal{A}\mathbf{x}^{k-1}\right)_i = 0 = \lambda(\mathcal{A})x_i^{k-1}.$$

Thus, if $\lambda(A) \neq 0$, then for each $i \in \{2, ..., n\}$

 $x_i = 0.$

Hence, by

$$\lambda(\mathcal{A})x_1^{k-1} = \sum_{j=1}^d x_{j_2}x_{j_3}\dots x_{j_k} = 0,$$

we have that

 $x_1 = 0.$

Thus, $\mathbf{x} = \mathbf{0}$, which contradicts with $\mathbf{x} \neq \mathbf{0}$. Consequently, $\lambda(\mathcal{A}) = 0$. By the same methods, we can prove that $\mu(\mathcal{A}) = 0$. Since $\lambda(\mathcal{A})$ (respectively, $\mu(\mathcal{A})$) is the largest (respectively, smallest) *H*-eigenvalue of *G*, then all *H*-eigenvalues of the adjacency tensor for *G* are 0.

And we have the following propositions about *H*-eigenvalues of the Laplacian tensor and signless Laplacian tensor for a uniform directed hyperstar.

PROPOSITION 6.2 Let G = (V, E) be a k-uniform directed hyperstar of size d > 0 and \mathcal{L} be its Laplacian tensor. Then $\frac{1}{k-1}$ is a H-eigenvalues of the Laplacian tensor for G.

Proof Suppose, without loss of generality, that $d_1 = d$. Let $\mathbf{x} \in \mathbb{R}^n$ be a nonzero vector such that $x_1 = \alpha \in \mathbb{R}$, and $x_2 = \cdots = x_n = 1$. Then, we see that

$$\left(\mathcal{L}\mathbf{x}^{k-1}\right)_1 = d\alpha^{k-1} - d1^{k-1} = d\alpha^{k-1} - d,$$

and for $i \in \{2, ..., n\}$

$$\left(\mathcal{L}\mathbf{x}^{k-1}\right)_i = \frac{1}{k-1}\mathbf{1}^{k-1} - \mathbf{0} = \frac{1}{k-1}.$$

Thus, **x** is an *H*-eigenvector of \mathcal{L} corresponding to an *H*-eigenvalue λ if and only if

$$d\alpha^{k-1} - d = \lambda \alpha^{k-1}$$
, and $\frac{1}{k-1} = \lambda 1^{k-1} = \lambda$.

It can be calculated that

$$\lambda = \frac{1}{k-1}$$
, and $\alpha = \sqrt[k-1]{\frac{d}{d-\frac{1}{k-1}}}$.

Hence, $\mathbf{x} = (\sqrt[k-1]{\frac{d}{d-\frac{1}{k-1}}}, 1, \dots, 1)^T$ is an *H*-eigenvector of Laplacian tensor for *G* corresponding to *H*-eigenvalue $\frac{1}{k-1}$. The result follows.

PROPOSITION 6.3 Let k be even and G = (V, E) be a k-uniform directed hyperstar of size d > 0 and Q be its signless Laplacian tensor. Then $\frac{1}{k-1}$ is a H-eigenvalues of the signless Laplacian tensor for G.

Proof Suppose, without loss of generality, that $d_1 = d$. Let $\mathbf{x} \in \mathbb{R}^n$ be a nonzero vector such that $x_1 = \alpha \in \mathbb{R}$, and $x_2 = \cdots = x_n = 1$. Then, we see that

$$\left(\mathcal{Q}\mathbf{x}^{k-1}\right)_1 = d\alpha^{k-1} + d1^{k-1} = d\alpha^{k-1} + d,$$

and for $i \in \{2, ..., n\}$

$$\left(\mathcal{L}\mathbf{x}^{k-1}\right)_i = \frac{1}{k-1}\mathbf{1}^{k-1} + \mathbf{0} = \frac{1}{k-1}$$

Thus, **x** is an *H*-eigenvector of \mathcal{L} corresponding to an *H*-eigenvalue λ if and only if

$$d\alpha^{k-1} + d = \lambda \alpha^{k-1}$$
, and $\frac{1}{k-1} = \lambda 1^{k-1} = \lambda$

When k is even, it can be calculated that

$$\lambda = \frac{1}{k-1}$$
, and $\alpha = -\sqrt[k-1]{\frac{d}{d-\frac{1}{k-1}}}$.

Hence, $\mathbf{x} = \left(-\frac{k-1}{\sqrt{\frac{d}{d-\frac{1}{k-1}}}}, 1, \dots, 1\right)^T$ is an *H*-eigenvector of signless Laplacian tensor for *G* corresponding to *H*-eigenvalue $\frac{1}{k-1}$. The result follows.

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