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$\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors

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ABSTRACT

The M-matrix is an important concept in matrix theory, and has many applications. Recently, this concept has been extended to higher order tensors [18]. In this paper, we establish some important properties of $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors. An $\mathcal{M}$-tensor is a $\mathcal{Z}$-tensor. We show that a $\mathcal{Z}$-tensor is a nonsingular $\mathcal{M}$-tensor if and only if it is semi-positive. Thus, a nonsingular $\mathcal{M}$-tensor has all positive diagonal entries; an $\mathcal{M}$-tensor, regarding as the limit of a sequence of nonsingular $\mathcal{M}$-tensors, has all nonnegative diagonal entries. We introduce even-order monotone tensors and present their spectral properties. In matrix theory, a Z-matrix is a nonsingular M-matrix if and only if it is monotone. This is no longer true in the case of higher order tensors. We show that an even-order monotone $\mathcal{Z}$-tensor is an even-order nonsingular $\mathcal{M}$-tensor, but not vice versa. An example of an even-order nontrivial monotone $\mathcal{Z}$-tensor is also given.

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1. Introduction

Several topics in multi-linear algebra have attracted considerable attention in recent years, especially on the eigenvalues of tensors [11–14] and higher order tensor decompositions [10]. Tensors (or hypermatrices) generalize the concept of matrices in linear algebra. The main difficulty in tensor problems is that they are generally nonlinear. Therefore, large amounts of results for matrices are never pervasive for higher order tensors. However, there are still some results preserved in the case of higher order tensors.

As an important example, some properties of spectra (or eigenvalues) in linear algebra remain true to tensors. Qi generalizes the concept of eigenvalues to higher order tensors in [13] by defining the tensor-vector product as

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,i_3,\dots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m},$$

where a multi-array $\mathcal{A}$ is an m-order n-dimensional tensor in $\mathbb{C}^{n\times n\times\cdots\times n}$ and x is a vector in $\mathbb{C}^n$. We call λ as an eigenvalue of tensor $\mathcal{A}$, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1})^{\mathrm{T}}$ denotes the componentwise $(m-1)$-th power of x. Further, we call λ as an H-eigenvalue, H$^+$-eigenvalue, or H^{++}-eigenvalue if $x \in \mathbb{R}^n$, $x \in \mathbb{R}_+^n$ $(x \geqslant 0)$, or $x \in \mathbb{R}_{++}^n$ $(x > 0)$, respectively. Also, Qi [13] introduces another kind of eigenvalues for higher order tensors. We call λ an E-eigenvalue of tensor $\mathcal{A}$, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x, \qquad x^{\mathrm{T}}x = 1;$$

and we call λ a Z-eigenvalue if $x \in \mathbb{R}^n$. There have been extensive studies and applications of both kinds of eigenvalues for tensors.

M-matrices are an important class of matrices and have been well studied (cf. [1]). They are closely related with spectral graph theory, the stationary distribution of Markov chains and the convergence of iterative methods for linear equations. Zhang et al. extend M-matrices to $\mathcal{M}$-tensors in [18] and study their properties. The main result in their paper is that every eigenvalue of an $\mathcal{M}$-tensor has a positive real part, which is the same as the M-matrix.

The main motivation of this paper is that there are no less than fifty equivalent definitions of nonsingular M-matrices [1]. We extend the other two definitions of nonsingular M-matrices, semi-positivity and monotonicity [1], to higher order tensors. How can they be generalized into higher order case? Are they still the equivalent conditions of higher order nonsingular $\mathcal{M}$-tensors? What properties of $\mathcal{M}$-tensors can they indicate? We will answer these questions in our paper.

In this way, we obtain some important properties of $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors. An $\mathcal{M}$-tensor is a $\mathcal{Z}$-tensor. We prove that a $\mathcal{Z}$-tensor is a nonsingular $\mathcal{M}$-tensor if and only if it is semi-positive. Thus, a nonsingular $\mathcal{M}$-tensor has all positive diagonal entries; an $\mathcal{M}$-tensor, regarded as the limit of a sequence of nonsingular $\mathcal{M}$-tensors, has all nonnegative diagonal entries. We introduce even-order monotone tensors and establish their spectral properties. In matrix theory, a Z-matrix is a nonsingular M-matrix if and only if it is monotone [1]. It is no longer true in the case of higher order tensors. We show that an even-order monotone $\mathcal{Z}$-tensor is an even-order nonsingular $\mathcal{M}$-tensor, but not vice versa. An example of an even-order nontrivial monotone $\mathcal{Z}$-tensor is also given.

An outline of this paper is as follows. Some preliminaries about tensors and M-matrices are presented in Section 2. We investigate semi-positive $\mathcal{Z}$-tensors and monotone $\mathcal{Z}$-tensors in Sections 3 and 4, respectively. We discuss $\mathcal{H}$-tensors, an extension of $\mathcal{M}$-tensors, in Section 5. Finally, we draw some conclusions in the last section.

2. Preliminaries

We present some preliminaries about the Perron–Frobenius theorem for nonnegative tensors and M-matrices.

2.1. Nonnegative tensor

Because of the difficulties in studying the properties of a general tensor, researchers focus on some *structured* tensors. The nonnegative tensor is one of the most well studied tensors. A tensor is said to be nonnegative, if all its entries are nonnegative.

The Perron–Frobenius theorem is the most famous result for nonnegative matrices (cf. [1]), which investigates the spectral radius of nonnegative matrices. Researchers also propose the similar results for nonnegative tensors, and refer them as the Perron–Frobenius theorem for nonnegative tensors. This theorem also studies the spectral radius of a nonnegative tensor $\mathcal{B}$,

$$\rho(\mathcal{B}) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \mathcal{B}\}.$$

Before stating the Perron–Frobenius theorem, we briefly introduce the conceptions of irreducible and weakly irreducible tensors.

Definition 1 *(Irreducible tensor).* (See [2].) A tensor $\mathcal{B}$ is called *reducible*, if there exists a non-empty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$b_{i_1 i_2 \cdots i_m} = 0, \quad \forall i_1 \in I, \ \forall i_2, i_3, \ldots, i_m \notin I.$$

Otherwise, we say $\mathcal{B}$ is *irreducible*.

Definition 2 *(Weakly irreducible nonnegative tensor).* (See [4].) We call a nonnegative matrix $GM(\mathcal{B})$ the *representation* associated to a nonnegative tensor $\mathcal{B}$, if the (i, j)-th entry of $GM(\mathcal{B})$ is defined to be the summation of $b_{i i_2 i_3 \cdots i_m}$ with indices $\{i_2, i_3, \ldots, i_m\} \ni j$. We call a tensor $\mathcal{B}$ *weakly reducible*, if its representation $GM(\mathcal{B})$ is reducible. If $\mathcal{B}$ is not weakly reducible, then it is called *weakly irreducible*.

Now with these conceptions, we can recall several results of the Perron–Frobenius theorem for nonnegative tensors that we will use in this paper.

Theorem 1 *(The Perron–Frobenius theorem for nonnegative tensors). If $\mathcal{B}$ is a nonnegative tensor of order m and dimension n, then $\rho(\mathcal{B})$ is an eigenvalue of $\mathcal{B}$ with a nonnegative eigenvector $x \in \mathbb{R}_+^n$ [16,17].*

If furthermore $\mathcal{B}$ is strictly nonnegative, then $\rho(\mathcal{B}) > 0$ [6].

If furthermore $\mathcal{B}$ is weakly irreducible, then $\rho(\mathcal{B})$ is an eigenvalue of $\mathcal{B}$ with a positive eigenvector $x \in \mathbb{R}_{++}^n$ [4].

Suppose that furthermore $\mathcal{B}$ is irreducible. If λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(\mathcal{B})$ [2].

2.2. M-matrix

M-matrix arises frequently in scientific computations [1], and we briefly introduce its definition and properties in this section.

A matrix is called a *Z-matrix* if all its off-diagonal entries are non-positive. It is apparent that a Z-matrix A can be written as [1]

$$A = sI - B,$$

where B is a nonnegative matrix ($B \geqslant 0$) and $s > 0$; When $s \geqslant \rho(B)$, we call A as an *M-matrix*; And further when $s > \rho(B)$, we call A as a *nonsingular M-matrix* [1].

There are more than fifty conditions in the literature that are equivalent to the definition of nonsingular M-matrix. We just list eleven of them here, which will be involved in our paper (cf. [1]).

If A is a Z-matrix, then the following conditions are equivalent:

(C1) A is a nonsingular M-matrix;
(C2) $A + D$ is nonsingular for each nonnegative diagonal matrix D;

(C3) Every real eigenvalue of *A* is positive;

(C4) *A* is *positive stable*; that is, the real part of each eigenvalue of *A* is positive;

(C5) *A* is *semi-positive*; that is, there exists $x > 0$ with $Ax > 0$;

(C6) There exists $x \geqslant 0$ with $Ax > 0$;

(C7) *A* has all positive diagonal entries and there exists a positive diagonal matrix *D* such that *AD* is strictly diagonally dominant;

(C8) *A* has all positive diagonal entries and there exists a positive diagonal matrix *D* such that *DAD* is strictly diagonally dominant;

(C9) *A* is *monotone*; that is, $Ax \geqslant 0$ implies $x \geqslant 0$;

(C10) There exist an inverse-positive matrix *B* and a nonsingular *M*-matrix *C* such that $A = BC$;

(C11) *A* has a *convergent regular splitting*; that is, *A* has a representation $A = M - N$, where $M^{-1} \geqslant 0$, $N \geqslant 0$, and $\rho(M^{-1}N) < 1$;

(C12) $\cdots$

2.3. $\mathcal{M}$-tensor

Zhang et al. define the $\mathcal{M}$-tensor following the definition of *M*-matrix [18]. In this section, we will introduce their results for $\mathcal{M}$-tensors. First, they define the $\mathcal{Z}$-tensor and $\mathcal{M}$-tensor as follows. We call a tensor the unit tensor and denote it $\mathcal{I}$ [10], if all of its diagonal entries are 1 and all of its off-diagonal entries are 0.

Definition 3 (*$\mathcal{Z}$-tensor*). We call a tensor $\mathcal{A}$ as a *$\mathcal{Z}$-tensor*, if all of its off-diagonal entries are non-positive, which is equivalent to write $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $s > 0$ and $\mathcal{B}$ is a nonnegative tensor ($\mathcal{B} \geqslant 0$).

Definition 4 (*$\mathcal{M}$-tensor*). We call a $\mathcal{Z}$-tensor $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ ($\mathcal{B} \geqslant 0$) as an *$\mathcal{M}$-tensor* if $s \geqslant \rho(\mathcal{B})$; We call it as a *nonsingular $\mathcal{M}$-tensor* if $s > \rho(\mathcal{B})$.

Unlike an *M*-matrix, the concept of $\mathcal{M}$-*tensor* not only depends on the tensor itself, but the eigenvalue problem associated with the tensor as well, that is, we can define other $\mathcal{M}$-tensors by applying other eigenvalues of a tensor, such as E-eigenvalues [13].

The main results in their paper show two equivalent conditions for the definition of nonsingular $\mathcal{M}$-tensor, which are the extensions of the conditions (C3) and (C4). If $\mathcal{A}$ is a $\mathcal{Z}$-tensor, then the following conditions are equivalent:

(D1) $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor;

(D2) Every real eigenvalue of $\mathcal{A}$ is positive;

(D3) The real part of each eigenvalue of $\mathcal{A}$ is positive.

2.4. Notation

We adopt the following notation in this paper. The calligraphy letters $\mathcal{A}, \mathcal{B}, \mathcal{D}, \dots$ denote the tensors; the capital letters $A, B, D, \dots$ represent the matrices; the lowercase letters $a, x, y, \dots$ refer to the vectors; and the Greek letters $\alpha, \beta, \lambda, \dots$ designate the scalars. Usually, the tensors in our paper are of order *m* and dimension *n*. When we write $\mathcal{A} \geqslant 0$, $A \geqslant 0$, or $x \geqslant 0$, we mean that every entry of $\mathcal{A}$, *A*, or *x* is nonnegative; when we write $\mathcal{A} > 0$, $A > 0$, or $x > 0$, we mean that every entry of $\mathcal{A}$, *A*, or *x* is positive.

The product of a tensor $\mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$ and a matrix $X \in \mathbb{R}^{n \times n}$ on mode-*k* [10] is defined as

$$(\mathcal{A} \times_k X)_{i_1 \cdots j_k \cdots i_m} = \sum_{i_k=1}^{n} a_{i_1 \cdots i_k \cdots i_m} x_{i_k j_k};$$

then denote

$$\mathcal{A}X^{m-1} = \mathcal{A} \times_2 X \times_3 \cdots \times_m X, \quad \text{and} \quad X_1^{\mathrm{T}} \mathcal{A} X_2^{m-1} = \mathcal{A} \times_1 X_1 \times_2 X_2 \times_3 \cdots \times_m X_2;$$

Finally, we introduce the "composite" of a diagonal tensor $\mathcal{D}$ and another tensor $\mathcal{A}$ as

$$(\mathcal{D}\mathcal{A})_{i_1 i_2 \cdots i_m} = d_{i_1 i_1 \cdots i_1} \cdot a_{i_1 i_2 \cdots i_m},$$

which indicates that $(\mathcal{D}\mathcal{A})x^{m-1} = \mathcal{D}((\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]})^{m-1}$; the "inverse" of a diagonal tensor $\mathcal{D}$ without zero diagonal entries as [10],

$$\left(\mathcal{D}^{-1}\right)_{i_1 i_1 \cdots i_1} = d_{i_1 i_1 \cdots i_1}^{-1}, \quad \text{and} \quad \text{otherwise } 0,$$

which indicates that $(\mathcal{D}^{-1}\mathcal{D})x^{m-1} = (\mathcal{D}\mathcal{D}^{-1})x^{m-1} = \mathcal{I}x^{m-1} = x^{[m-1]}$.

3. Semi-positivity and semi-nonnegativity

In this section, we will propose an equivalent definition of nonsingular $\mathcal{M}$-tensors following the conditions (C5) and (C6) in Section 2.2.

3.1. Definitions

First, we extend the *semi-positivity* [1] from matrices to tensors.

Definition 5 (*Semi-positive tensor*). We call a tensor $\mathcal{A}$ as a *semi-positive tensor*, if there exists $x > 0$ such that $\mathcal{A}x^{m-1} > 0$.

Because of the continuity of the tensor-vector product on the entries of the vector, the requirement $x > 0$ in the first definition can be relaxed into $x \geqslant 0$. We verify this statement as follows.

Theorem 2. *A tensor $\mathcal{A}$ is semi-positive if and only if there exists $x \geqslant 0$ such that $\mathcal{A}x^{m-1} > 0$.*

Proof. Define a map $T_{\mathcal{A}}(x) = (\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}$, then $x \mapsto T_{\mathcal{A}}(x)$ is continuous and bounded [3].
If $\mathcal{A}$ is semi-positive, then it is trivial that there is $x \geqslant 0$ such that $\mathcal{A}x^{m-1} > 0$ according to the definition.
If there exists $x \geqslant 0$ with $\mathcal{A}x^{m-1} > 0$, then there must be a closed ball $B(T_{\mathcal{A}}(x), \varepsilon)$ in $\mathbb{R}_{++}^n$, where $B(c, r) := \{v \mid \|v - c\| \leqslant r\}$. Since $T_{\mathcal{A}}$ is continuous, there exists $\delta > 0$ such that $T_{\mathcal{A}}(y) \in B(T_{\mathcal{A}}(x), \varepsilon)$ for all $y \in B(x, \delta)$. Let d be a zero-one vector with $d_i = 1$ if $x_i = 0$ and $d_i = 0$ if $x_i > 0$. Take $y = x + \frac{\delta}{\|d\|}d \in B(x, \delta)$. Then $y > 0$ and $T_{\mathcal{A}}(y) > 0$. Therefore, $\mathcal{A}$ is semi-positive. $\quad\square$

It is well known that a Z-matrix is a nonsingular M-matrix if and only if it is semi-positive [1]. Furthermore, we come to a similar conclusion for nonsingular $\mathcal{M}$-tensors.

Theorem 3. *A $\mathcal{Z}$-tensor is a nonsingular $\mathcal{M}$-tensor if and only if it is semi-positive.*

The proof of this theorem will be presented at the end of this section, after studying some properties of semi-positive $\mathcal{Z}$-tensors.

3.2. Semi-positive $\mathcal{Z}$-tensors

The first property is about the diagonal entries of a semi-positive $\mathcal{Z}$-tensor.

Proposition 4. *A semi-positive $\mathcal{Z}$-tensor has all positive diagonal entries.*

Proof. When $\mathcal{A}$ is a semi-positive $\mathcal{Z}$-tensor, there exists $x > 0$ such that $\mathcal{A}x^{m-1} > 0$. Consider $\mathcal{A}x^{m-1}$, we have

$$\left(\mathcal{A}x^{m-1}\right)_i = a_{ii\cdots i}x_i^{m-1} + \sum_{(i_2,i_3,\ldots,i_m)\neq(i,i,\ldots,i)} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m} > 0,$$

for $i = 1, 2, \ldots, n$. From $x_j > 0$ and $a_{ii_2i_3\cdots i_m} \leqslant 0$ for $(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)$, we can conclude that $a_{ii\cdots i} > 0$ for $i = 1, 2, \ldots, n$. $\quad\square$

Moreover we have a series of equivalent conditions of semi-positive $\mathcal{Z}$-tensors, following the conditions (C7), (C8), (C10), and (C11) in Section 2.2.

Proposition 5. *A $\mathcal{Z}$-tensor $\mathcal{A}$ is semi-positive if and only if $\mathcal{A}$ has all positive diagonal entries and there exists a positive diagonal matrix D such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant.*

Proof. Let $D = \text{diag}(d_1, d_2, \ldots, d_n)$. Then $\mathcal{A}D^{m-1}$ is strictly diagonally dominant means

$$\left|a_{ii\cdots i}d_i^{m-1}\right| > \sum_{(i_2,i_3,\ldots,i_m)\neq(i,i,\ldots,i)} |a_{ii_2\cdots i_m}d_{i_2}\cdots d_{i_m}|, \quad i = 1, 2, \ldots, n.$$

If $\mathcal{A}$ is a semi-positive $\mathcal{Z}$-tensor, then we know that $a_{ii\cdots i} > 0$ for $i = 1, 2, \ldots, n$ from Proposition 4, $a_{ii_2\cdots i_m} \leqslant 0$ for $(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)$, and there is $x > 0$ with $\mathcal{A}x^{m-1} > 0$. Let $D = \text{diag}(x)$, it is easy to conclude that D is positive diagonal and $\mathcal{A}D^{m-1}$ is strictly diagonally dominant.

If $\mathcal{A}$ has all positive diagonal entries, and there exists a positive diagonal matrix D such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant, let $x = \text{diag}(D) > 0$, then $\mathcal{A}x^{m-1} > 0$ since $a_{ii\cdots i} > 0$ for $i = 1, 2, \ldots, n$ and $a_{ii_2\cdots i_m} \leqslant 0$ for $(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)$. Thus, $\mathcal{A}$ is a semi-positive tensor. $\quad\square$

Proposition 6. *A $\mathcal{Z}$-tensor $\mathcal{A}$ is semi-positive if and only if $\mathcal{A}$ has all positive diagonal entries and there exist two positive diagonal matrices D_1 and D_2 such that $D_1\mathcal{A}D_2^{m-1}$ is strictly diagonally dominant.*

Proof. Notice that $D_1\mathcal{A}D_2^{m-1}$ is strictly diagonally dominant if and only if $\mathcal{A}D_2^{m-1}$ is strictly diagonally dominant in sake of the positivity of D_1's diagonal entries. Therefore, this proposition is a direct corollary of Proposition 5. $\quad\square$

Proposition 7. *A $\mathcal{Z}$-tensor $\mathcal{A}$ is semi-positive if and only if there exist a positive diagonal tensor $\mathcal{D}$ and a semi-positive $\mathcal{Z}$-tensor $\mathcal{C}$ with $\mathcal{A} = \mathcal{D}\mathcal{C}$.*

Proof. Let $\mathcal{D}$ be the diagonal tensor of $\mathcal{A}$ and $\mathcal{C} = \mathcal{D}^{-1}\mathcal{A}$. Clearly, $\mathcal{A} = \mathcal{D}\mathcal{C}$.

If $\mathcal{A}$ is semi-positive $\mathcal{Z}$-tensor, then $\mathcal{D}$ is positive diagonal and there exists $x > 0$ with $\mathcal{A}x^{m-1} > 0$. Thus, the vector $\mathcal{C}x^{m-1} = \mathcal{D}^{-1}((\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]})^{m-1}$ is also positive. So $\mathcal{C}$ is also a semi-positive $\mathcal{Z}$-tensor.

If $\mathcal{C}$ is a semi-positive $\mathcal{Z}$-tensor and $\mathcal{D}$ is positive diagonal, then there exists $x > 0$ with $\mathcal{C}x^{m-1} > 0$. Then the vector $\mathcal{A}x^{m-1} = \mathcal{D}((\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]})^{m-1}$ is also positive. Thus, $\mathcal{A}$ is a semi-positive $\mathcal{Z}$-tensor. $\quad\square$

Remark 8. After we prove Theorem 3, Proposition 7 can be restated as: A $\mathcal{Z}$-tensor $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor if and only if there exist a positive diagonal tensor $\mathcal{D}$ and a nonsingular $\mathcal{M}$-tensor $\mathcal{C}$ with $\mathcal{A} = \mathcal{D}\mathcal{C}$.

Proposition 9. *A $\mathcal{Z}$-tensor $\mathcal{A}$ is semi-positive if and only if there exist a positive diagonal tensor $\mathcal{D}$ and a nonnegative tensor $\mathcal{E}$ such that $\mathcal{A} = \mathcal{D} - \mathcal{E}$ and there exists $x > 0$ with $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} < x^{[m-1]}$.*

Proof. Let $\mathcal{D}$ be the diagonal tensor of $\mathcal{A}$ and $\mathcal{E} = \mathcal{D} - \mathcal{A}$. Clearly, $\mathcal{A} = \mathcal{D} - \mathcal{E}$ and $\mathcal{D}^{-1}\mathcal{E} = \mathcal{I} - \mathcal{D}^{-1}\mathcal{A}$.

If $\mathcal{A}$ is a semi-positive $\mathcal{Z}$-tensor, then $\mathcal{D}$ is positive diagonal and there exists $x > 0$ with $\mathcal{A}x^{m-1} > 0$. Then $\mathcal{D}x^{m-1} > \mathcal{E}x^{m-1}$, and thus $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} < x^{[m-1]}$.

If there exists $x > 0$ with $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} < x^{[m-1]}$, then $\mathcal{E}x^{m-1} < \mathcal{D}x^{m-1}$, thus $\mathcal{A}x^{m-1} > 0$. Therefore, $\mathcal{A}$ is a semi-positive $\mathcal{Z}$-tensor. $\quad\square$

Remark 10. It follows from [17, Lemma 5.4] that a semi-positive $\mathcal{Z}$-tensor can be split into $\mathcal{A} = \mathcal{D} - \mathcal{E}$, where $\mathcal{D}$ is a positive diagonal tensor and $\mathcal{E}$ is a nonnegative tensor with $\rho(\mathcal{D}^{-1}\mathcal{E}) < 1$.

3.3. Examples

Next we present some examples of nontrivial semi-positive $\mathcal{Z}$-tensors.

Proposition 11. *A strictly diagonally dominant $\mathcal{Z}$-tensor with nonnegative diagonal entries is a semi-positive $\mathcal{Z}$-tensor.*

Proof. Use **e** to denote the all ones vector. It is direct to show that a $\mathcal{Z}$-tensor $\mathcal{A}$ with all nonnegative diagonal entries is strictly diagonally dominant is equivalent to $\mathcal{A}\mathbf{e}^{m-1} > 0$. Since $\mathbf{e} > 0$, the result follows the definition of semi-positive $\mathcal{Z}$-tensors. □

Proposition 12. *A weakly irreducible nonsingular $\mathcal{M}$-tensor is a semi-positive $\mathcal{Z}$-tensor.*

Proof. When $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, we write $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B} \geqslant 0$ and $s > \rho(\mathcal{B})$. Since $\mathcal{A}$ is weakly irreducible, so is $\mathcal{B}$. Then there exists $x > 0$ such that $\mathcal{B}x^{m-1} = \rho(\mathcal{B})x^{[m-1]}$ from the Perron–Frobenius Theorem for nonnegative tensors (cf. Theorem 1), thus

$$\mathcal{A}x^{m-1} = \big(s - \rho(\mathcal{B})\big)x^{[m-1]} > 0.$$

Therefore, $\mathcal{A}$ is a semi-positive tensor. □

3.4. Proof of Theorem 3

Our aim is to prove the equality relation between the following two sets:

{semi-positive $\mathcal{Z}$-tensors} = {nonsingular $\mathcal{M}$-tensors}.

The first step is to verify the "$\subseteq$" part, which is relatively simple.

Lemma 13. *A semi-positive $\mathcal{Z}$-tensor is also a nonsingular $\mathcal{M}$-tensor.*

Proof. When $\mathcal{A}$ is semi-positive, there exists $x > 0$ with $\mathcal{A}x^{m-1} > 0$. We write $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ since $\mathcal{A}$ is a $\mathcal{Z}$-tensor, where $\mathcal{B} \geqslant 0$. Then

$$\min_{1 \leqslant i \leqslant n} \frac{(\mathcal{B}x^{m-1})_i}{x_i^{m-1}} \leqslant \rho(\mathcal{B}) \leqslant \max_{1 \leqslant i \leqslant n} \frac{(\mathcal{B}x^{m-1})_i}{x_i^{m-1}}.$$

Thus, $s - \rho(\mathcal{B}) \geqslant s - \max_{1 \leqslant i \leqslant n} \frac{(\mathcal{B}x^{m-1})_i}{x_i^{m-1}} = \min_{1 \leqslant i \leqslant n} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} > 0$, since both x and $\mathcal{A}x^{m-1}$ are positive. Therefore, $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor. □

The second step is to prove the "$\supseteq$" part employing a partition of general nonnegative tensors into weakly irreducible leading sub-tensors.

Lemma 14. *A nonsingular $\mathcal{M}$-tensor is also a semi-positive $\mathcal{Z}$-tensor.*

Proof. Assume that a nonsingular $\mathcal{M}$-tensor $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ is weakly reducible, otherwise we have proved a weakly irreducible nonsingular $\mathcal{M}$-tensor is also a semi-positive $\mathcal{Z}$-tensor in Proposition 12. Then $\mathcal{B}$ is also weakly reducible. Following [6, Theorem 5.2], the index set $I = \{1, 2, \dots, n\}$ can be partitioned into $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_k$ (here $A = A_1 \sqcup A_2$ means that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$) such that

(1) $\mathcal{B}_{I_t I_t \cdots I_t}$ is weakly irreducible,

(2) $b_{i_1 i_2 \cdots i_m} = 0$ for $i_1 \in I_t$ and $\{i_2, i_3, \ldots, i_m\} \nsubseteq I_t \sqcup I_{t+1} \sqcup \cdots \sqcup I_k$, $t = 1, 2, \ldots, k$.

Without loss of generality, we can assume that

$$I_1 = \{1, 2, \ldots, n_1\},$$

$$I_2 = \{n_1 + 1, n_1 + 2, \ldots, n_2\},$$

$$\cdots$$

$$I_k = \{n_{k-1} + 1, n_{k-1} + 2, \ldots, n\}.$$

We introduce the following denotations

$$\mathcal{B}_{(t,a)} := \mathcal{B}_{I_t I_{a_1} \cdots I_{a_{m-1}}}$$

and

$$\mathcal{B}_{(t,a)} z_a^{m-1} := \mathcal{B}_{(t,a)} \times_{a_1} z_{a_1} \times \cdots \times_{a_{m-1}} z_{a_{m-1}},$$

where a is an index vector of length $m-1$ and z_j's are column vectors. We also apply $\mathcal{B}[J]$ to denote the leading sub-tensor $(b_{i_1 i_2 \cdots i_m})_{i_j \in J}$, where J is an arbitrary index set. Since $s > \rho(\mathcal{B}) \geqslant \rho(\mathcal{B}[I_t])$, the leading sub-tensors $s\mathcal{I} - \mathcal{B}[I_t]$ are irreducible nonsingular $\mathcal{M}$-tensors. Hence they are also semi-positive, that is, there exists $x_t > 0$ with $sx_t^{[m-1]} - \mathcal{B}[I_t]x_t^{m-1} > 0$ for all $t = 1, 2, \ldots, k$.

Consider the leading sub-tensor $\mathcal{B}[I_1 \sqcup I_2]$ first. For all vectors z_1 of length n_1 and z_2 of length $n_2 - n_1$, we write

$$\mathcal{B}[I_1 \sqcup I_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^{m-1} = \begin{bmatrix} \mathcal{B}[I_1]z_1^{m-1} + \sum_{a \neq (1,1,\ldots,1)} \mathcal{B}_{(1,a)} z_a^{m-1} \\ \mathcal{B}[I_2]z_2^{m-1} \end{bmatrix},$$

where the entries of a only contain 1 and 2. Take $z_1 = x_1$ and $z_2 = \varepsilon x_2$, where $\varepsilon \in (0, 1)$ satisfies

$$\varepsilon \cdot \sum_{a \neq (1,1,\ldots,1)} \mathcal{B}_{(1,a)} x_a^{m-1} < sx_1^{[m-1]} - \mathcal{B}[I_1]x_1^{m-1}.$$

Since $\sum_{a \neq (1,1,\ldots,1)} \mathcal{B}_{(1,a)} z_a^{m-1} \leqslant \varepsilon (\sum_{a \neq (1,1,\ldots,1)} \mathcal{B}_{(1,a)} x_a^{m-1})$, it can be ensured that $\mathcal{B}[I_1]z_1^{m-1} + \sum_{a \neq (1,1,\ldots,1)} \mathcal{B}_{(1,a)} z_a^{m-1} < sz_1^{[m-1]}$. Therefore, we obtain

$$\begin{bmatrix} x_1 \\ \varepsilon x_2 \end{bmatrix} > 0 \quad \text{and} \quad s \begin{bmatrix} x_1 \\ \varepsilon x_2 \end{bmatrix}^{[m-1]} - \mathcal{B} \begin{bmatrix} x_1 \\ \varepsilon x_2 \end{bmatrix}^{m-1} > 0,$$

so $\mathcal{A}[I_1 \sqcup I_2] = s\mathcal{I} - \mathcal{B}[I_1 \sqcup I_2]$ is a semi-positive $\mathcal{Z}$-tensor.

Assume that $\mathcal{A}[I_1 \sqcup I_2 \sqcup \cdots \sqcup I_t]$ is a semi-positive $\mathcal{Z}$-tensor. Consider the leading sub-tensor $\mathcal{A}[I_1 \sqcup I_2 \sqcup \cdots \sqcup I_{t+1}]$ next. Substituting the index sets I_1 and I_2 above with $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_t$ and I_{t+1}, respectively, we can conclude that $\mathcal{A}[I_1 \sqcup I_2 \sqcup \cdots \sqcup I_{t+1}]$ is also a semi-positive $\mathcal{Z}$-tensor. Thus, by induction, we can prove that the weakly reducible nonsingular $\mathcal{M}$-tensor $\mathcal{A}$ is a semi-positive $\mathcal{Z}$-tensor as well. □

Combining Lemma 13 and Lemma 14, we finish the proof of Theorem 3. Thus, all the properties of semi-positive $\mathcal{Z}$-tensors we investigate above are the same with nonsingular $\mathcal{M}$-tensors, and vice versa. So the semi-positivity can be employed to study the nonsingular $\mathcal{M}$-tensors.

3.5. General $\mathcal{M}$-tensors

We discuss the nonsingular $\mathcal{M}$-tensors above. However, the general $\mathcal{M}$-tensors are also useful. The examples can be found in the literature. For instance, the Laplacian tensor $\mathcal{L}$ of a hypergraph (cf. [7,9,8,15]) is an $\mathcal{M}$-tensor, but is not a nonsingular $\mathcal{M}$-tensor.

An $\mathcal{M}$-tensor can be written as $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is nonnegative and $s \geqslant \rho(\mathcal{B})$. It is easy to verify that the tensor $\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon\mathcal{I}$ ($\varepsilon > 0$) is then a nonsingular $\mathcal{M}$-tensor and $\mathcal{A}$ is the limit of a sequence of $\mathcal{A}_\varepsilon$ when $\varepsilon \to 0$. Since all the diagonal entries of a semi-positive $\mathcal{Z}$-tensor, i.e., a nonsingular $\mathcal{M}$-tensor, are positive. Therefore, the diagonal entries of a general $\mathcal{M}$-tensor, as the limit of a sequence of nonsingular $\mathcal{M}$-tensor, must be nonnegative. Thus, we prove the following proposition.

Proposition 15. *A general $\mathcal{M}$-tensor has all nonnegative diagonal entries.*

The conception semi-positivity [1] can be extended as follows.

Definition 6 *(Semi-nonnegative tensor).* We call a tensor $\mathcal{A}$ as a *semi-nonnegative tensor*, if there exists $x > 0$ such that $\mathcal{A}x^{m-1} \geqslant 0$.

Unlike the semi-positive case, a tensor being a semi-nonnegative $\mathcal{Z}$-tensor is not equivalent to being a general $\mathcal{M}$-tensor. Actually, a semi-nonnegative $\mathcal{Z}$-tensor must be an $\mathcal{M}$-tensor, but the converse is not true. The proof of the first statement is analogous to Lemma 13, so we have the next theorem.

Theorem 16. *A semi-nonnegative $\mathcal{Z}$-tensor is also an $\mathcal{M}$-tensor.*

Conversely, we can give a counterexample to show that there exists an $\mathcal{M}$-tensor which is not semi-nonnegative. Let $\mathcal{B}$ be a nonnegative tensor of size $2 \times 2 \times 2 \times 2$, and the entries are as follows:

$$b_{1111} = 2, \qquad b_{1122} = b_{2222} = 1, \quad \text{and} \quad b_{i_1 i_2 i_3 i_4} = 0, \quad \text{otherwise.}$$

Then the spectral radius of $\mathcal{B}$ is apparently 2. Let $\mathcal{A} = 2\mathcal{I} - \mathcal{B}$, then $\mathcal{A}$ is an $\mathcal{M}$-tensor with entries

$$a_{1122} = -1, \qquad a_{2222} = 1, \quad \text{and} \quad a_{i_1 i_2 i_3 i_4} = 0, \quad \text{otherwise.}$$

Thus, for every positive vector x, the first component of $\mathcal{A}x^3$ is always negative and the second one is positive, which is to say that there is no such a positive vector x with $\mathcal{A}x^3 \geqslant 0$. Therefore, $\mathcal{A}$ is an $\mathcal{M}$-tensor but is not semi-nonnegative. However, there are still some special $\mathcal{M}$-tensors are semi-nonnegative.

Proposition 17. *The following tensors are semi-nonnegative*:

(1) *A diagonally dominant $\mathcal{Z}$-tensor with nonnegative diagonal entries is semi-nonnegative*;
(2) *A weakly irreducible $\mathcal{M}$-tensor is semi-nonnegative*;
(3) *Let $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ be a weakly reducible $\mathcal{M}$-tensor, where $\mathcal{B} \geqslant 0$ and $s = \rho(\mathcal{B})$, and $I_1, I_2, \ldots, I_k$ be the same as in Lemma 14. If*

$$\rho(\mathcal{B}[I_t]) \begin{cases} < s, & t = 1, 2, \ldots, k_1, \\ = s, & t = k_1 + 1, k_1 + 2, \ldots, k \end{cases}$$

and the entries of $\mathcal{B}[I_{k_1+1} \sqcup I_{k_1+2} \sqcup \cdots \sqcup I_k]$ are all zeros except the ones in the leading sub-tensors $\mathcal{B}[I_{k_1+1}], \mathcal{B}[I_{k_1+2}], \ldots, \mathcal{B}[I_k]$, then $\mathcal{A}$ is semi-nonnegative;
(4) *A symmetric $\mathcal{M}$-tensor is semi-nonnegative.*

The proofs of (1)–(3) are similar with the semi-positive case and (4) is a direct corollary of (3), therefore we omit them.

4. Monotonicity

Following the condition (C9) in Section 2.2, we generalize *monotone* [1] from nonsingular M-matrices to higher order tensors.

4.1. Definitions

Definition 7 *(Monotone tensor).* We call a tensor $\mathcal{A}$ as a *monotone tensor*, if $\mathcal{A}x^{m-1} \geqslant 0$ implies $x \geqslant 0$.

It is easy to show that the set of all monotone tensors is not empty, since the even-order diagonal tensors with all positive diagonal entries belong to this set. However, an odd-order tensor is never a monotone tensor. Since when m is odd, $\mathcal{A}x^{m-1} \geqslant 0$ implies $\mathcal{A}(-x)^{m-1} \geqslant 0$ as well, thus we cannot guarantee that x is nonnegative. Therefore, we refer to even-order tensors only in this section.

Sometimes we will use another equivalent definition of monotone tensor for convenience.

Lemma 18. *An even-order tensor $\mathcal{A}$ is a monotone tensor if and only if $\mathcal{A}x^{m-1} \leqslant 0$ implies $x \leqslant 0$.*

Proof. Suppose that $\mathcal{A}$ is a monotone tensor. Since $\mathcal{A}x^{m-1} \leqslant 0$ and $m-1$ is odd, we have

$$\mathcal{A}(-x)^{m-1} = -\mathcal{A}x^{m-1} \geqslant 0.$$

Then $-x \geqslant 0$ by the definition, which is equivalent to $x \leqslant 0$.

If $\mathcal{A}x^{m-1} \leqslant 0$ implies $x \leqslant 0$. When $\mathcal{A}y^{m-1} \geqslant 0$, we have

$$\mathcal{A}(-y)^{m-1} = -\mathcal{A}y^{m-1} \leqslant 0.$$

Therefore, $-y \leqslant 0$, which is equivalent to $y \geqslant 0$. Thus, $\mathcal{A}$ is a monotone tensor. $\square$

4.2. Properties

We shall prove that a monotone $\mathcal{Z}$-tensor is also a nonsingular $\mathcal{M}$-tensor. Before that, we need some lemmas.

Lemma 19. *An even-order monotone tensor has no zero H-eigenvalue.*

Proof. Let $\mathcal{A}$ be an even-order monotone tensor. If $\mathcal{A}$ has a zero H-eigenvalue, that is, there is a nonzero vector $x \in \mathbb{R}^n$ such that $\mathcal{A}x^{m-1} = 0$, then $\mathcal{A}(\alpha x)^{m-1} = 0$ as well for all $\alpha \in \mathbb{R}$. Thus, we cannot ensure that $\alpha x \geqslant 0$, which contradicts the definition of a monotone tensor. Therefore, $\mathcal{A}$ has no zero H-eigenvalue. $\square$

Lemma 20. *Every H^+-eigenvalue of an even-order monotone tensor is nonnegative.*

Proof. Let $\mathcal{A}$ be an even-order monotone tensor and λ be an H^+-eigenvalue of $\mathcal{A}$, that is, there is a nonzero vector $x \in \mathbb{R}^n_+$ such that $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$. Then we have $\mathcal{A}(-x)^{m-1} = -\lambda x^{[m-1]}$, since $m-1$ is odd. If $\lambda < 0$, then $\mathcal{A}(-x)^{m-1} = -\lambda x^{[m-1]} \geqslant 0$, which indicates $-x \geqslant 0$ as well as $x \leqslant 0$. It contradicts that x is nonzero and nonnegative. Then $\lambda \geqslant 0$. $\square$

The next theorem follows directly from Lemma 19 and Lemma 20.

Theorem 21. *Every H^+-eigenvalue of an even-order monotone tensor is positive.*

By applying this result, we can now reveal the relationship of the set of even-order monotone $\mathcal{Z}$-tensors and that of nonsingular $\mathcal{M}$-tensors.

Theorem 22. *An even-order monotone $\mathcal{Z}$-tensor is also a nonsingular $\mathcal{M}$-tensor.*

Proof. Let $\mathcal{A}$ be an even-order monotone $\mathcal{Z}$-tensor. We write $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ since $\mathcal{A}$ is a $\mathcal{Z}$-tensor, where $\mathcal{B} \geqslant 0$. Then $\rho(\mathcal{B})$ is an H$^+$-eigenvalue of $\mathcal{B}$ by Perron–Frobenius theorem for nonnegative tensors; that is, there is a nonzero vector $x \geqslant 0$ with $\mathcal{B}x^{m-1} = \rho(\mathcal{B})x^{[m-1]}$. Then

$$\mathcal{A}x^{m-1} = (s\mathcal{I} - \mathcal{B})x^{m-1} = \big(s - \rho(\mathcal{B})\big)x^{[m-1]},$$

which is to say that $s - \rho(\mathcal{B})$ is an H$^+$-eigenvalue of $\mathcal{A}$. From Theorem 21, the H$^+$-eigenvalue $s - \rho(\mathcal{B}) > 0$, which indicates $s > \rho(\mathcal{B})$. So $\mathcal{A}$ is also a nonsingular $\mathcal{M}$-tensor. $\square$

This theorem tells us that

$$\{\text{Monotone } \mathcal{Z}\text{-tensors of even order}\} \subseteq \{\text{Nonsingular } \mathcal{M}\text{-tensors of even order}\}.$$

However, we will show that not every nonsingular $\mathcal{M}$-tensor is monotone in the following sub-section. The equivalent relation in matric situations between these two conditions is no longer true, when the order is larger than 2.

Next, we will present some properties of monotone $\mathcal{Z}$-tensors.

Proposition 23. *An even-order monotone $\mathcal{Z}$-tensor has all positive diagonal entries.*

Proof. Let $\mathcal{A}$ be an even-order monotone $\mathcal{Z}$-tensor. Consider $\mathcal{A}e_i^{m-1}$ $(i = 1, 2, \ldots, n)$, where e_i denotes the vector with only one nonzero entry 1 in the i-th position. We have

$$\big(\mathcal{A}e_i^{m-1}\big)_i = a_{ii\cdots i}, \quad i = 1, 2, \ldots, n,$$
$$\big(\mathcal{A}e_i^{m-1}\big)_j = a_{ji\cdots i} \leqslant 0, \quad j \neq i.$$

If $a_{ii\cdots i} \leqslant 0$, then $\mathcal{A}e_i^{m-1} \leqslant 0$, which indicates $e_i \leqslant 0$ by Lemma 18, but it is impossible. So we have $a_{ii\cdots i} > 0$ for $i = 1, 2, \ldots, n$. $\square$

The next proposition shows some rows of a monotone $\mathcal{Z}$-tensor is strictly diagonally dominant.

Proposition 24. *Let $\mathcal{A}$ be an even-order monotone $\mathcal{Z}$-tensor. Then*

$$a_{ii\cdots i} > \sum_{(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)} |a_{ii_2\cdots i_m}|$$

for some $i \in \{1, 2, \ldots, n\}$.

Proof. Consider $\mathcal{A}\mathbf{e}^{m-1}$ $(i = 1, 2, \ldots, n)$, where $\mathbf{e}$ denotes the all ones vector. We have

$$\big(\mathcal{A}\mathbf{e}^{m-1}\big)_i = a_{ii\cdots i} + \sum_{(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)} a_{ii_2\cdots i_m} = a_{ii\cdots i} - \sum_{(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)} |a_{ii_2\cdots i_m}|,$$

since $a_{ii_2\cdots i_m} \leqslant 0$ for $(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)$.

If $a_{ii\cdots i} \leqslant \sum_{(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)} |a_{ii_2\cdots i_m}|$ for all $i = 1, 2, \ldots, n$, then $\mathcal{A}\mathbf{e}^{m-1} \leqslant 0$, which indicates $\mathbf{e} \leqslant 0$ by Lemma 18, and it is impossible. So $a_{ii\cdots i} > \sum_{(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)} |a_{ii_2\cdots i_m}|$ for some i. $\square$

Proposition 25. *Let $\mathcal{A}$ be an even-order monotone $\mathcal{Z}$-tensor. Then $\mathcal{A} + \mathcal{D}$ has all positive H$^+$-eigenvalues for each nonnegative diagonal tensor $\mathcal{D}$.*

Proof. If $\mathcal{A} + \mathcal{D}$ has a non-positive H$^+$-eigenvalue, that is, there exists a nonzero vector $x \geqslant 0$ such that $(\mathcal{A} + \mathcal{D})x^{m-1} = \lambda x^{[m-1]}$ and $\lambda \leqslant 0$, then $\mathcal{A}x^{m-1} = \lambda x^{[m-1]} - \mathcal{D}x^{m-1} \leqslant 0$, since x and $\mathcal{D}$ are non-negative and λ is non-positive, which implies $x \leqslant 0$ from the definition of monotone $\mathcal{Z}$-tensors. This is a contradiction. Therefore, $\mathcal{A} + \mathcal{D}$ has no non-positive H$^+$-eigenvalues for all nonnegative diagonal tensor $\mathcal{D}$. $\square$

Furthermore, the monotone $\mathcal{Z}$-tensor also has the following two equivalent definitions, following the condition (C10) and (C11).

Proposition 26. *A $\mathcal{Z}$-tensor $\mathcal{A}$ is monotone if and only if there exist a positive diagonal tensor $\mathcal{D}$ and a monotone $\mathcal{Z}$-tensor $\mathcal{C}$ such that $\mathcal{A} = \mathcal{D}\mathcal{C}$.*

Proof. Let $\mathcal{D}$ be the diagonal tensor of $\mathcal{A}$ and $\mathcal{C} = \mathcal{D}^{-1}\mathcal{A}$. Clearly, $\mathcal{A} = \mathcal{D}\mathcal{C}$.

If $\mathcal{A}$ is a monotone $\mathcal{Z}$-tensor, then $\mathcal{D}$ is positive diagonal and $\mathcal{A}x^{m-1} \geqslant 0$ implies $x \geqslant 0$. When $\mathcal{C}x^{m-1} \geqslant 0$, the vector $\mathcal{A}x^{m-1} = \mathcal{D}((\mathcal{C}x^{m-1})^{[\frac{1}{m-1}]})^{m-1}$ is also nonnegative, thus $x \geqslant 0$. Since $\mathcal{C}$ is also a $\mathcal{Z}$-tensor, then $\mathcal{C}$ is a monotone $\mathcal{Z}$-tensor.

If $\mathcal{C}$ is a monotone $\mathcal{Z}$-tensor and $\mathcal{D}$ is positive diagonal, then $\mathcal{C}x^{m-1} \geqslant 0$ implies $x \geqslant 0$. When $\mathcal{A}x^{m-1} \geqslant 0$, the vector $\mathcal{C}x^{m-1} = \mathcal{D}^{-1}((\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]})^{m-1}$ is also nonnegative, thus $x \geqslant 0$. So $\mathcal{A}$ is a monotone $\mathcal{Z}$-tensor. $\square$

Proposition 27. *A $\mathcal{Z}$-tensor $\mathcal{A}$ is monotone if and only if there exist a positive diagonal tensor $\mathcal{D}$ and a non-negative tensor $\mathcal{E}$ such that $\mathcal{A} = \mathcal{D} - \mathcal{E}$ and $\mathcal{D}^{-1}\mathcal{E}$ satisfying $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} \leqslant x^{[m-1]}$ implies $x \geqslant 0$.*

Proof. Let $\mathcal{D}$ be the diagonal tensor of $\mathcal{A}$ and $\mathcal{E} = \mathcal{D} - \mathcal{A}$. Clearly, $\mathcal{A} = \mathcal{D} - \mathcal{E}$ and $\mathcal{D}^{-1}\mathcal{E} = \mathcal{I} - \mathcal{D}^{-1}\mathcal{A}$.

If $\mathcal{A}$ is a monotone $\mathcal{Z}$-tensor, then $\mathcal{D}$ is positive diagonal and $\mathcal{A}x^{m-1} \geqslant 0$ implies $x \geqslant 0$. When $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} \leqslant x^{[m-1]}$, we have $\mathcal{E}x^{m-1} \leqslant \mathcal{D}x^{m-1}$, thus $\mathcal{A}x^{m-1} \geqslant 0$. This indicates $x \geqslant 0$.

If $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} \leqslant x^{[m-1]}$ implies $x \geqslant 0$. When $\mathcal{A}x^{m-1} \geqslant 0$, we have $\mathcal{D}x^{m-1} \geqslant \mathcal{E}x^{m-1}$, thus $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} \leqslant x^{[m-1]}$, which indicates $x \geqslant 0$. Since $\mathcal{A}$ is also a $\mathcal{Z}$-tensor, then $\mathcal{A}$ is a monotone $\mathcal{Z}$-tensor. $\square$

4.3. A counterexample

We will give a 4-order counterexample in this section to show that the set of all monotone $\mathcal{Z}$-tensors is a proper subset of the set of all nonsingular $\mathcal{M}$-tensors, when the order is larger than 2.

The denotation of the Kronecker product [5] for $\mathcal{A} = X \otimes Y$ means $a_{i_1 i_2 i_3 i_4} = x_{i_1 i_2} \cdot y_{i_3 i_4}$. Let $\mathcal{J} = I_n \otimes I_n$, where I_n denotes the $n \times n$ identity matrix. It is obvious that the spectral radius $\rho(\mathcal{J}) = n$, since the sum of each rows of $\mathcal{J}$ equals n. Take

$$\mathcal{A} = s\mathcal{I} - \mathcal{J} \quad (s > n) \quad \text{and} \quad x = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -\delta \end{bmatrix} \quad (0 < \delta < 1).$$

Then $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor and

$$\mathcal{A}x^3 = sx^{[3]} - (x^{\mathrm{T}}x)x = s\begin{bmatrix} 1 \\ \vdots \\ 1 \\ -\delta^3 \end{bmatrix} - (n - 1 + \delta^2)\begin{bmatrix} 1 \\ \vdots \\ 1 \\ -\delta \end{bmatrix} = \begin{bmatrix} s - n + 1 - \delta^2 \\ \vdots \\ s - n + 1 - \delta^2 \\ (n - 1 + (1 - s)\delta^2)\delta \end{bmatrix}.$$

When $\delta \leqslant \sqrt{\frac{n-1}{s-1}}$, the vector $\mathcal{A}x^3$ is nonnegative while x is not nonnegative. Therefore, $\mathcal{A}$ is not a monotone $\mathcal{Z}$-tensor, although it is a nonsingular $\mathcal{M}$-tensor.

4.4. An example

Let $\mathcal{B} = (a^{[2k-1]}b^{\mathrm{T}}) \otimes (bb^{\mathrm{T}}) \otimes \cdots \otimes (bb^{\mathrm{T}})$ be a tensor of order $2k$, where a and b are nonnegative vectors. It is direct to compute that $\rho(\mathcal{B}) = (b^{\mathrm{T}}a)^{2k-1}$. Then $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ $(s > (b^{\mathrm{T}}a)^{2k-1})$ is a nonsingular $\mathcal{M}$-tensor. For each x, we get

$$\mathcal{A}x^{2k-1} = sx^{[2k-1]} - a^{[2k-1]}(b^{\mathrm{T}}x)^{2k-1}.$$

When $\mathcal{A}x^{2k-1} \geqslant 0$, we have

$$s^{\frac{1}{2k-1}} \cdot x \geqslant a \cdot (b^{\mathrm{T}}x),$$

thus

$$s^{\frac{1}{2k-1}} \cdot (b^{\mathrm{T}}x) \geqslant (b^{\mathrm{T}}a)(b^{\mathrm{T}}x).$$

Since $s > (b^{\mathrm{T}}a)^{2k-1}$, we can conclude $b^{\mathrm{T}}x \geqslant 0$. So $x \geqslant a \cdot (b^{\mathrm{T}}x) \cdot s^{-\frac{1}{2k-1}} \geqslant 0$, which indicates that $\mathcal{A}$ is a monotone $\mathcal{Z}$-tensor.

5. An extension of $\mathcal{M}$-tensors

Inspired by the conception of H-matrix [1], we can extend $\mathcal{M}$-tensors to $\mathcal{H}$-tensors. First, we define the comparison tensor.

Definition 8. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a tensor of order m and dimension n. We call another tensor $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \cdots i_m})$ as the *comparison tensor* of $\mathcal{A}$ if

$$m_{i_1 i_2 \cdots i_m} = \begin{cases} +|a_{i_1 i_2 \cdots i_m}|, & \text{if } (i_2, i_3, \ldots, i_m) = (i_1, i_1, \ldots, i_1), \\ -|a_{i_1 i_2 \cdots i_m}|, & \text{if } (i_2, i_3, \ldots, i_m) \neq (i_1, i_1, \ldots, i_1). \end{cases}$$

Then we can state what is an $\mathcal{H}$-tensor.

Definition 9. We call a tensor an $\mathcal{H}$-*tensor*, if its comparison tensor is an $\mathcal{M}$-tensor; we call it as a *nonsingular $\mathcal{H}$-tensor*, if its comparison tensor is a nonsingular $\mathcal{M}$-tensor.

Nonsingular $\mathcal{H}$-tensors have a property called quasi-strictly diagonally dominant, which can be proved directly from the properties of nonsingular $\mathcal{M}$-tensors. Therefore, we omit the proof.

Proposition 28. *A tensor $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor if and only if it is* quasi-strictly diagonally dominant, *that is, there exist n positive real numbers $d_1, d_2, \ldots, d_n$ such that*

$$|a_{ii\cdots i}|d_i^{m-1} > \sum_{(i_2, i_3, \ldots, i_m) \neq (i,i,\ldots,i)} |a_{ii_2 \cdots i_m}|d_{i_2} \cdots d_{i_m}, \quad i = 1, 2, \ldots, n.$$

Similarly to the nonsingular $\mathcal{M}$-tensor, nonsingular $\mathcal{H}$-tensor has other equivalent definitions.

Proposition 29. *The following conditions are equivalent*:

(E1) *A tensor $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor*;
(E2) *There exists a positive diagonal matrix D such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant*;
(E3) *There exist two positive diagonal matrix D_1 and D_2 such that $D_1\mathcal{A}D_2^{m-1}$ is strictly diagonally dominant.*

6. Conclusions

In this paper, we give the proofs or the counterexamples to show these three relations between different sets of tensors:

$$\{\text{semi-positive } \mathcal{Z}\text{-tensors}\} = \{\text{nonsingular } \mathcal{M}\text{-tensors}\},$$

$$\{\text{semi-nonnegative } \mathcal{Z}\text{-tensors}\} \subsetneqq \{\text{general } \mathcal{M}\text{-tensors}\},$$

$$\{\text{even-order monotone } \mathcal{Z}\text{-tensors}\} \subsetneqq \{\text{even-order nonsingular } \mathcal{M}\text{-tensors}\}.$$

Applying these relations, we investigate the properties of nonsingular and general $\mathcal{M}$-tensors. Along with the results in Zhang et al. [18], the equivalent conditions of nonsingular $\mathcal{M}$-tensors until now are listed as follows.

If $\mathcal{A}$ is a $\mathcal{Z}$-tensor, then the following conditions are equivalent:

(D1) $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor;
(D2) Every real eigenvalue of $\mathcal{A}$ is positive [18];
(D3) The real part of each eigenvalue of $\mathcal{A}$ is positive [18];
(D4) $\mathcal{A}$ is semi-positive; that is, there exists $x > 0$ with $\mathcal{A}x^{m-1} > 0$;
(D5) There exists $x \geqslant 0$ with $\mathcal{A}x^{m-1} > 0$;
(D6) $\mathcal{A}$ has all positive diagonal entries, and there exists a positive diagonal matrix D such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant;
(D7) $\mathcal{A}$ has all positive diagonal entries, and there exist two positive diagonal matrices D_1 and D_2 such that $D_1\mathcal{A}D_2^{m-1}$ is strictly diagonally dominant;
(D8) There exist a positive diagonal tensor $\mathcal{D}$ and a nonsingular $\mathcal{M}$-tensor $\mathcal{C}$ with $\mathcal{A} = \mathcal{D}\mathcal{C}$;
(D9) There exist a positive diagonal tensor $\mathcal{D}$ and a nonnegative tensor $\mathcal{E}$ such that $\mathcal{A} = \mathcal{D} - \mathcal{E}$ and there exists $x > 0$ with $(\mathcal{D}^{-1}\mathcal{E})x^{m-1} < x^{[m-1]}$.

Acknowledgements

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