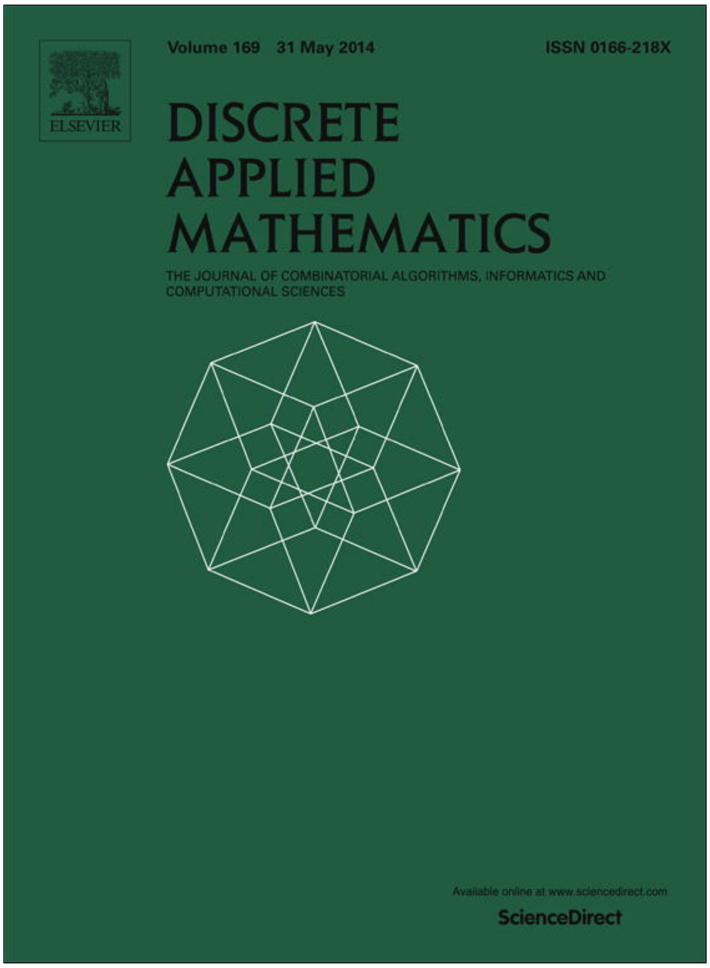
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The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph

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ABSTRACT

In this paper, we show that the eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a k-uniform hypergraph are closely related to some configured components of that hypergraph. We show that the components of an eigenvector associated with the zero eigenvalue of the Laplacian or signless Laplacian tensor have the same modulus. Moreover, under a *canonical* regularization, the phases of the components of these eigenvectors only can take some uniformly distributed values in $\{\exp(\frac{2j\pi}{k}) \mid j \in [k]\}$. These eigenvectors are divided into H-eigenvectors and N-eigenvectors. Eigenvectors with maximal support are called *maximal*. The maximal canonical H-eigenvectors characterize the even (odd)-bipartite connected components of the hypergraph and vice versa, and maximal canonical N-eigenvectors characterize some multi-partite connected components of the hypergraph and vice versa.

1. Introduction

In this paper, we study the eigenvectors associated with the zero eigenvalues of the Laplacian tensor and the signless Laplacian tensor proposed by Qi [23]. It turns out that these eigenvectors are characterized by some configured components of the underlying hypergraph and vice versa. This work is motivated by the classical results about bipartite graphs [5,2]. Our analysis makes use of the recent rapid developments on both spectral hypergraph theory [17,10,6,23,19,15,27,26,32,31,30,16,12] and spectral theory of tensors [3,4,8,9,16,17,15,11,18,20–22,24,28,33,34].

The study of Laplacian-type tensors for a uniform hypergraph has become an active research frontier in spectral hypergraph theory recently [10,15,32,30,23,12]. Notably, Qi [23] proposed a simple and natural definition $\mathcal{D}-\mathcal{A}$ for the Laplacian tensor and $\mathcal{D}+\mathcal{A}$ for the signless Laplacian tensor. Here $\mathcal{A}=(a_{i_1\ldots i_k})$ is the adjacency tensor of a k-uniform hypergraph and $\mathcal{D}=(d_{i_1\ldots i_k})$ the diagonal tensor with its diagonal elements being the degrees of the vertices. Following this, Hu and Qi proposed the normalized Laplacian tensor (or simply Laplacian) and made some explorations on it [12], which is the analogue of the spectral graph theory investigated by Chung [5].

In spectral graph theory, it is well known that the multiplicity of the zero eigenvalue of the Laplacian matrix is equal to the number of connected components of the graph, and the multiplicity of the zero eigenvalue of the signless Laplacian matrix is equal to the number of bipartite connected components of the graph [2]. In this paper, we investigate these matrices' analogues in spectral hypergraph theory. It turns out that, on the one hand, the situation is much more complicated, and, on the other hand, the results are more abundant. We show the following (please see Section 2 for basic definitions):

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(i) Let k be even and G be a k-uniform hypergraph.
   - (Proposition 5.1) The number of maximal canonical H-eigenvectors associated with the zero eigenvalue of the signless Laplacian tensor equals the number of odd-bipartite connected components of G.
   - (Proposition 5.2) The number of maximal canonical H-eigenvectors associated with the zero eigenvalue of the Laplacian tensor equals the sum of the number of even-bipartite connected components of G and the number of connected components of G, minus the number of singletons of G.

(ii) Let k be odd and G be a k-uniform hypergraph.
   - (Proposition 5.3) The number of maximal canonical H-eigenvectors associated with the zero eigenvalue of the Laplacian tensor equals the number of connected components of G.
   - (Corollary 5.1) The number of maximal canonical H-eigenvectors associated with the zero eigenvalue of the signless Laplacian tensor is equal to the number of singletons of G.

When we turn to N-eigenvectors, we have the following (the definitions for the various multi-partite connected components are given in Section 6):

(i) (Proposition 6.1) Let $G = (V, E)$ be a 3-uniform hypergraph. Then the number of maximal canonical conjugate N-eigenvector pairs associated with the zero eigenvalue of the Laplacian tensor equals the number of tripartite connected components of G.

(ii) Let $G = (V, E)$ be a 4-uniform hypergraph.
   - (Proposition 6.2) The number of maximal canonical conjugate N-eigenvector pairs associated with the zero eigenvalue of the Laplacian tensor equals the number of L-quadripartite connected components of G.
   - (Proposition 6.3) The number of maximal canonical conjugate N-eigenvector pairs of the zero eigenvalue of the signless Laplacian tensor equals the number of sL-quadripartite connected components of G.

(iii) (Proposition 6.4) Let $G = (V, E)$ be a 5-uniform hypergraph. Then the number of maximal canonical conjugate N-eigenvector pairs of the zero eigenvalue of the Laplacian tensor equals the number of pentapartite connected components of G.

The results related with N-eigenvectors can be generalized to any order $k \geq 6$. But it is somewhat complicated to describe the corresponding configured components. Hence, only $k = 3, 4, 5$ are presented in this paper.

On top of the above results, we show in Proposition 3.1 that an hm-bipartite hypergraph has a symmetric spectrum, and in Proposition 3.2 that when k is even the spectrum of the Laplacian tensor and the spectrum of the signless Laplacian tensor of an hm-bipartite hypergraph are equal. We also show that zero is not an eigenvalue of the signless Laplacian tensor of a connected k-uniform hypergraph with odd k (Proposition 4.1).

The rest of this paper is organized as follows. Some definitions on eigenvalues of tensors and hypergraphs are presented in the next section. We discuss in Section 3 some spectral properties of hm-bipartite hypergraphs. Then we characterize the eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph in Section 4. In Section 5, we establish the connection between maximal canonical H-eigenvectors associated with the eigenvalue zero and some configured components of the hypergraph. The discussion is extended to N-eigenvectors in Section 6. Some final remarks are made in the last section.

2. Preliminaries

Some preliminary definitions of eigenvalues of tensors and uniform hypergraphs are presented in this section.

2.1. Eigenvalues of tensors

In this subsection, some basic facts about eigenvalues and eigenvectors of tensors are reviewed. For comprehensive references, see [20–22,8] and references therein.

Let $\mathbb{C}$ ($\mathbb{R}$) be the field of complex (real) numbers and $\mathbb{C}^n$ ($\mathbb{R}^n$) the n-dimensional complex (real) space. For integers $k \geq 3$ and $n \geq 2$, a real tensor $\mathcal{T} = (t_{i_1\dots i_k})$ of order k and dimension n refers to a multiway array (also called hypermatrix) with entries $t_{i_1\dots i_k}$ such that $t_{i_1\dots i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1, \dots, n\}$ and $j \in [k]$. Tensors referred to are always k-th order real tensors in this paper, and the dimensions will be clear from context. Given a vector $\mathbf{x} \in \mathbb{C}^n$, define an n-dimensional vector $\mathcal{T}\mathbf{x}^{k-1}$ with its ith element being $\sum_{i_2,\dots,i_k\in[n]} t_{ii_2\dots i_k} x_{i_2} \cdots x_{i_k}$ for all $i \in [n]$. Let $\mathcal{I}$ be the identity tensor of appropriate dimension, e.g., $i_{i_1\dots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise when the dimension is n. The following definition was introduced by Qi [20].

Definition 2.1. Let $\mathcal{T}$ be a k-th order n-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})\,\mathbf{x}^{k-1} = 0$ has a solution $\mathbf{x} \in \mathbb{C}^n \setminus \{0\}$, then λ is called an eigenvalue of the tensor $\mathcal{T}$ and $\mathbf{x}$ an eigenvector of $\mathcal{T}$ associated with λ. If an eigenvalue λ has an eigenvector $\mathbf{x} \in \mathbb{R}^n$, then λ is called an H-eigenvalue and $\mathbf{x}$ an H-eigenvector. If an eigenvector $\mathbf{x} \in \mathbb{C}^n$ cannot be scaled to be real,[1] then it is called an N-eigenvector.

[1] We see that when $\mathbf{x}$ is an eigenvector, then $\alpha\mathbf{x}$ is still an eigenvector for all nonzero $\alpha \in \mathbb{C}$. An eigenvector $\mathbf{x}$ can be scaled to be real means that there is a nonzero $\alpha \in \mathbb{C}$ such that $\alpha\mathbf{x} \in \mathbb{R}^n$. In this situation, we prefer to study this $\alpha\mathbf{x}$, which is an H-eigenvector, rather than the others in the orbit $\{\gamma\mathbf{x} \mid \gamma \in \mathbb{C} \setminus \{0\}\}$.

It is easy to see that an H-eigenvalue is real. However, an H-eigenvalue may still be associated with some N-eigenvectors.

We may rescale the eigenvector such that the components have maximum modulus one. We call such an eigenvector *canonical*. In the following, unless stated otherwise, all eigenvectors referred to are canonical eigenvectors. This convention does not introduce any restrictions, since the eigenvector defining equations are homogeneous. An eigenvector $\mathbf{x}$ of the eigenvalue zero is called *maximal* if there does not exist another eigenvector of the eigenvalue zero such that its support is strictly contained in that of $\mathbf{x}$.

The *algebraic multiplicity* of an eigenvalue is defined as the multiplicity of this eigenvalue as a root of the characteristic polynomial $\chi_{\mathcal{T}}(\lambda)$. To give the definition of the characteristic polynomial, determinant theory is needed. For determinant theory of a tensor, see [8].

Definition 2.2. Let $\mathcal{T}$ be a k-th order n-dimensional real tensor and λ be an indeterminate variable. The determinant $\mathrm{Det}(\lambda\mathcal{I} - \mathcal{T})$ of $\lambda\mathcal{I} - \mathcal{T}$, which is a polynomial in $\mathbb{C}[\lambda]$ and denoted by $\chi_{\mathcal{T}}(\lambda)$, is called the *characteristic polynomial* of the tensor $\mathcal{T}$.

It is known that the set of eigenvalues of $\mathcal{T}$ equals the set of roots of $\chi_{\mathcal{T}}(\lambda)$; see [8, Theorem 2.3]. If λ is a root of $\chi_{\mathcal{T}}(\lambda)$ of multiplicity s, then we call s the algebraic multiplicity of the eigenvalue λ. Let $c(n, k) = n(k-1)^{n-1}$. By [8, Theorem 2.3], $\chi_{\mathcal{T}}(\lambda)$ is a monic polynomial of degree $c(n, k)$. The set of all the eigenvalues of $\mathcal{T}$ (with algebraic multiplicities) is the *spectrum* of $\mathcal{T}$.

Sub-tensors are involved in this paper. For more discussions on this, see [8].

Definition 2.3. Let $\mathcal{T}$ be a k-th order n-dimensional real tensor and $s \in [n]$. Consider a subset $S = \{j_1, \ldots, j_s\}$ of $[n]$ with $j_k \in [n]$ for $k \in [s]$. The k-th order s-dimensional tensor $\mathcal{U}$ with entries $u_{i_1\ldots i_k} = t_{j_{i_1}\ldots j_{i_k}}$ for all $i_1, \ldots, i_k \in [s]$ is called the *sub-tensor* of $\mathcal{T}$ associated to the subset S. We usually denote $\mathcal{U}$ as $\mathcal{T}(S)$.

For a subset $S \subseteq [n]$, we denote by $|S|$ its cardinality. For $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}(S)$ is defined as an $|S|$-dimensional sub-vector of $\mathbf{x}$ with its entries being x_i for $i \in S$, and $\mathrm{supp}(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$ is its *support*.

2.2. Uniform hypergraphs

In this subsection, we present some basic concepts concerning uniform hypergraphs which will be used presently. Please refer to [1,5,2,12,23] for comprehensive references.

In this paper, unless stated otherwise, a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V, which is labeled as $[n] = \{1, \ldots, n\}$, and edge set E. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. Throughout this paper, $k \geq 3$ and $n \geq k$. Moreover, since the trivial hypergraph (i.e., $E = \emptyset$) is of little interest, we consider only hypergraphs having at least one edge in this paper.

For a subset $S \subset [n]$, we denoted by E_S the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. This is the set of edges containing the vertex i, i.e., $E_i := \{e \in E \mid i \in e\}$. The cardinality $|E_i|$ of the set E_i is defined as the *degree* of the vertex i, which is denoted by d_i. Then we have that $k|E| = \sum_{i\in[n]} d_i$. If $d_i = 0$, then we say that the vertex i is *isolated* or it is a *singleton*. Two different vertices i and j are *connected* to each other (or the pair i and j is connected), if there is a sequence of edges $(e_1, \ldots, e_m)$ such that $i \in e_1$, $j \in e_m$ and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [m-1]$. A hypergraph is called *connected*, if every pair of different vertices of G is connected. A set $S \subseteq V$ is a *connected component* of G, if every two vertices of S are connected and there is no vertex in $V \setminus S$ that is connected to any vertex in S. For convenience, an isolated vertex is regarded as a connected component as well. Then, it is easy to see that for every hypergraph G, there is a partition of V into pairwise disjoint subsets $V = V_1 \cup \cdots \cup V_r$ such that every V_i is a connected component of G. Let $S \subseteq V$, the hypergraph with vertex set S and edge set $\{e \in E \mid e \subseteq S\}$ is called the *sub-hypergraph* of G induced by S. We will denoted it by G_S.

The following definition for the Laplacian tensor and signless Laplacian tensor was proposed by Qi [23].

Definition 2.4. Let $G = (V, E)$ be a k-uniform hypergraph. The *adjacency tensor* of G is defined as the k-th order n-dimensional tensor $\mathcal{A}$ whose $(i_1 i_2 \cdots i_k)$-entry is:

$$a_{i_1 i_2 \ldots i_k} := \begin{cases} \dfrac{1}{(k-1)!} & \text{if } \{i_1, i_2, \ldots, i_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}$ be a k-th order n-dimensional diagonal tensor with its diagonal element $d_{i\ldots i}$ being d_i, the degree of vertex i, for all $i \in [n]$. Then $\mathcal{D} - \mathcal{A}$ is the *Laplacian tensor* of the hypergraph G, and $\mathcal{D} + \mathcal{A}$ is the *signless Laplacian tensor* of the hypergraph G.

Let $G = (V, E)$ be a hypergraph with connected components $V = V_1 \cup \cdots \cup V_r$ for $r \geq 1$. By reordering the indices if necessary, $\mathcal{A} = (a_{i_1\cdots i_n})$ can be represented by a block diagonal structure according to $V_1, \ldots, V_r$, i.e., $a_{i_1\cdots i_k} \equiv 0$ unless $i_j \in V_l$ for $j \in [k]$ and some $l \in [r]$. By Definition 2.1, the spectrum of $\mathcal{A}$ does not change when reordering the indices. Thus, in the sequel, we assume that $\mathcal{A}$ is in the block diagonal structure with its ith block tensor being the sub-tensor of $\mathcal{A}$ associated to V_i for $i \in [r]$. It is easy to see that $\mathcal{A}(V_i)$ is the adjacency tensor of the sub-hypergraph G_{V_i} for all $i \in [r]$. Similar conventions are assumed to the signless Laplacian tensor and the Laplacian tensor. By similar discussions as [12, Lemma 3.3], the spectra of the adjacency tensor, the signless Laplacian tensor, and the Laplacian tensor are the unions of the spectra of its diagonal blocks respectively, with adequate multiplicity consideration; see [6,8,12,23].

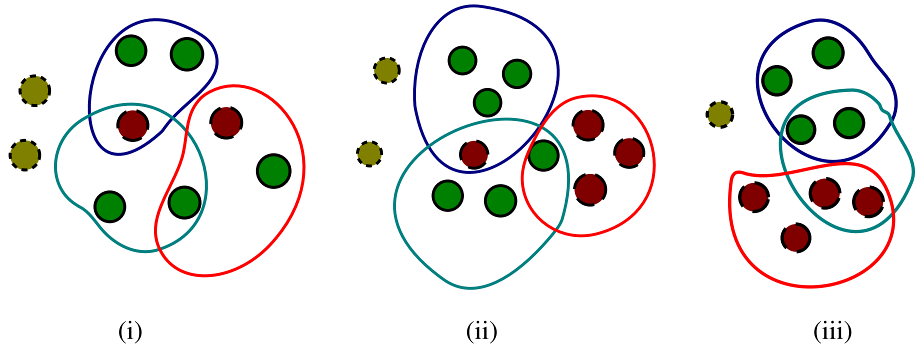


Fig. 1. Examples of the three kinds of bipartite hypergraphs in Definitions 2.5–2.7. (i) is an hm-bipartite 3-uniform hypergraph; (ii) is an odd-bipartite 4-uniform hypergraph; and (iii) is an even-bipartite 4-uniform hypergraph. An edge is pictured as a closed curve with the containing solid disks the vertices in that edge. Different edges are in different curves with different colors. A solid disk in dotted margin is a singleton of the hypergraph. The bipartition is clear from the different colors (also the dashed margins from the solid ones) of the disks in the connected component.

In the following, we introduce three kinds of bipartite hypergraphs. It will be shown that the first class of uniform hypergraphs has symmetric spectra, and the latter two classes of uniform hypergraphs characterize maximal canonical H-eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors.

Definition 2.5. Let $G = (V, E)$ be a k-uniform hypergraph. It is called *hm-bipartite* if there is a disjoint partition of the vertex set V as $V = V_1 \cup V_2$ such that $V_1, V_2 \neq \emptyset$ and every edge in E intersects V_1 in exactly one vertex and V_2 the other $k-1$ vertices.

We use the name *hm-bipartite*, since a *h*ead is selected from every edge and the rest is the *m*ass.

Definition 2.6. Let k be even and $G = (V, E)$ be a k-uniform hypergraph. It is called *odd-bipartite* if there is a disjoint partition of the vertex set V as $V = V_1 \cup V_2$ such that $V_1, V_2 \neq \emptyset$ and every edge in E intersects V_1 in an odd number of vertices.

Definition 2.7. Let $k \geq 4$ be even and $G = (V, E)$ be a k-uniform hypergraph. It is called *even-bipartite* if there is a disjoint partition of the vertex set V as $V = V_1 \cup V_2$ such that $V_1, V_2 \neq \emptyset$ and every edge in E intersects V_1 in an even number of vertices.

The idea of even-bipartite hypergraphs appeared in [12, Corollary 6.5]. In Fig. 1, we give preliminary examples on the three kinds of bipartite hypergraphs defined above.

When G is an ordinary graph, i.e., $k = 2$, Definitions 2.5 and 2.6 reduce to the classic definition for bipartite graphs [2]. When $k > 2$, "bipartite" has various meaningful generalizations. In this paper, we investigate the proposed three generalizations.

We note that unlike its graph counterpart, an even (odd)-bipartite connected component of an even-uniform hypergraph G may have several bipartitions witnessing the bipartiteness. In counting the total number of even (odd)-bipartite connected components, each connected component is counted with multiplicity equal to the number of decompositions into the appropriate type of bipartition. Likewise, for a connected component V_0 of a hypergraph G, if it has two bipartitions of the same type as $S_1 \cup T_1 = V_0$ and $S_2 \cup T_2 = V_0$, unless $S_1 = S_2$ or $S_1 = T_2$, the two bipartitions are regarded different. See the following example.

Example 2.1. Let $G = (V, E)$ be a 4-uniform hypergraph with vertex set $[6]$ and edge set $E = \{\{1, 2, 3, 4\}, \{1, 3, 5, 6\}, \{1, 2, 3, 6\}\}$. Then, G is connected and can be viewed as an even-bipartite hypergraph with bipartition $V_1 := \{1, 2, 5\}$ and $V_2 := \{3, 4, 6\}$ by Definition 2.7. Meanwhile, G is an even-bipartite hypergraph associated to the bipartitions $V_1 := \{2, 3, 5\}$ and $V_2 := \{1, 4, 6\}$; and $V_1 := \{1, 3\}$ and $V_2 := \{2, 4, 5, 6\}$. Hence the number of even-bipartite connected components of the hypergraph G is three. The three even-bipartitions of the hypergraph are pictured in Fig. 2.

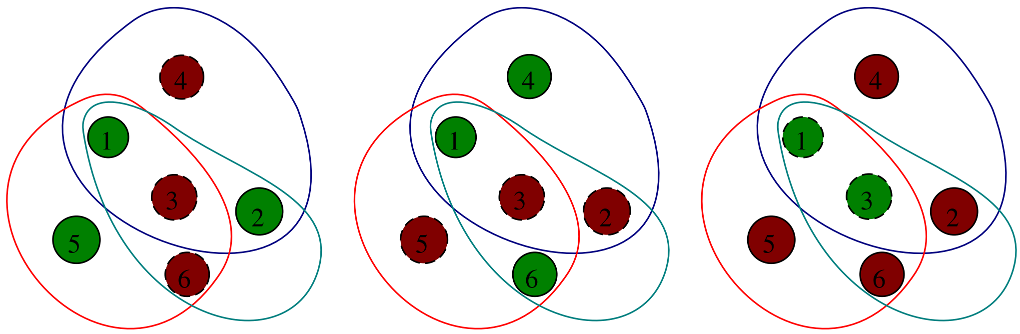
3. HM-bipartite hypergraphs

This section presents some basic facts about the spectra of hm-bipartite hypergraphs.

The next proposition says that the spectrum of an hm-bipartite hypergraph is symmetric. The meaning of the symmetry is clear from the statement of this proposition.

Proposition 3.1. *Let G be a k-uniform hm-bipartite hypergraph. Then the spectrum of $\mathcal{A}$ is invariant under the multiplication by any k-th root of unity.*

Proof. As we assumed at the beginning of the paper, $E \neq \emptyset$. Since G is hm-bipartite, let V_1 and V_2 be a bipartition of V such that $|e \cap V_1| = 1$ for all $e \in E$.



(i) (ii) (iii)

Fig. 2. The three even-bipartitions for the hypergraph in Example 2.1. The legend of the pictures is similar to that of Fig. 1.

Let α be any k-th root of unity. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ with an eigenvector being $\mathbf{x} \in \mathbb{C}^n$. Then, let $\mathbf{y} \in \mathbb{C}^n$ be a vector such that $y_i = \alpha x_i$ whenever $i \in V_1$ and $y_i = x_i$ for the others. For $i \in V_1$, we have

$$(\mathcal{A}\mathbf{y}^{k-1})_i = \sum_{e\in E_i}\prod_{j\in e\setminus\{i\}} y_j = \sum_{e\in E_i}\prod_{j\in e\setminus\{i\}} x_j = \lambda x_i^{k-1} = (\alpha\lambda)(\alpha x_i)^{k-1} = (\alpha\lambda)y_i^{k-1},$$

where the second equality follows from the fact that G is hm-bipartite which implies that exactly the rest of the vertices of every $e \in E_i$ other than i belong to V_2, and the third from the eigenvalue equation for $(\lambda, \mathbf{x})$. For $i \in V_2$, we have

$$(\mathcal{A}\mathbf{y}^{k-1})_i = \sum_{e\in E_i}\prod_{j\in e\setminus\{i\}} y_j = \alpha\sum_{e\in E_i}\prod_{j\in e\setminus\{i\}} x_j = \alpha\lambda x_i^{k-1} = (\alpha\lambda)y_i^{k-1},$$

where the second equality follows from the fact that G is hm-bipartite which implies that exactly one vertex other than i from every $e \in E_i$ belongs to V_1, and the third from the eigenvalue equation for $(\lambda, \mathbf{x})$.

Hence, by Definition 2.1, $\alpha\lambda$ is an eigenvalue of $\mathcal{A}$. The result follows. □

A hypergraph is called k-partite if there is a pairwise disjoint partition of $V = V_1 \cup \cdots \cup V_k$ such that every edge $e \in E$ intersects V_i nontrivially (i.e., $e \cap V_i \neq \emptyset$) for all $i \in [k]$. Obviously, k-partite hypergraphs are hm-bipartite. Thus, Proposition 3.1 generalizes [6, Theorem 4.2].

The next proposition establishes the connection of the spectra of the signless Laplacian tensor and the Laplacian tensor for an hm-bipartite hypergraph.

Proposition 3.2. *Let k be even and G be a k-uniform hm-bipartite hypergraph. Then the spectrum of the Laplacian tensor and the spectrum of the signless Laplacian tensor are equal.*

Proof. For a tensor $\mathcal{T}$ of order k and dimension n, its similar transformation[2] by a diagonal matrix P, denoted by $P^{-1}\cdot\mathcal{T}\cdot P$, is defined as a k-th order n-dimensional tensor with its entries being

$$(P^{-1}\cdot\mathcal{T}\cdot P)_{i_1\ldots i_k} := p_{i_1i_1}^{-k+1}t_{i_1\ldots i_k}p_{i_2i_2}\cdots p_{i_ki_k}, \quad \forall i_s \in [n],\ s \in [k].$$

Consequently, $P^{-1}\cdot\mathcal{I}\cdot P = \mathcal{I}$. We then have

$$\begin{aligned}\mathrm{Det}(P^{-1}\cdot(\lambda\mathcal{I}-\mathcal{T})\cdot P) &= \mathrm{Det}(\lambda P^{-1}\cdot\mathcal{I}\cdot P - P^{-1}\cdot\mathcal{T}\cdot P)\\ &= \mathrm{Det}(\lambda\mathcal{I} - P^{-1}\cdot\mathcal{T}\cdot P).\end{aligned}$$

By [7, Propositions 4.3 and 4.4], we get that

$$\begin{aligned}\mathrm{Det}(P^{-1}\cdot(\lambda\mathcal{I}-\mathcal{T})\cdot P) &= [\mathrm{Det}(P^{-1})]^{(k-1)^n}\mathrm{Det}(\lambda\mathcal{I}-\mathcal{T})[\mathrm{Det}(P)]^{(k-1)^n}\\ &= \mathrm{Det}(\lambda\mathcal{I}-\mathcal{T}).\end{aligned}$$

These two facts, together with [8, Theorem 2.3], imply that $\mathcal{T}$ and $P^{-1}\cdot\mathcal{T}\cdot P$ have the same spectrum for any invertible diagonal matrix P, see also [28].

By the diagonal block structure of $\mathcal{D}-\mathcal{A}$ and $\mathcal{D}+\mathcal{A}$, we can assume that $E \neq \emptyset$ and G is connected. Since G is hm-bipartite, let V_1 and V_2 be a bipartition of V such that $|e\cap V_1| = 1$ for all $e \in E$. Let P be a diagonal matrix with its ith diagonal entry being 1 if $i \in V_1$ and -1 if $i \in V_2$. By direct computation, we have that

$$P^{-1}\cdot(\mathcal{D}-\mathcal{A})\cdot P = \mathcal{D}+\mathcal{A}.$$

Then, the conclusion follows from the preceding discussion since P is invertible. □

[2] This is a special case of matrix-tensor multiplication. Please refer to [7,28] for more properties of general matrix-tensor multiplication.

When k is odd and G is nontrivial, we do not have $P^{-1} \cdot (\mathcal{D} - \mathcal{A}) \cdot P = \mathcal{D} + \mathcal{A}$. Thus, it is unknown whether the spectrum of the Laplacian tensor and the spectrum of the signless Laplacian tensor are equal or not in this situation. On the other hand, when $k \geq 4$ is even, at present we are not able to prove the converse of Proposition 3.2, which is well-known in spectral graph theory as: a graph is bipartite if and only if the spectrum of the Laplacian matrix and the spectrum of the signless Laplacian matrix are equal [2].

There are fruitful connections between the structure of hypergraphs and their spectra [12,23]. In the sequel, the discussion concerns the eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors. To this end, we establish some basic facts about the eigenvectors associated with the eigenvalue zero in the next section. For the Laplacian, there are similar results in [12, Section 6].

4. Eigenvectors associated with the zero eigenvalue

The next lemma characterizes the eigenvectors associated with the zero Laplacian and signless Laplacian eigenvalues of a uniform hypergraph.

Lemma 4.1. *Let G be a k-uniform hypergraph and $V_i,\ i \in [s]$ be its connected components with $s > 0$. If $\mathbf{x}$ is an eigenvector associated with the zero eigenvalue of the Laplacian or signless Laplacian tensor, then $\mathbf{x}(V_i)$ is an eigenvector of $(\mathcal{D} - \mathcal{A})(V_i)$ or $(\mathcal{D} + \mathcal{A})(V_i)$ corresponding to the eigenvalue zero whenever $\mathbf{x}(V_i) \neq 0$. Furthermore, in this situation, $\mathrm{supp}(\mathbf{x}(V_i)) = V_i$, and $x_j = \gamma \exp(\frac{2\alpha_j\pi}{k}\sqrt{-1})$ for some nonnegative integer α_j for all $j \in V_i$ and some $\gamma \in \mathbb{C} \setminus \{0\}$.*

Proof. The proof for the signless Laplacian tensor and that for the Laplacian tensor are similar. Hence, only the former is given.

Similar to the proof of [12, Lemma 3.3], we have that for every connected component V_i of G, $\mathbf{x}(V_i)$ is an eigenvector of $(\mathcal{D} + \mathcal{A})(V_i)$ corresponding to the eigenvalue zero whenever $\mathbf{x}(V_i) \neq 0$.

Suppose that $\mathbf{x}(V_i) \neq 0$. The case for V_i being a singleton is trivial. In the following, we assume that V_i has more than two vertices. We can always scale $\mathbf{x}(V_i)$ with some nonzero $\gamma \in \mathbb{C}$ such that $\frac{x_j}{\gamma}$ is positive and of the maximum modulus 1 for some $j \in V_i$. Thus, without loss of generality, we assume that $\mathbf{x}(V_i)$ is a canonical eigenvector of $(\mathcal{D} + \mathcal{A})(V_i)$ and $x_j = 1$ for some $j \in V_i$. Then the jth eigenvalue equation is

$$0 = \left[(\mathcal{D} + \mathcal{A})\mathbf{x}^{k-1}\right]_j = d_j x_j^{k-1} + \sum_{e \in E_j} \prod_{t \in e \setminus \{j\}} x_t = d_j + \sum_{e \in E_j} \prod_{t \in e \setminus \{j\}} x_t. \tag{1}$$

Since $d_j = |\{e \mid e \in E_j\}|$, we must have that

$$\prod_{t \in e \setminus \{j\}} x_t = -1, \quad \forall e \in E_j.$$

This implies that

$$\prod_{t \in e} x_t = -1, \quad \forall e \in E_j. \tag{2}$$

Since the maximum modulus is 1, $x_t = \exp(\theta_t\sqrt{-1})$ for some $\theta_t \in [0, 2\pi]$ for all $t \in e$ with $e \in E_j$. For another vertex s which shares an edge with j, we have

$$0 = \left[(\mathcal{D} + \mathcal{A})\mathbf{x}^{k-1}\right]_s = d_s x_s^{k-1} + \sum_{e \in E_s} \prod_{t \in e \setminus \{s\}} x_t.$$

Similarly, we have

$$x_s^{k-1} = -\prod_{t \in e \setminus \{s\}} x_t, \quad \forall e \in E_s.$$

Thus,

$$x_s^k = -\prod_{t \in e} x_t, \quad \forall e \in E_s.$$

The fact that s and j share one edge, together with (2), implies that $x_s^k = 1$. Similarly, we have that

$$x_s^k = 1, \quad \forall s \in e,\ e \in E_j.$$

As G_{V_i} is connected, by induction, we can show that $x_s^k = 1$ for all $s \in V_i$. Consequently, $\theta_t = \frac{2\alpha_t}{k}\pi$ for some integers α_t for all $t \in V_i$. $\square$

With Lemma 4.1, parallel results as those in [12, Section 6] can be established for the signless Laplacian tensor and the Laplacian tensor. In particular, we have the following result.

Theorem 4.1. *Let G be a k-uniform connected hypergraph.*

(i) *A nonzero vector $\mathbf{x}$ is an eigenvector of the Laplacian tensor $\mathcal{D} - \mathcal{A}$ corresponds to the zero eigenvalue if and only if there exist nonzero $\gamma \in \mathbb{C}$ and integers α_i such that $x_i = \gamma \exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})$ for $i \in [n]$, and*

$$\sum_{j\in e}\alpha_j = \sigma_e k, \quad \forall e \in E, \tag{3}$$

for some integer σ_e with $e \in E$.

(ii) *A nonzero vector $\mathbf{x}$ is an eigenvector of the signless Laplacian tensor $\mathcal{D} + \mathcal{A}$ corresponds to the zero eigenvalue if and only if there exist nonzero $\gamma \in \mathbb{C}$ and integers α_i such that $x_i = \gamma \exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})$ for $i \in [n]$, and*

$$\sum_{j\in e}\alpha_j = \sigma_e k + \frac{k}{2}, \quad \forall e \in E, \tag{4}$$

for some integer σ_e associated with each $e \in E$.

An immediate consequence of Theorem 4.1 is that the signless Laplacian tensor of a k-uniform connected hypergraph with odd k does not have zero as an eigenvalue, since $\frac{k}{2}$ is a half-integer, and (4) can never be fulfilled in this situation.

Proposition 4.1. *Let k be odd and G be a k-uniform connected hypergraph. Then zero is not an eigenvalue of the signless Laplacian tensor.*

When we restrict the discussion of Lemma 4.1 to H-eigenvectors, we get the following corollary. We state it explicitly here for convenience, since it will be used in the next section.

Corollary 4.1. *Let G be a k-uniform hypergraph and V_i, $i \in [s]$ be its connected components with $s > 0$. If $\mathbf{x}$ is an H-eigenvector associated with the zero eigenvalue of the Laplacian or signless Laplacian tensor, then $\mathbf{x}(V_i)$ is an H-eigenvector of $(\mathcal{D} - \mathcal{A})(V_i)$ or $(\mathcal{D} + \mathcal{A})(V_i)$ corresponding to the eigenvalue zero whenever $\mathbf{x}(V_i) \neq 0$. Furthermore, in this situation, $\mathrm{supp}(\mathbf{x}(V_i)) = V_i$, and up to a real scalar multiplication $x_j = \pm 1$ for $j \in V_i$.*

By Corollary 4.1, we get that a maximal canonical H-eigenvector $\mathbf{x} \in \mathbb{R}^n$, associated with the zero eigenvalue of the Laplacian or signless Laplacian tensor must be of this form: x_i is 1, -1 or 0 for all $i \in [n]$.

5. H-eigenvectors

In this section, we establish the connection between the H-eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors and the even (odd)-connected components of the underlying hypergraph.

We note that canonical eigenvectors are considered in this section. Then, when we do the number count, we always consider a maximal canonical H-eigenvector $\mathbf{x}$ and its reciprocal $-\mathbf{x}$ as the same.

The next proposition, together with Corollary 4.1, generalizes [2, Theorem 1.3.9] which says that the multiplicity of the eigenvalue zero of the signless Laplacian matrix of a graph is equal to the number of bipartite connected components of this graph.

Proposition 5.1. *Let k be even and G be a k-uniform hypergraph. Then the number of maximal canonical H-eigenvectors associated with the zero signless Laplacian eigenvalue equals the number of odd-bipartite connected components of G.*

Proof. Suppose that $V_1 \subseteq V$ is an odd-bipartite connected component of G. If V_1 is a singleton, then 1 is a maximal canonical H-eigenvector of $(\mathcal{D} + \mathcal{A})(V_1) = 0$ by definition. In the following, we assume that G_{V_1} has at least one edge. Let $V_1 = S \cup T$ be an odd-bipartition of the sub-hypergraph G_{V_1} such that every edge of G_{V_1} intersects with S in an odd number of vertices. Then $S, T \neq \emptyset$, since k is even. Let $\mathbf{y} \in \mathbb{R}^{|V_1|}$ be a vector such that $y_i = 1$ whenever $i \in S$ and $y_i = -1$ whenever $i \in T$. Then, for $i \in S$,

$$\begin{aligned}\left[(\mathcal{D}+\mathcal{A})\mathbf{y}^{k-1}\right]_i &= d_i y_i^{k-1} + \sum_{e\in E_i}\prod_{j\in e\setminus\{i\}} y_j\\ &= d_i - d_i\\ &= 0.\end{aligned}$$

Here the second equality follows from the fact that for every $e \in E_i$, the number of $y_j = -1$ is odd.

Next, for $i \in T$,

$$\begin{aligned}\left[(\mathcal{D}+\mathcal{A})\mathbf{y}^{k-1}\right]_i &= d_i y_i^{k-1} + \sum_{e\in E_i}\prod_{j\in e\setminus\{i\}} y_j\\ &= -d_i + d_i\\ &= 0.\end{aligned}$$

Here the second equality follows from the fact that for every $e \in E_i$, the number of $y_j = -1$ other than y_i is even.

Thus, for every odd-bipartite connected component of G, we can associate it with a canonical H-eigenvector corresponding to the eigenvalue zero. Since $d_i = |\{e \mid e \in E_i\}|$, it is easy to see that this vector $\mathbf{y}$ is a maximal canonical H-eigenvector. Otherwise, suppose that $\mathbf{x}$ with $\mathrm{supp}(\mathbf{x}) \subset \mathrm{supp}(\mathbf{y})$ is a canonical H-eigenvector of $(\mathcal{D}+\mathcal{A})(V_1)$ corresponding to the eigenvalue zero. Since V_1 is a nontrivial connected component of G and $\mathbf{x} \neq 0$, we must have a $j \in V_1$ such that $x_j \neq 0$ and there is an edge containing both j and s with $x_s = 0$. The jth eigenvalue equation is

$$0 = \left[(\mathcal{D}+\mathcal{A})\mathbf{y}^{k-1}\right]_j = d_j x_j^{k-1} + \sum_{e\in E_j}\prod_{t\in e\setminus\{j\}} x_t.$$

We have $|\prod_{t\in e\setminus\{j\}} x_t| \leq 1$ and $|x_j| = 1$. Since $x_s = 0$ and there is an edge containing both s and j, we have $|\sum_{e\in E_j}\prod_{t\in e\setminus\{j\}} x_t| \leq |\{e \mid e \in E_j\}| - 1 < |\{e \mid e \in E_j\}| = d_j$. Thus, this results a contradiction to the eigenvalue equation. Hence, $\mathbf{y}$ is maximal. Obviously, if $S_1 \cup T_1 = V_1$ and $S_2 \cup T_2 = V_1$ are two different odd-bipartitions of the connected component V_1, then the constructed maximal canonical H-eigenvectors are different. So the number of odd-bipartite connected components of G is not greater than the number of maximal canonical H-eigenvectors associated with the zero eigenvalue of the signless Laplacian tensor.

Conversely, suppose that $\mathbf{x} \in \mathbb{R}^n$ is a maximal canonical H-eigenvector corresponding to the eigenvalue zero, then $\mathrm{supp}(\mathbf{x})$ is a connected component of G by Corollary 4.1. Denote by V_0 this connected component of G. If V_0 is a singleton, then it is an odd-bipartite connected component by Definition 2.6. In the following, we assume that V_0 has more than one vertex.

For all $j \in V_0$,

$$0 = \left[(\mathcal{D}+\mathcal{A})\mathbf{y}^{k-1}\right]_j = d_j x_j^{k-1} + \sum_{e\in E_j}\prod_{s\in e\setminus\{j\}} x_s. \tag{5}$$

Let $S \cup T = V_0$ be a bipartition of V_0 such that $x_s > 0$ whenever $s \in S$ and $x_s < 0$ whenever $s \in T$. Since $\mathbf{x}$ is canonical and $|V_0| > 1$, we must have $S \neq \emptyset$. This, together with (5), implies that $T \neq \emptyset$. From (5), we see that for every edge $e \in E_j$ with $j \in S$, $|e \cap T|$ must be an odd number; and for every edge $e \in E_j$ with $j \in T$, $|e \cap T|$ must be an odd number as well. Then V_0 is an odd-bipartite component of G. Hence, every maximal canonical H-eigenvector corresponding to the eigenvalue zero determines an odd-bipartite connected component of G. Obviously, the odd-bipartite connected components determined by a maximal canonical H-eigenvector $\mathbf{x}$ and its reciprocal $-\mathbf{x}$ are the same.

Combining the above results, the conclusion follows. □

The next proposition is an analogue of Proposition 5.1 for the Laplacian tensor.

Proposition 5.2. *Let k be even and G be a k-uniform hypergraph. Then the number of maximal canonical H-eigenvectors associated with the zero Laplacian eigenvalue equals the sum of the number of even-bipartite connected components of G and the number of connected components of G, minus the number of singletons of G.*

Proof. By similar proof of that for Proposition 5.1, we see that if V_0 is an even-bipartite connected component of G, then we can construct a maximal canonical H-eigenvector of $\mathcal{D}-\mathcal{A}$. By the proof and the definition of even-bipartite hypergraph, we see that this maximal canonical H-eigenvector contains negative components whenever this connected component is not a singleton. Meanwhile, by Definition 2.1 and Corollary 4.1, for every connected component of G, the vector of all ones is a maximal canonical H-eigenvector corresponding to the eigenvalue zero. The two systems of maximal canonical H-eigenvectors have common members at the singletons of G. Thus, the number of maximal canonical H-eigenvectors associated with the zero Laplacian eigenvalue is not smaller than the sum of the number of even-bipartite connected components of G and the number of connected components of G, minus the number of singletons of G.

Conversely, suppose that $\mathbf{x}$ is a maximal canonical H-eigenvector associated with the zero eigenvalue of the Laplacian tensor. Then, by Corollary 4.1 and a similar proof to that in Proposition 5.1, $\mathrm{supp}(\mathbf{x})$ is a connected component of G. If $\mathbf{x}$ contains both positive and negative components, then $\mathrm{supp}(\mathbf{x})$ must be an even-bipartite connected component. When $\mathbf{x}$ contains only positive components, we can only conclude that $\mathrm{supp}(\mathbf{x})$ is a connected component. When $\mathrm{supp}(\mathbf{x})$ is a singleton, the two systems of maximal canonical H-eigenvectors have common members. Thus, we can see that the number of maximal canonical H-eigenvectors associated with the zero Laplacian eigenvalue is not greater than the sum of the number of even-bipartite connected components of G and the number of connected components of G, minus the number of singletons of G.

Combining these two conclusions, the result follows. □

The next proposition is an analogue of Proposition 5.2 for k odd.

Proposition 5.3. *Let k be odd and G be a k-uniform hypergraph. Then the number of maximal canonical H-eigenvectors associated with the zero Laplacian eigenvalue equals the number of connected components of G.*

Proof. By a similar proof of that in Proposition 5.2, we can see that a maximal canonical H-eigenvector of the eigenvalue zero can only have positive components, since k is odd. Then, the result follows. □

The next proposition follows from Proposition 4.1. We highlight it here to make the statements in this section more complete.

Proposition 5.4. *Let k be odd and G be a k-uniform connected hypergraph. Then the signless Laplacian tensor has no zero H-eigenvalue.*

The next corollary is an analogue of Proposition 5.3 for the signless Laplacian tensor, which follows from Proposition 5.4.

Corollary 5.1. *Let k be odd and G be a k-uniform hypergraph. Then the number of maximal canonical H-eigenvectors associated with the zero signless Laplacian eigenvalue is equal to the number of singletons of G.*

6. N-eigenvectors

In this section, we establish the connection between maximal canonical N-eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors and some multi-partite connected components of the underlying hypergraph.

Canonical N-eigenvectors are considered in this section. In counting the number of such vectors, we always consider a maximal canonical eigenvector $\mathbf{x}$ and its multiplication $\exp(\theta\sqrt{-1})\mathbf{x}$ by any $\theta \in [0, 2\pi]$ as the same. Actually, by Lemma 4.1, only $\theta = \frac{2\alpha\pi}{k}$ for integers α are involved.

6.1. 3-uniform hypergraphs

In this subsection, 3-uniform hypergraphs are discussed.

Definition 6.1. Let $G = (V, E)$ be a 3-uniform hypergraph. If there exists a partition of $V = V_1 \cup V_2 \cup V_3$ such that $V_1, V_2, V_3 \neq \emptyset$, and for every edge $e \in E$, either $e \subseteq V_i$ for some $i \in [3]$ or e intersects V_i nontrivially for all $i \in [3]$, then G is called *tripartite*.

We prefer to use *tripartite* to distinguish it from the concept of 3-*partite*. Note that if a connected component V_0 of G is tripartite, then $|V_0| \geq 3$. As for bipartite hypergraphs introduced in Section 2, if a connected tripartite hypergraph G has two tripartitions as $V = V_1 \cup V_2 \cup V_3 = S_1 \cup S_2 \cup S_3$, unless (with a possible renumbering of the subscripts of $S_i,\ i \in [3]$) $S_j = V_j,\ j \in [3]$, the two tripartitions are regarded different. Please see Fig. 3 as an example. In that figure, the partition (i) and the partition (iii) are considered different.

The next lemma helps to establish the connection.

Lemma 6.1. *Let $G = (V, E)$ be a k-uniform hypergraph. Then $\mathbf{x} \in \mathbb{C}^n$ is a canonical N-eigenvector associated with the zero eigenvalue of the Laplacian or signless Laplacian tensor if and only if the conjugate $\mathbf{x}^*$ of $\mathbf{x}$ is a canonical N-eigenvector of the zero eigenvalue of the Laplacian or signless Laplacian tensor.*

Proof. Since the Laplacian tensor and the signless Laplacian tensor of a uniform hypergraph are real, the conclusion follows immediately from the definition of eigenvalues. □

With the above lemmas and the definitions, we are in a position to present the following result.

Proposition 6.1. *Let $G = (V, E)$ be a 3-uniform hypergraph. Then the number of maximal canonical conjugate N-eigenvector pairs of the zero Laplacian eigenvalue equals the number of tripartite connected components of G.*

Proof. If V_0 is a tripartite connected component of G, then let $V_0 = R \cup S \cup T$ be a tripartition of it. Let vector $\mathbf{y} \in \mathbb{C}^{|V_0|}$ be such that $y_i = 1$ whenever $i \in R$; $y_i = \exp(\frac{2\pi}{3}\sqrt{-1})$ whenever $i \in S$; and $y_i = \exp(\frac{4\pi}{3}\sqrt{-1})$ whenever $i \in T$. By Definition 6.1 and Theorem 4.1(i), we see that $\mathbf{y}$ is an N-eigenvector of $(\mathcal{D} - \mathcal{A})(V_0)$ corresponding to the eigenvalue zero. By Lemma 4.1, we see that the image of $\mathbf{y}$ under the natural inclusion map $\mathbb{C}|V_0| \to \mathbb{C}^n$ is a maximal canonical N-eigenvector. By Lemma 6.1, the conjugate of $\mathbf{y}$ is also a maximal canonical N-eigenvector. Moreover, the maximal canonical N-eigenvector constructed in this way from the tripartition $V_0 = R \cup S \cup T$ only can be either $\mathbf{y}$ or $\mathbf{y}^*$. Hence the number of maximal canonical conjugate N-eigenvector pairs associated with the zero eigenvalue of the Laplacian tensor is not less than the number of tripartite connected components of G.

Conversely, assume that $\mathbf{x} \in \mathbb{C}^n$ is a maximal canonical N-eigenvector associated with the zero eigenvalue of the Laplacian tensor. By Lemma 4.1, $V_0 := \mathrm{supp}(\mathbf{x})$ is a connected component of G. Since $\mathbf{x}$ is an N-eigenvector, V_0 is a nontrivial connected component of G, i.e., $|V_0| > 1$. This, together with Lemma 4.1, implies that we can get a tripartition of V_0 as $V_0 = R \cup S \cup T$ such that R consists of the indices such that the corresponding components of $\mathbf{x}$ are one; S consists of the indices such that the corresponding components of $\mathbf{x}$ are $\exp(\frac{2\pi}{3}\sqrt{-1})$; and T consists of the indices such that the corresponding components of $\mathbf{x}$ are $\exp(\frac{4\pi}{3}\sqrt{-1})$.

It is easy to see that $R \neq \emptyset$, and $S \cup T \neq \emptyset$ since $\mathbf{x}$ is an N-eigenvector. The fact that $\mathbf{x}$ is an eigenvector corresponding to the eigenvalue zero, together with Theorem 4.1(i), implies that both S and T are nonempty. Otherwise, the equations in (3) cannot be fulfilled. By (3) again, we must have that for every edge $e \in E(G_{V_0})$ (*i.e.*, E_{V_0}) either $e \subseteq R,\ S$ or T; or e intersects $R,\ S$ and T nontrivially. Consequently, V_0 is a tripartite connected component of G. We also see from Definition 6.1, Lemma 6.1 that the tripartite connected component of G determined by $\mathbf{x}^*$ is the same as that of $\mathbf{x}$. Hence, the number of maximal canonical conjugate N-eigenvector pairs of the zero eigenvalue of the Laplacian tensor is not greater than the number of tripartite connected components of G.

Combining the above two conclusions, the result follows. □

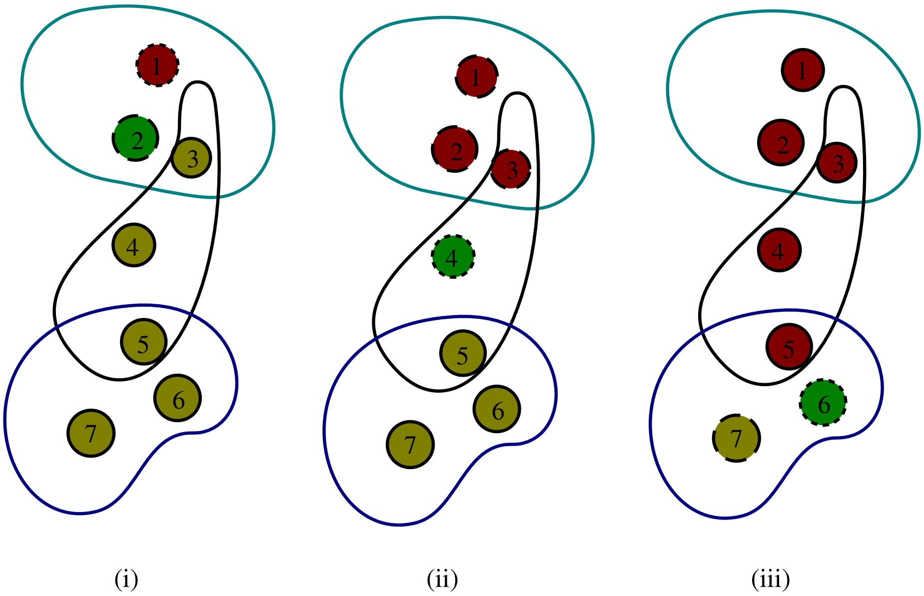


Fig. 3. The three tripartitions of the 3-uniform hypergraph in Example 6.1. The tripartitions are clear from the groups of disks in different colors (also with dotted, dashed and solid margins).

In the following, we give an example to illustrate the result obtained.

Example 6.1. Let $G = (V, E)$ be a 3-uniform hypergraph with $V = [7]$ and

$$E = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 7\}\}.$$

By Definition 6.1, there are three tripartitions of G as

$$\{1\}, \{2\}, \{3, 4, 5, 6, 7\}; \ \{1, 2, 3\}, \{4\}, \{5, 6, 7\}; \ \text{and } \{1, 2, 3, 4, 5\}, \{6\}, \{7\}.$$

These tripartitions are pictured in Fig. 3. By Proposition 6.1, there are three maximal canonical conjugate N-eigenvector pairs associated with the zero eigenvalue of the Laplacian tensor. By Proposition 5.3, there is only one maximal H-eigenvector associated with the zero eigenvalue of the Laplacian tensor, i.e., the vector of all ones, since G is connected.

We note that the tripartitions in (i) and (iii) of Fig. 3 are essentially the same. The numbering which is used to distinguish the tripartitions (i) and (iii) above was added manually. Then, there is a gap between Proposition 6.1 and the complete characterization of the intrinsic tripartite connected components of a uniform hypergraph.

We try to generalize the result in this subsection to k-uniform hypergraphs with $k \geq 4$. However, the definitions for multi-partite connected components of k-uniform hypergraphs with bigger k are somewhat complicated. Thus, only $k = 4$ and $k = 5$ are presented in the next two subsections respectively. Neater statements are expected in the future.

6.2. 4-uniform hypergraphs

In this subsection, 4-uniform hypergraphs are discussed.

Definition 6.2. Let $G = (V, E)$ be a 4-uniform hypergraph. If there exists a partition of $V = V_1 \cup V_2 \cup V_3 \cup V_4$ such that at least two of V_i, $i \in [4]$ are nonempty, and for every edge $e \in E$, either $e \subseteq V_i$ for some $i \in [4]$ or one of the following situations happens:

(i) $|e \cap V_1| = 2$ and $|e \cap V_3| = 2$;
(ii) $|e \cap V_2| = 2$ and $|e \cap V_4| = 2$;
(iii) $|e \cap V_1| = 2$, $|e \cap V_2| = 1$, and $|e \cap V_4| = 1$;
(iv) $|e \cap V_3| = 2$, $|e \cap V_2| = 1$, and $|e \cap V_4| = 1$,

then G is called *L-quadripartite*.

Similarly, we prefer to use *L-quadripartite* to distinguish it from the concept of 4-*partite*. Here the prefix "L" is for the Laplacian tensor.

Definition 6.3. Let $G = (V, E)$ be a 4-uniform hypergraph. If there exists a partition of $V = V_1 \cup V_2 \cup V_3 \cup V_4$ such that at least two of V_i, $i \in [4]$ are nonempty, and for every edge $e \in E$, either $e \subseteq V_i$ for some $i \in [4]$ or one of the following situations happens:

(i) $|e \cap V_1| = 3$ and $|e \cap V_3| = 1$;
(ii) $|e \cap V_3| = 3$ and $|e \cap V_1| = 1$;
(iii) $|e \cap V_2| = 3$ and $|e \cap V_4| = 1$;
(iv) $|e \cap V_4| = 3$ and $|e \cap V_2| = 1$;
(v) $|e \cap V_1| = 2$ and $|e \cap V_2| = 2$;
(vi) $|e \cap V_2| = 2$ and $|e \cap V_3| = 2$;
(vii) $|e \cap V_3| = 2$ and $|e \cap V_4| = 2$;
(viii) e intersects V_i nontrivially for all $i \in [4]$,

then G is called *sL-quadripartite*.

Here the prefix "sL" is for the signless Laplacian tensor. For a connected L(sL)-quadripartite hypergraph G which has two L(sL)-quadripartitions $V = V_1 \cup \cdots \cup V_4 = S_1 \cup \cdots \cup S_4$, unless (with a possible renumbering of the subscripts of $S_i,\ i \in [4]$) $S_j = V_j,\ j \in [4]$, the two L(sL)-quadripartitions are regarded different.

With similar proofs as that for Proposition 6.1, we can get the following two propositions. However, we give the key points of the proof for the next proposition as an illustration.

Proposition 6.2. *Let $G = (V, E)$ be a 4-uniform hypergraph. Then the number of maximal canonical conjugate N-eigenvector pairs of the zero Laplacian eigenvalue equals the number of L-quadripartite connected components of G.*

Proof. Let V_0 be an L-quadripartite connected component of G with the L-quadripartition being $S_1 \cup S_2 \cup S_3 \cup S_4$. Without loss of generality, we can assume that $S_1 \neq \emptyset$. Then, we can associate it with a maximal canonical N-eigenvector $\mathbf{x} \in \mathbb{C}^n$, defined by

$$x_i := \exp\left(\frac{2(j-1)\pi}{4}\sqrt{-1}\right), \quad \text{whenever } i \in S_j,\ j \in [4].$$

By Definition 6.3, we can conclude that the other maximal canonical N-eigenvectors $\mathbf{y}$ with $y_i = 1$ for $i \in S_1$ can only be $\mathbf{y} = \mathbf{x}^*$. Consequently, an L-quadripartite connected component of G determines a maximal canonical conjugate N-eigenvector pair. The converse is obvious. Then the result follows from a similar proof as that of Proposition 6.1. □

Proposition 6.3. *Let $G = (V, E)$ be a 4-uniform hypergraph. Then the number of maximal canonical conjugate N-eigenvector pairs of the zero signless Laplacian eigenvalue equals the number of sL-quadripartite connected components of G.*

6.3. 5-uniform hypergraphs

In this subsection, 5-uniform hypergraphs are discussed.

Definition 6.4. Let $G = (V, E)$ be a 5-uniform hypergraph. If there exists a partition of $V = V_1 \cup \cdots \cup V_5$ such that at least three of $V_i,\ i \in [5]$ are nonempty, and for every edge $e \in E$, either $e \subseteq V_i$ for some $i \in [5]$ or one of the following situations happens:

(i) $|e \cap V_2| = 2,\ |e \cap V_5| = 2$, and $|e \cap V_1| = 1$;
(ii) $|e \cap V_3| = 2,\ |e \cap V_4| = 2$, and $|e \cap V_1| = 1$;
(iii) $|e \cap V_1| = 3,\ |e \cap V_2| = 1$, and $|e \cap V_5| = 1$;
(iv) $|e \cap V_1| = 3,\ |e \cap V_3| = 1$, and $|e \cap V_4| = 1$;
(v) $|e \cap V_2| = 3,\ |e \cap V_4| = 1$, and $|e \cap V_5| = 1$;
(vi) $|e \cap V_3| = 3,\ |e \cap V_1| = 1$, and $|e \cap V_4| = 1$;
(vii) $|e \cap V_4| = 3,\ |e \cap V_1| = 1$, and $|e \cap V_2| = 1$;
(viii) $|e \cap V_5| = 3,\ |e \cap V_2| = 1$, and $|e \cap V_3| = 1$;
(ix) e intersects V_i nontrivially for all $i \in [5]$,

then G is called *pentapartite*.

For a connected pentapartite hypergraph G which has two pentapartitions $V = V_1 \cup \cdots \cup V_5 = S_1 \cup \cdots \cup S_5$, unless (with a possible renumbering of the subscripts of $S_i,\ i \in [5]$) $S_j = V_j,\ j \in [5]$, the two pentapartitions are regarded as different. With a similar proof as that of Propositions 6.1 and 6.2, we obtain the next proposition.

Proposition 6.4. *Let $G = (V, E)$ be a 5-uniform hypergraph. Then the number of maximal canonical conjugate N-eigenvector pairs of the zero Laplacian eigenvalue equals the number of pentapartite connected components of G.*

7. Final remarks

In this paper, the relations of the eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph, with some configured components of that hypergraph, are discussed. It is different from the recent work [23,12] which mainly concentrates on the discussions of H^+-eigenvalues of the Laplacian tensor, the signless Laplacian tensor and the Laplacian. H-eigenvectors and, more importantly, N-eigenvectors are discovered to be applicable in spectral hypergraph theory. It is shown that the H-eigenvectors and N-eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors can characterize some intrinsic structures of the underlying hypergraph. More work on the other eigenvalues of these Laplacian-type tensors are expected in the future. In particular, a general statement on the connection between the N-eigenvectors and the configured components of a hypergraph is our next topic.

After the first draft of this paper, several new results related with this paper appeared. The concept of odd-bipartite hypergraph introduced in this paper has found further applications. It was proved in [23] that the largest H-eigenvalue of the Laplacian tensor of a k-uniform hypergraph G is always less than or equal to the largest H-eigenvalue of the signless Laplacian tensor of G. In [14], it was proved that if G is connected, then equality holds here if and only if k is even and G is odd-bipartite. This generalized a classical result in spectral graph theory: the largest eigenvalue of the Laplacian matrix of a graph G is always less than or equal to the largest eigenvalue of the signless Laplacian matrix of G, and when G is connected, equality holds if and only if G is bipartite. In [13], a k-uniform hypergraph G is called a cored hypergraph if each edge of G has at least one vertex with degree one. It was proved there that an even-uniform cored hypergraph is odd-bipartite. Sunflowers, loose and general loose s-paths with $1 \le s < \frac{k}{2}$, loose cycles and general loose s-cycles with $1 \le s < \frac{k}{2}$, hypertress and squids are examples of cored hypergraphs. In [25], it was proved that all the even-uniform s-paths and all the even-uniform non-regular s-cycles are odd-bipartite for $1 \le s \le k-1$, though for $s \ge \frac{k}{2}$, s-paths and s-cycles are not cored hypergraphs. An s-cycle is regular if and only if $k = q(k-s)$ for some integer q. A sufficient and necessary condition for an even-uniform regular s-cycle to be odd-bipartite was also given in [25]. Finally, a question raised at the end of Section 4 of this paper about the relation between the Laplacian and signless Laplacian spectra of a k-uniform hypergraph when k is odd, was answered in [29].

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