Front. Math. China 2016, 11(3): 539–556
DOI 10.1007/s11464-016-0520-4

RESEARCH ARTICLE

Spectral properties of odd-bipartite *Z*-tensors and their absolute tensors

Haibin CHEN, Liqun QI

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China

Abstract Stimulated by odd-bipartite and even-bipartite hypergraphs, we define odd-bipartite (weakly odd-bipartie) and even-bipartite (weakly even-bipartite) tensors. It is verified that all even order odd-bipartite tensors are irreducible tensors, while all even-bipartite tensors are reducible no matter the parity of the order. Based on properties of odd-bipartite tensors, we study the relationship between the largest H-eigenvalue of a Z-tensor with nonnegative diagonal elements, and the largest H-eigenvalue of absolute tensor of that Z-tensor. When the order is even and the Z-tensor is weakly irreducible, we prove that the largest H-eigenvalue of the Z-tensor and the largest H-eigenvalue of the absolute tensor of that Z-tensor are equal, if and only if the Z-tensor is weakly odd-bipartite. Examples show the authenticity of the conclusions. Then, we prove that a symmetric Z-tensor with nonnegative diagonal entries and the absolute tensor of the Z-tensor are diagonal similar, if and only if the Z-tensor has even order and it is weakly odd-bipartite. After that, it is proved that, when an even order symmetric Z-tensor with nonnegative diagonal entries is weakly irreducible, the equality of the spectrum of the Z-tensor and the spectrum of absolute tensor of that Z-tensor, can be characterized by the equality of their spectral radii.

Keywords H-Eigenvalue, Z-tensor, odd-bipartite tensor, absolute tensor
MSC 90C30, 15A06

1 Introduction

Since the early work of [12,17], more and more researchers are interested in studying eigenvalue problems of tensors in the past several years [1,3–6,9,10, 14–16,18–21,24,25]. In [17], two kinds of eigenvalues were defined for real

Received March 21, 2015; accepted January 22, 2016
Corresponding author: Haibin CHEN, E-mail: chenhaibin508@163.com

symmetric tensors: eigenvalues and E-eigenvalues. An eigenvalue (resp. E-eigenvalue) with a real eigenvector (resp. E-eigenvector) is called an H-eigenvalue (resp. Z-eigenvalue). When a symmetric tensor has even order, H-eigenvalues and Z-eigenvalues always exist. An even order symmetric tensor is positive definite (resp. semi-definite) if and only if all of its H-eigenvalues or all of its Z-eigenvalues are positive (resp. nonnegative). Based upon this property, an H-eigenvalue method for the positive definiteness (resp. positive semi-definiteness) identification problem is developed.

The main difficulty in tensor problems is that they are generally nonlinear. Because of the difficulties in studying the properties of a general tensor, researchers focus on some structured tensors. Z-tensors are an important class of structured tensors and have been well studied [7,13,26]. They are closely related with spectral graph theory, the stationary distribution of Markov chains, and the convergence of iterative methods for linear systems.

Recently, Hu et al. [10] considered the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue of a k-uniform connected hypergraph. When the order is even and the hypergraph is odd-bipartite, they proved that the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue are equal. For the odd order case, it is proved that the largest Laplacian H-eigenvalue is strictly less than the largest signless Laplacian H-eigenvalue [10]. Later, Shao et al. [23] gave several spectral characterizations of the connected odd-bipartite hypergraphs. They proved that the spectrum of the Laplacian tensor and the spectrum of the signless Laplacian tensor of a uniform hypergraph are equal if and only if the hypergraph is an even order connected odd-bipartite hypergraph. Since the Laplacian tensor is a special case of Z-tensors and the signless Laplacian tensor is a special case of the absolute tensors of Z-tensors, questions comes naturally: what is the relation between the largest H-eigenvalue of a general Z-tensor and the largest H-eigenvalue of the Z-tensor's absolute tensor? What is the relation between spectrums of a general Z-tensor and its absolute tensor? These constitute main motivations of the paper.

In this article, some spectral properties of Z-tensors with nonnegative diagonal entries, and absolute tensors of Z-tensors are studied. The rest of this paper is organized as follows. In Section 2, some basic notions and preliminaries of tensors are presented. In Section 3, stimulated by odd-bipartite and even-bipartite hypergraphs [9], odd-bipartite (resp. weakly odd-bipartite) and even-bipartite (resp. weakly even-bipartite) tensors are defined. Odd-bipartite (resp. even-bipartite) tensors are weakly odd-bipartite (resp. weakly even-bipartite) tensors. Examples show that the converse, generally, may not hold. A square odd-bipartite matrix is irreducible. For high order tensors, we prove that an even order odd-bipartite tensor is irreducible, while a tensor is reducible if it is even-bipartite no matter the parity of the order.

In Section 4, we study the relation between the largest H-eigenvalue of a Z-tensor with nonnegative diagonal entries and the largest H-eigenvalue of the Z-tensor's absolute tensor. For an even order Z-tensor with nonnegative diagonal entries, if it is weakly irreducible, we show that the largest H-eigenvalues of the Z-tensor and its absolute tensor are equal if and only if the Z-tensor is weakly odd-bipartite. For the odd order case, sufficient conditions for the equality of these largest H-eigenvalues are given. Examples show the authenticity of the conclusions. In Section 5, we prove that, when an even order symmetric Z-tensor with nonnegative diagonal entries is weakly irreducible, its spectrum and the spectrum of its absolute tensor are equal if and only if the Z-tensor is odd-bipartite. Furthermore, it is shown that the equality of the spectrum of a symmetric Z-tensor with nonnegative diagonal entries and the spectrum of the absolute tensor of that Z-tensor, can be characterized by the equality of their spectral radii. We conclude this paper with some final remarks in Section 6.

By the end of the introduction, we add some comments on notation that will be used in the sequel. Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and the set consisting of all natural numbers is denoted by $\mathbb{N}$. Suppose that $m, n \in \mathbb{N}$ are two natural numbers. Denote $[n] = \{1, 2, \ldots, n\}$. Vectors are denoted by italic lowercase letters, i.e., $x, y, \ldots$, and tensors are written as calligraphic capitals such as $\mathscr{A}, \mathscr{T}, \ldots$. The i-th unit coordinate vector in $\mathbb{R}^n$ is denoted by e_i. Let $|V|$ denote the number of elements when the symbol $|\cdot|$ be used on a subset $V \subseteq \mathbb{N}$. If the symbol $|\cdot|$ is used on a tensor

$$\mathscr{A} = (a_{i_1 i_2 \cdots i_m})_{1 \leqslant i_j \leqslant n}, \quad j = 1, 2, \ldots, m,$$

we get another tensor

$$|\mathscr{A}| = (|a_{i_1 i_2 \cdots i_m}|)_{1 \leqslant i_j \leqslant n}, \quad j = 1, 2, \ldots, m.$$

If both

$$\mathscr{A} = (a_{i_1 i_2 \cdots i_m})_{1 \leqslant i_j \leqslant n}, \quad \mathscr{B} = (b_{i_1 i_2 \cdots i_m})_{1 \leqslant i_j \leqslant n}, \quad j = 1, 2, \ldots, m,$$

are real mth order n dimensional tensors, then $\mathscr{A} \leqslant \mathscr{B}$ means

$$a_{i_1 i_2 \cdots i_m} \leqslant b_{i_1 i_2 \cdots i_m}, \quad \forall\, i_1, i_2 \ldots, i_m \in [n].$$

2 Preliminaries

In this section, we will review some basic notions of tensors. For more details, see [17] and references therein.

A real mth order n-dimensional tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ is a multi-array of real entries $a_{i_1 i_2 \cdots i_m}$, where $i_j \in [n]$ for $j \in [m]$. If the entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of their indices, then tensor $\mathscr{A}$ is called a symmetric tensor.

The following definition on eigenvalue-eigenvector comes from [17].

Definition 1 Let $\mathbb{C}$ be the complex field. A pair $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ is called an eigenvalue-eigenvector pair of $\mathscr{T}$, if they satisfy

$$\mathscr{T} x^{m-1} = \lambda x^{[m-1]}, \tag{1}$$

where $\mathscr{T}x^{m-1}$ and $x^{[m-1]}$ are all n-dimensional column vectors such as

$$\mathscr{T}x^{m-1} = \Bigg(\sum_{i_2,i_3,\ldots,i_m=1}^{n} t_{ii_2i_3\cdots i_m} x_{i_2}x_{i_3}\cdots x_{i_m}\Bigg)_{1\leqslant i\leqslant n}, \quad x^{[m-1]} = (x_i^{m-1})_{1\leqslant i\leqslant n}.$$

For real tensor $\mathscr{T}$ and $x \in \mathbb{R}^n$ in (1), λ is a real number since $\lambda = (\mathscr{T}x^{m-1})_j/x_j^{m-1}$ for some j with $x_j \neq 0$. In this case, λ is called an H-eigenvalue of $\mathscr{T}$ and x is its corresponding H-eigenvector [17].

Next, we present a fundamental result which will be much used in the sequel.

Proposition 1 [17] *Suppose that $\mathscr{T} = a(\mathscr{B} + b\mathscr{I})$, where a and b are two real numbers. Then μ is an eigenvalue (resp. H-eigenvalue) of tensor $\mathscr{T}$ if and only if $\mu = a(\lambda + b)$, where λ is an eigenvalue (resp. H-eigenvalue) of tensor $\mathscr{B}$. In this case, they have the same eigenvectors (resp. H-eigenvectors).*

The spectral radius of tensor $\mathscr{T}$ is denoted by

$$\rho(\mathscr{T}) = \max\{|\lambda| \colon \lambda \text{ is an eigenvalue of } \mathscr{T}\}.$$

All eigenvalues of tensor $\mathscr{T}$ construct the spectrum denoted by $\sigma(\mathscr{T})$.

3 Odd-bipartite and even-bipartite tensors

In this section, we first define odd-bipartite tensors and even-bipartite tensors. Then, some special characteristics of this kinds of tensors are shown.

Definition 2 Assume that $\mathscr{A} = (a_{i_1i_2\cdots i_m})$ is a tensor with order m and dimension n. If there is a nonempty proper index subset $V \subset [n]$ such that

$$a_{i_1i_2\cdots i_m} \begin{cases} \neq 0, & |V \cap \{i_1, i_2, \ldots, i_m\}| \text{ odd}, \\ = 0, & \text{otherwise}, \end{cases}$$

then $\mathscr{A}$ is called an odd-bipartite tensor corresponding to set V or $\mathscr{A}$ is odd-bipartite for simple.

Here, we should note that indices of an edge $\{i_1, i_2, \ldots, i_m\}$ in hypergraph [9] are different from each other, which is a notable distinction to general tensors. So, in this paper, we define that $|V \cap \{i_1, i_2, \ldots, i_m\}|$ is the number of indices $V \cap \{i_1, i_2, \ldots, i_m\}$, and duplicate indices should be calculated. For example, suppose that $V = \{1, 2, 3\}$ and $\mathscr{A}$ is a 4th order 6 dimensional tensor, then

$$|V \cap \{1, 1, 3, 3\}| = 4, \quad |V \cap \{1, 2, 3, 5\}| = 3, \quad |V \cap \{4, 6, 4, 5\}| = 0.$$

Definition 3 Assume that $\mathscr{A} = (a_{i_1i_2\cdots i_m})$ is a tensor with order m and dimension n. $\mathscr{A}$ is called weakly odd-bipartite if there is a nonempty proper index subset $V \subset [n]$ such that

$$a_{i_1i_2\cdots i_m} = 0$$

when $|V \cap \{i_1, i_2, \ldots, i_m\}|$ is even.

From Definitions 2 and 3, even-bipartite and weakly even-bipartite tensors can be defined similarly. Furthermore, we can easily prove that, if $\mathscr{A}$ is odd-bipartite (resp. even-bipartite), then $\mathscr{A}$ is weakly odd-bipartite (resp. weakly even-bipartite), but not vice versa. For example, suppose that $\mathscr{A}$ is a 3rd order 2 dimensional tensor with entries such that $a_{222} = 1$ and $a_{i_1 i_2 i_3} = 0$ for the others. It is easy to check that $\mathscr{A}$ is weakly odd-bipartite corresponding to the index set $V = \{2\}$ but not odd-bipartite corresponding to $\{1\}$ or $\{2\}$.

When m is odd, for all $i_1, i_2, \ldots, i_m \in [n]$ and a nonempty proper index subset $V \subset [n]$, it holds that $|\{i_1, i_2, \ldots, i_m\} \cap V|$ is odd if and only if $|\{i_1, i_2, \ldots, i_m\} \cap \overline{V}|$ is even, where $\overline{V} = [n] \backslash V$. So, by Definitions 2 and 3, we can readily obtain the following conclusion.

Lemma 1 *Let $\mathscr{A}$ be a tensor with order m and dimension n. Assume that m is odd. Then $\mathscr{A}$ is odd-bipartite* (*resp. weakly odd-bipartite*) *corresponding to nonempty proper index subset $V \subset [n]$ if and only if $\mathscr{A}$ is even-bipartite* (*resp. weakly even-bipartite*) *corresponding to the nonempty proper index subset $\overline{V} = [n] \backslash V$.*

Irreducible tensors are a class of important and useful tensors, which have been repeatedly used in the Perron Frobenius Theorem for nonnegative tensors [2,24,25]. Next, we will study the relation between irreducible tensors and odd-bipartite tensors. To do this, we first list the corresponding definition below.

Definition 4 [2] For a tensor $\mathscr{T}$ with order m and dimension n. We say that $\mathscr{T}$ is reducible if there is a nonempty proper index subset $V \subset [n]$ such that

$$t_{i_1 i_2 \cdots i_m} = 0, \quad \forall\, i_1 \in V, \forall\, i_2, i_3, \ldots, i_m \notin V.$$

Otherwise, we say that $\mathscr{T}$ is irreducible.

Theorem 1 *Let m be even. Assume that tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ with order m and dimension n is odd-bipartite. Then $\mathscr{A}$ is irreducible.*

Proof Since $\mathscr{A}$ is odd-bipartite, there exists a nonempty proper index subset $V \subset [n]$ satisfying

$$a_{i_1 i_2 \cdots i_m} \neq 0, \quad |V \cap \{i_1, i_2, \ldots, i_m\}| \text{ odd.} \tag{2}$$

By contradiction, suppose that $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ is reducible. Then there is a nonempty proper index subset $V_1 \subset [n]$ such that

$$a_{i_1 i_2 \cdots i_m} = 0, \quad \forall\, i_1 \in V_1, \forall\, i_2, i_3, \ldots, i_m \notin V_1. \tag{3}$$

We will break the proof into four cases.

(i) When $V_1 \subseteq V$, let $i_1 \in V_1, i_2, i_3, \ldots, i_m \notin V$. Here, several indices in $i_2, i_3, \ldots, i_m$ may equal to each other when the number of elements in $[n] \backslash V$ is strictly less than $m - 1$. Then, by (3), we have

$$a_{i_1 i_2 \cdots i_m} = 0,$$

which contradicts (2) since $|V \cap \{i_1, i_2, \ldots, i_m\}| = 1$ is odd.

(ii) When $V \subseteq V_1$, let $i_1 \in V$, $i_2, i_3, \ldots, i_m \notin V_1$. Then, by (3), one has

$$a_{i_1 i_2 \cdots i_m} = 0,$$

which is a contradiction with (2).

(iii) When $V \cap V_1 \neq \emptyset$ and neither $V \subseteq V_1$ nor $V_1 \subseteq V$, let $i_1 \in V_1 \setminus V$, $i_2, i_3, \ldots, i_m \in V \setminus V_1$. Then it follows that

$$a_{i_1 i_2 \cdots i_m} = 0,$$

which also contradicts (2), since $|V \cap \{i_1, i_2, \ldots, i_m\}| = m-1$ is an odd number.

(iv) When $V \cap V_1 = \emptyset$, let $i_1 \in V_1$, $i_2, i_3, \ldots, i_m \in V$. By Definition 4, we have

$$a_{i_1 i_2 \cdots i_m} = 0.$$

Since $|V \cap \{i_1, i_2, \ldots, i_m\}| = m - 1$ is odd, by (2), one has

$$a_{i_1 i_2 \cdots i_m} \neq 0,$$

which is a contradiction.

From (i)–(iv), we conclude that $\mathscr{A}$ is not reducible and the desired results follow. □

When a tensor $\mathscr{A}$ is even-bipartite, no matter the order of $\mathscr{A}$ is odd or even, we have the following result.

Theorem 2 *Assume that tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ with order m and dimension n is even-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. Then $\mathscr{A}$ is reducible corresponding to V.*

Proof By definitions of reducible tensors and even-bipartite tensors, the conclusion obviously holds. □

Suppose that an even order Z-tensor and its absolute tensor are defined such that

$$\mathscr{A} = \mathscr{D} - \mathscr{C}, \quad |\mathscr{A}| = \mathscr{D} + \mathscr{C}, \tag{4}$$

where $\mathscr{D}$ is a nonnegative diagonal tensor and $\mathscr{C}$ is a nonnegative tensor with zero diagonal entries. From Theorem 2, if $\mathscr{C}$ is odd-bipartite, then tensors $\mathscr{A}$ and $|\mathscr{A}|$ are irreducible. Combining this with [8, Theorem 3.1], we have the following result.

Corollary 1 *Let m be even. Suppose that tensor $\mathscr{A} = \mathscr{D} - \mathscr{C}$ with order m and dimension n is defined as in (4). Then, $\mathscr{A}$ and its absolute tensor $|\mathscr{A}|$ are both weakly irreducible if nonnegative tensor $\mathscr{C}$ is odd-bipartite.*

By the Perron-Frobenius Theorem on nonnegative tensors [2] and by [8, Theorem 4.1], the following result follows.

Corollary 2 *Let m be even. Assume that tensor $\mathscr{A}$ is defined as in Corollary 1. If $\mathscr{C}$ is odd-bipartite, then the largest H-eigenvalue of $|\mathscr{A}|$ is $\rho(|\mathscr{A}|)$. Furthermore, there exists a positive n-dimensional eigenvector $x \in \mathbb{R}^n$ such that*

$$|\mathscr{A}|x^{m-1} = \rho(|\mathscr{A}|)x^{[m-1]}.$$

4 Relation between largest H-eigenvalues of a Z-tensor and its absolute tensor

In this section, suppose that an order m dimension n Z-tensor $\mathscr{A}$ with nonnegative diagonal elements has format

$$\mathscr{A} = \mathscr{D} - \mathscr{C}, \tag{5}$$

where $\mathscr{D}$ is a nonnegative diagonal tensor and $\mathscr{C}$ is a nonnegative tensor with zero diagonal elements. Then the absolute format of $\mathscr{A}$ is

$$|\mathscr{A}| = \mathscr{D} + \mathscr{C}.$$

In the following analysis, entries of $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{D}$ are always defined as

$$\mathscr{A} = (a_{i_1 i_2 \cdots i_m}), \quad \mathscr{C} = (c_{i_1 i_2 \cdots i_m}), \quad \mathscr{D} = (d_{i_1 i_2 \cdots i_m}), \quad i_1, i_2, \ldots, i_m \in [n].$$

For the sake of simple, let $d_{ii\cdots i} = d_i$, $i \in [n]$.

During this part, we mainly study the relationship between the largest H-eigenvalue of a Z-tensor $\mathscr{A}$ in (5), and the largest H-eigenvalue of the absolute tensor of $\mathscr{A}$. Sufficient and necessary conditions or sufficient conditions to guarantee the equality of these largest H-eigenvalues are shown. It should be noted that all even order nonnegative tensors always have H-eigenvalues [24]. To proceed, we make an assumption in advance, all tensors considered in this part always have H-eigenvalues.

The largest H-eigenvalues of $\mathscr{A}$ and $|\mathscr{A}|$ are denoted by $\lambda(\mathscr{A})$ and $\lambda(|\mathscr{A}|)$, respectively. From Corollary 2, we know that

$$\lambda(|\mathscr{A}|) = \rho(|\mathscr{A}|).$$

Theorem 3 *Let m be even. Suppose that $\mathscr{A} = \mathscr{D} - \mathscr{C}$ is defined as* (5)*. Then,*

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$$

if $\mathscr{C}$ is odd-bipartite.

Proof By [19, Lemma 13], we have

$$\lambda(\mathscr{A}) \leqslant \rho(\mathscr{A}) \leqslant \rho(|\mathscr{A}|) = \lambda(|\mathscr{A}|).$$

Thus, in order to prove the conclusion, we only need to prove

$$\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A}).$$

Since $\mathscr{C}$ is odd-bipartite, there exists a nonempty proper index subset $V \subset [n]$ satisfying

$$c_{i_1 i_2 \cdots i_m} \begin{cases} \neq 0, & |V \cap \{i_1, i_2, \ldots, i_m\}| \text{ odd}, \\ = 0, & \text{otherwise}. \end{cases}$$

So, for all entries of $\mathscr{A}$, it follows that

$$a_{i_1 i_2 \cdots i_m} \neq 0$$

if $|V \cap \{i_1, i_2, \ldots, i_m\}|$ is odd, and $a_{i_1 i_2 \cdots i_m} = 0$ for the others except the diagonal entries $a_{ii\cdots i}$, $i \in [n]$. By Theorem 2, we know that $\mathscr{C}$, $\mathscr{A}$, and $|\mathscr{A}|$ are all irreducible tensors. From [8, Theorem 4.1] and Definition 1, there is a vector $x \in \mathbb{R}^n$, $x > 0$, satisfying

$$|\mathscr{A}|x^{m-1} = \lambda(|\mathscr{A}|)x^{[m-1]}.$$

Let $y \in \mathbb{R}^n$ be defined with $y_i = x_i$ whenever $i \in V$ and $y_i = -x_i$ for the others. When $i \in V$, we have

$$\begin{aligned} (\mathscr{A} y^{m-1})_i &= [(\mathscr{D} - \mathscr{C})y^{m-1}]_i \\ &= d_i y_i^{m-1} - \sum_{i_2, i_3, \ldots, i_m \in [n]} c_{i i_2 i_3 \cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m} \\ &= d_i y_i^{m-1} - \sum_{i_2, i_3, \ldots, i_m \in [n],\, |V \cap \{i, i_2, i_3, \ldots, i_m\}| \text{ odd}} c_{i i_2 i_3 \cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m} \\ &= d_i x_i^{m-1} + \sum_{i_2, i_3, \ldots, i_m \in [n],\, |V \cap \{i, i_2, i_3, \ldots, i_m\}| \text{ odd}} c_{i i_2 i_3 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\ &= [(\mathscr{D} + \mathscr{C})x^{m-1}]_i \\ &= \lambda(|\mathscr{A}|) x_i^{m-1} \\ &= \lambda(|\mathscr{A}|) y_i^{m-1}. \end{aligned} \tag{6}$$

Here, the fourth equality follows the fact that m is even and exactly odd number indices take negative values for each $\{i_2, i_3, \ldots, i_m\} \subseteq [n]$. When $i \notin V$, we have

$$\begin{aligned} (\mathscr{A} y^{m-1})_i &= [(\mathscr{D} - \mathscr{C})y^{m-1}]_i \\ &= d_i y_i^{m-1} - \sum_{i_2, i_3, \ldots, i_m \in [n]} c_{i i_2 i_3 \cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m} \\ &= d_i y_i^{m-1} - \sum_{i_2, i_3, \ldots, i_m \in [n],\, |V \cap \{i, i_2, i_3, \ldots, i_m\}| \text{ odd}} c_{i i_2 i_3 \cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m} \\ &= -d_i x_i^{m-1} - \sum_{i_2, i_3, \ldots, i_m \in [n],\, |V \cap \{i, i_2, i_3, \ldots, i_m\}| \text{ odd}} c_{i i_2 i_3 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\ &= -[(\mathscr{D} + \mathscr{C})x^{m-1}]_i \\ &= -\lambda(|\mathscr{A}|) x_i^{m-1} \\ &= \lambda(|\mathscr{A}|) y_i^{m-1}. \end{aligned} \tag{7}$$

Here, the fourth equality follows the fact that m is even and exactly even number indices take negative values for each $\{i_2, i_3, \ldots, i_m\} \subseteq [n]$. The last equality of (7) follows from the definition of $y_i = -x_i$ when $i \notin V$. Thus, by (6), (7), and Definition 1, $\lambda(|\mathscr{A}|)$ is an H-eigenvalue of $\mathscr{A}$ with H-eigenvector y. So, we have

$$\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A}),$$

and the desired result follows. □

Here, in the proof of Theorem 3, odd-bipartite property of $\mathscr{C}$ guarantees that $|\mathscr{A}|$ has a positive H-eigenvector. Actually, if the H-eigenvector is nonnegative, one can obtain the same result. Before proving this, we first cite an useful conclusion.

Lemma 2 [24] *If $\mathscr{A}$ is a nonnegative tensor with order m and dimension n, then $\rho(\mathscr{A})$ is an eigenvalue of $\mathscr{A}$ with a nonnegative eigenvector $y \neq 0$.*

Theorem 4 *Let m be even. Suppose that $\mathscr{A}$ is defined as in Theorem 3. If $\mathscr{C}$ is weakly odd-bipartite, then it holds that*

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|).$$

Proof Since tensor $\mathscr{C}$ is weakly odd-bipartite, there is a nonempty proper index subset $V \subseteq [n]$ such that

$$c_{i_1 i_2 \cdots i_m} = 0$$

when $|\{i_1, i_2, \ldots, i_m\} \cap V|$ is even, and $|\{i_1, i_2, \ldots, i_m\} \cap V|$ must be an odd number for nonzero entries $c_{i_1 i_2 \cdots i_m} \neq 0$, $i_1, i_2, \ldots, i_m \in [n]$.

On the other hand, by Lemma 2, there is a nonnegative H-eigenvector $x \geqslant 0$ of $|\mathscr{A}|$ corresponding to $\lambda(|\mathscr{A}|)$. Suppose that vector $y \in \mathbb{R}^n$ is defined such that $y_i = x_i$ whenever $i \in V$ and $y_i = -x_i$ for the others. Then, the remaining process is similar with the proof of Theorem 3. □

Now, we will give an example to show that the conditions in Theorem 4 is not necessary. For example, suppose 4th order 2 dimensional tensor $\mathscr{A}$ with entries such that

$$a_{1111} = a_{2222} = 1, \quad a_{1122} = -1,$$

and $a_{i_1 i_2 i_3 i_4} = 0$ for the others. After calculating the largest H-eigenvalues of $\mathscr{A}$ and $|\mathscr{A}|$, we obtain

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 1.$$

But, the nonnegative tensor $\mathscr{C}$ is not weakly odd-bipartite corresponding to any nonempty proper index subset of $\{1, 2\}$. In the following, sufficient and necessary conditions for the equality of the two largest H-eigenvalues are presented, and it is proved that the necessity of Theorem 4 holds when the nonnegative tensor $\mathscr{C}$ is weakly irreducible. Before doing this, we cite a definition.

Definition 5 [19] Assume that $\mathscr{T}$ is a tensor with order m and dimension n. Construct a graph $\hat{G} = (\hat{V}, \hat{E})$, where $\hat{V} = \cup_{j=1}^{d} V_j$ and V_j are subsets of $\{1, 2, \ldots, n\}$ for $j = 1, 2, \ldots, d$. Suppose that $i_j \in V_j$, $i_l \in V_l$, $j \neq l$. $(i_j, i_l) \in \hat{E}$ if and only if $t_{i_1 i_2 \cdots i_m} \neq 0$ for some $m-2$ indices $\{i_1, i_2, \ldots, i_m\} \backslash \{i_j, i_l\}$. Then, tensor $\mathscr{T}$ is called weakly irreducible if $\hat{G}$ is connected.

As observed in [8], an irreducible tensor must be always weakly irreducible.

Theorem 5 *Let $\mathscr{A}$ be defined as in Theorem* 4. *Assume that $\mathscr{C}$ is weakly irreducible. Then,*

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$$

if and only if $\mathscr{C}$ is weakly odd-bipartite.

Proof The sufficient condition has been proved in Theorem 4, and we only need to prove the necessary part.

Suppose that $x \in \mathbb{R}^n$ is an H-eigenvector of $\mathscr{A}$ corresponding to $\lambda(\mathscr{A})$ such that

$$\sum_{i=1}^{n} x_i^m = 1.$$

Assume that $y \in \mathbb{R}^n$ is defined by $y_i = |x_i|$ for $i \in [n]$. Since m is even, one has

$$\sum_{i=1}^{n} y_i^m = 1.$$

By [11, Lemma 3.1], we have

$$\begin{aligned}
\lambda(\mathscr{A}) &= \mathscr{A} x^m \\
&= (\mathscr{D} - \mathscr{C}) x^m \\
&= \sum_{i=1}^{n} d_i x_i^m - \sum_{i_1, i_2, \ldots, i_m \in [n]} c_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\
&\leqslant \sum_{i=1}^{n} d_i y_i^m + \sum_{i_1, i_2, \ldots, i_m \in [n]} c_{i_1 i_2 \cdots i_m} y_{i_1} y_{i_2} \cdots y_{i_m} \\
&= (\mathscr{D} + \mathscr{C}) y^m \\
&\leqslant \lambda(|\mathscr{A}|). \qquad (8)
\end{aligned}$$

Hence, by the fact that $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$, all inequalities in (8) should be equalities, which implies that y is an H-eigenvector of $|\mathscr{A}|$ corresponding to $\lambda(|\mathscr{A}|)$. Since $\mathscr{C}$ is weakly irreducible, $|\mathscr{A}|$ is also weakly irreducible. According to [8, Theorem 4.1], it holds that $y > 0$, i.e., all elements in y are positive. Let

$$V = \{i \in [n] \mid x_i > 0\}, \quad \overline{V} = \{i \in [n] \mid x_i < 0\}.$$

Then $V \cup \overline{V} = [n]$. By (8), we obtain

$$\sum_{i_1,i_2,\dots,i_m \in [n]} c_{i_1 i_2 \cdots i_m}(|x_{i_1}|\,|x_{i_2}| \cdots |x_{i_m}| + x_{i_1} x_{i_2} \cdots x_{i_m}) = 0,$$

which implies that

$$c_{i_1 i_2 \cdots i_m}(|x_{i_1}|\,|x_{i_2}| \cdots |x_{i_m}| + x_{i_1} x_{i_2} \cdots x_{i_m}) = 0, \quad \forall\, i_1, i_2, \dots, i_m \in [n],$$

since $\mathscr{C}$ is nonnegative. When $|\{i_1, i_2, \dots, i_m\} \cap V|$ is even, we have

$$|x_{i_1}|\,|x_{i_2}| \cdots |x_{i_m}| + x_{i_1} x_{i_2} \cdots x_{i_m} > 0,$$

which implies $c_{i_1 i_2 \cdots i_m} = 0$. When $|\{i_1, i_2, \dots, i_m\} \cap V|$ is odd, we have

$$|x_{i_1}|\,|x_{i_2}| \cdots |x_{i_m}| + x_{i_1} x_{i_2} \cdots x_{i_m} = 0.$$

In this case, the value $c_{i_1 i_2 \cdots i_m}$ may be zero or may not be zero. Thus, from Definition 3, it follows that $\mathscr{C}$ is weakly odd-bipartite corresponding to set V and the desired conclusion holds. □

Next, we study the relationship between a Z-tensor and its absolute tensor in the odd order case. Hu et al. [10] proved that the largest H-eigenvalue of an odd order Laplacian tensor is always strictly less than the largest H-eigenvalue of the signless Laplacian tensor corresponding to the Laplacian tensor. By definitions of Laplacian tensor and signless Laplacian tensor in connected hypergraphs, we know that their diagonal entries are positive, and subscripts of each nonzero element are mutually distinct. However, general Z-tensors (5) may not possess those advantages. Hence, for a general odd order Z-tensor (5), the largest H-eigenvalue of $\mathscr{A}$ may not be strictly less than the largest H-eigenvalue of $|\mathscr{A}|$ when the order is odd.

The following example shows that the largest H-eigenvalues of a Z-tensor (5) and its absolute tensor are equal.

Example 1 Let $\mathscr{A}$ be a 5th order 3 dimensional tensor. Its entries are given by

$$a_{11111} = a_{22222} = a_{33333} = 1, \quad a_{11122} = a_{22233} = -1,$$

and $a_{i_1 i_2 i_3 i_4 i_5} = 0$ for the others. Then the H-eigenvalue problems for $\mathscr{A}$ and $|\mathscr{A}|$ are

$$\begin{cases} x_1^4 - x_1^2 x_2^2 = \lambda x_1^4, \\ x_2^4 - x_2^2 x_3^2 = \lambda x_2^4, \\ x_3^4 = \lambda x_3^4, \end{cases} \qquad \begin{cases} x_1^4 + x_1^2 x_2^2 = \lambda x_1^4, \\ x_2^4 + x_2^2 x_3^2 = \lambda x_2^4, \\ x_3^4 = \lambda x_3^4. \end{cases}$$

After calculating the equation systems, we know that

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 1.$$

Theorem 6 *Let A be defined as* (5). *Assume that m is odd. Suppose that $\mathscr{C}$ is weakly odd-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. If for all $i \in V$, it satisfies*

$$c_{ii_2i_3\cdots i_m} = 0, \quad \forall\, i_2, i_3, \ldots, i_m \in [n],$$

then $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$.

Proof By the analysis in Theorems 3–5, from [19, Lemma 13] and Corollary 2, it follows that

$$\lambda(\mathscr{A}) \leqslant \rho(\mathscr{A}) \leqslant \rho(|\mathscr{A}|) = \lambda(|\mathscr{A}|).$$

Thus, we only need to prove

$$\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A}).$$

Let $x \in \mathbb{R}^n$ be a nonnegative H-eigenvector of $|\mathscr{A}|$ corresponding to $\lambda(|\mathscr{A}|)$. Then, for all $i \in [n]$, we have

$$(|\mathscr{A}|x^{m-1})_i = [(\mathscr{D} + \mathscr{C})x^{m-1}]_i = \lambda(|\mathscr{A}|)x_i^{m-1}. \tag{9}$$

Suppose that $y \in \mathbb{R}^n$ is defined as

$$y_i = \begin{cases} -x_i, & i \in V, \\ x_i, & i \notin V. \end{cases}$$

By conditions, $\mathscr{C}$ is weakly odd-bipartite corresponding to subset V, which means

$$c_{i_1i_2\cdots i_m} = 0, \quad i_1, i_2, \ldots, i_m \in [n],$$

when $|\{i_1, i_2, i_3, \ldots, i_m\} \cap V|$ is even. Then, for all $i \in [n]$, one has

$$\begin{aligned}
(\mathscr{A}y^{m-1})_i &= [(\mathscr{D} - \mathscr{C})y^{m-1}]_i \\
&= d_i y_i^{m-1} - \sum_{i_2,i_3,\ldots,i_m \in [n],\, |V \cap \{i,i_2,i_3,\ldots,i_m\}| \text{ odd}} c_{ii_2i_3\cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m} \\
&= d_i x_i^{m-1} - \sum_{i_2,i_3,\ldots,i_m \in [n],\, |V \cap \{i,i_2,i_3,\ldots,i_m\}| \text{ odd}} c_{ii_2i_3\cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m},
\end{aligned} \tag{10}$$

where the third equality follows $m-1$ is even and $y_i^{m-1} = x_i^{m-1}$. When $i \in V$, by the fact that $c_{ii_2i_3\cdots i_m} = 0$, $i_2, i_3, \ldots, i_m \in [n]$, and by (9), (10), we have

$$\begin{aligned}
(\mathscr{A}y^{m-1})_i &= [(\mathscr{D} - \mathscr{C})y^{m-1}]_i \\
&= d_i y_i^{m-1} - \sum_{i_2,i_3,\ldots,i_m \in [n]} c_{ii_2i_3\cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m} \\
&= d_i x_i^{m-1} \\
&= \lambda(|\mathscr{A}|) x_i^{m-1} \\
&= \lambda(|\mathscr{A}|) y_i^{m-1}.
\end{aligned} \tag{11}$$

Similarly, when $i \notin V$, it holds that

$$\begin{aligned}
(\mathscr{A}y^{m-1})_i &= [(\mathscr{D}-\mathscr{C})y^{m-1}]_i \\
&= d_i y_i^{m-1} - \sum_{i_2,i_3,\ldots,i_m\in[n],\,|V\cap\{i,i_2,i_3,\ldots,i_m\}|\ \text{odd}} c_{ii_2i_3\cdots i_m} y_{i_2}y_{i_3}\cdots y_{i_m} \\
&= d_i x_i^{m-1} + \sum_{i_2,i_3,\ldots,i_m\in[n],\,|V\cap\{i,i_2,i_3,\ldots,i_m\}|\ \text{odd}} c_{ii_2i_3\cdots i_m} x_{i_2}x_{i_3}\cdots x_{i_m} \\
&= d_i x_i^{m-1} + (\mathscr{C}x^{m-1})_i \\
&= [(\mathscr{D}+\mathscr{C})x^{m-1}]_i \\
&= (|\mathscr{A}|x^{m-1})_i \\
&= \lambda(|\mathscr{A}|)x_i^{m-1} \\
&= \lambda(|\mathscr{A}|)y_i^{m-1}, \qquad (12)
\end{aligned}$$

where the third equality follows the fact that m is odd and exactly odd indices take negative values. By (11) and (12), we know that $\lambda(|\mathscr{A}|)$ is an H-eigenvalue of $\mathscr{A}$. Hence, we have $\lambda(|\mathscr{A}|) \leqslant \lambda(\mathscr{A})$ and the desired result follows. □

Now, we present an example to verify the authenticity of Theorem 6.

Example 2 Set a 5th order 3 dimensional tensor $\mathscr{A}$ such that

$$a_{11111}=1,\quad a_{22222}=1,\quad a_{33333}=3,\quad a_{11333}=-1,\quad a_{22333}=-2,$$

and $a_{i_1i_2i_3i_4}=0$ for the others. Let $V=\{3\}$. Then $\mathscr{C}$ is weakly odd-bipartite corresponding to the set V and $c_{3i_2i_3i_4i_5}=0$, $\forall\, i_2,i_3,i_4,i_5\in[3]$.

The H-eigenvalue problems for $\mathscr{A}$ and $|\mathscr{A}|$ are to solve

$$\begin{cases} x_1^4 - x_1x_3^3 = \lambda x_1^4, \\ x_2^4 - 2x_2x_3^3 = \lambda x_2^4, \\ 3x_3^4 = \lambda x_3^4, \end{cases} \qquad \begin{cases} x_1^4 + x_1x_3^3 = \lambda x_1^4, \\ x_2^4 + 2x_2x_3^3 = \lambda x_2^4, \\ 3x_3^4 = \lambda x_3^4. \end{cases}$$

After calculating the largest H-eigenvalues of $\mathscr{A}$ and $|\mathscr{A}|$, we obtain

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 3.$$

The next example shows that the conditions in Theorem 6 are not necessary.

Example 3 Let $\mathscr{A}$ be a 5th order 3 dimensional tensor. Its entries are given by

$$a_{11111}=1,\quad a_{22222}=2,\quad a_{33333}=4,\quad a_{11122}=a_{11333}=-1,\quad a_{22233}=-2,$$

and $a_{i_1i_2i_3i_4i_5}=0$ for the others. Then the H-eigenvalue problems for $\mathscr{A}$ and $|\mathscr{A}|$ are

$$\begin{cases} x_1^4 - x_1^2x_2^2 - x_1x_3^3 = \lambda x_1^4, \\ 2x_2^4 - 2x_2^2x_3^2 = \lambda x_2^4, \\ 4x_3^4 = \lambda x_3^4, \end{cases} \qquad \begin{cases} x_1^4 + x_1^2x_2^2 + x_1x_3^3 = \lambda x_1^4, \\ 2x_2^4 + 2x_2^2x_3^2 = \lambda x_2^4, \\ 4x_3^4 = \lambda x_3^4. \end{cases}$$

After calculating these equation sets, we know that

$$\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|) = 4,$$

but the nonnegative tensor $\mathscr{C}$ is not weakly odd-bipartite corresponding to any nonempty proper index subset of $\{1, 2, 3\}$.

By Lemma 1 and Theorem 6, we have the following conclusion.

Corollary 3 *Let A be defined as in* (5). *Assume that m is odd. Suppose that $\mathscr{C}$ is weakly even-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. If for all $i \notin V$, it satisfies*

$$c_{ii_2i_3\cdots i_m} = 0, \quad \forall\, i_2, i_3, \ldots, i_m \in [n],$$

then $\lambda(\mathscr{A}) = \lambda(|\mathscr{A}|)$.

5 Relation between spectrums of a symmetric *Z*-tensor and its absolute tensor

In this section, we will study the relation between the spectrum of an even order symmetric Z-tensor with nonnegative diagonal entries and the spectrum of the absolute tensor of the Z-tensor. It is proved that, if the symmetric Z-tensor is weakly irreducible and odd-bipartite, then the two spectral sets equal. Furthermore, for a weakly irreducible symmetric Z-tensor with nonnegative diagonal entries, we show that the spectral sets of the Z-tensor and its absolute tensor equal if and only if their spectral radii equal. Before proving the conclusion, we first cite the definition of diagonal similar tensors [22], which is useful in the following analysis.

Definition 6 Let $\mathscr{A}$ and $\mathscr{B}$ be two order $m \geqslant 2$ dimension n tensors. If there exists a nonsingular diagonal matrix P of dimension n such that

$$\mathscr{B} = P^{-(m-1)}\mathscr{A}P,$$

then $\mathscr{A}$ and $\mathscr{B}$ are called diagonal similar.

Here, tensor $\mathscr{B} = P^{-(m-1)}\mathscr{A}P$ is defined by

$$b_{i_1i_2\cdots i_m} = \sum_{j_1,j_2,\ldots,j_m\in[n]} a_{j_1j_2\cdots j_m} p_{i_1j_1}^{m-1} p_{j_2i_2} p_{j_3i_3} \cdots p_{j_mi_m}, \quad i_1, i_2, \ldots, i_m \in [n].$$

Theorem 7 *Assume that order m dimension n symmetric Z-tensor $\mathscr{A}$ is defined as in* (5). *Suppose that $\mathscr{C}$ is weakly irreducible. Then, $\mathscr{A}$ and $|\mathscr{A}|$ are diagonal similar if and only if m is even and $\mathscr{C}$ is weakly odd-bipartite.*

Proof For necessity, from Definition 6, we know that there is a nonsingular diagonal matrix P satisfying

$$\mathscr{A} = P^{-(m-1)}|\mathscr{A}|P,$$

i.e.,

$$\mathscr{D}-\mathscr{C}=P^{-(m-1)}(\mathscr{D}+\mathscr{C})P.$$

Since $\mathscr{D}=P^{-(m-1)}\mathscr{D}P$, we have

$$-\mathscr{C}=P^{-(m-1)}\mathscr{C}P,$$

which implies that

$$-c_{i_1i_2\cdots i_m}=c_{i_1i_2\cdots i_m}p_{i_1i_1}^{-(m-1)}p_{i_2i_2}p_{i_3i_3}\cdots p_{i_mi_m}. \tag{13}$$

If $p_{11}=p_{22}=\cdots=p_{nn}$, by (13), we get $\mathscr{C}=0$, which is a contradiction to the fact that $\mathscr{C}$ is weakly irreducible. So there are at least two distinct diagonal entries in P.

When $c_{i_1i_2\cdots i_m}\neq 0$, by (13), one has

$$-p_{i_1i_1}^{m}=p_{i_1i_1}p_{i_2i_2}\cdots p_{i_mi_m}. \tag{14}$$

By (14), and by the fact that $\mathscr{C}$ is weakly irreducible, we obtain

$$p_{ii}^m=p_{jj}^m,\quad i,j\in[n],$$

which implies that m is even and

$$V=\{i\in[n]\mid p_{ii}<0\}\neq\emptyset,\quad \widetilde{V}=\{i\in[n]\mid p_{ii}>0\}\neq\emptyset.$$

Combining this with (13) and (14), we know that

$$c_{i_1i_2\cdots i_m}=0,\quad |\{i_1,i_2,\ldots,i_m\}\cap V|\ \text{even}.$$

Thus, tensor $\mathscr{C}$ is weakly odd-bipartite corresponding to V and the only if part holds.

For the if part, without loss of generality, suppose that $\mathscr{C}$ is weakly odd-bipartite corresponding to $\Omega\subset[n]$. Let P be a diagonal matrix with i-th diagonal entries being -1 when $i\in\Omega$ and 1 when $i\notin\Omega$. By a direct computation, one has

$$\mathscr{A}=P^{-(m-1)}|\mathscr{A}|P.$$

Apparently, P is a nonsingular diagonal matrix. From Definition 6, it follows that $\mathscr{A}$ and $|\mathscr{A}|$ are diagonal similar. □

It should be noted that diagonal similar tensors have the same characteristic polynomials, and thus, they have the same spectrum (see [22, Theorem 2.1]), which is similar to the matrix case.

Corollary 4 *Assume that tensor $\mathscr{A}$ is defined as in Theorem 7. Let m be even. Suppose that $\mathscr{C}$ is odd-bipartite. Then $\sigma(\mathscr{A})=\sigma(|\mathscr{A}|)$.*

Lemma 3 [25] *Let $\mathscr{A}$ and $\mathscr{B}$ be two order m dimension n tensors with $|\mathscr{B}|\leqslant\mathscr{A}$. Then*

(a) $\rho(\mathscr{B}) \leqslant \rho(\mathscr{A})$;

(b) *if* $\mathscr{A}$ *is weakly irreducible and* $\rho(\mathscr{B}) = \rho(\mathscr{A})$, *where* $\lambda = \rho(\mathscr{A})\mathrm{e}^{\mathrm{i}\psi}$ *is an eigenvalue of* $\mathscr{B}$ *with an eigenvector* y, *then*

(i) *all the components of* y *are nonzero*;

(ii) *when* $U = \mathrm{diag}(y_1/|y_1|, y_2/|y_2|, \ldots, y_n/|y_n|)$ *is a nonsingular diagonal matrix, we have*

$$\mathscr{B} = \mathrm{e}^{\mathrm{i}\psi} U^{-(m-1)} \mathscr{A} U.$$

Theorem 8 *Assume that order* m *dimension* n *symmetric* Z*-tensor* $\mathscr{A}$ *is defined as in* (5). *If* $\mathscr{C}$ *is weakly irreducible, then,* $\rho(\mathscr{A}) = \rho(|\mathscr{A}|)$ *if and only if* $\sigma(\mathscr{A}) = \sigma(|\mathscr{A}|)$.

Proof The sufficient condition is obvious. Now, we prove the only if part. Suppose that $\lambda = \rho(|\mathscr{A}|)\mathrm{e}^{\mathrm{i}\psi}$ is an eigenvalue of $\mathscr{A}$. Since $\mathscr{C}$ is weakly irreducible, from Lemma 3, we know that there exists a nonsingular diagonal matrix P such that

$$\mathscr{A} = \mathrm{e}^{\mathrm{i}\psi} P^{-(m-1)} |\mathscr{A}| P, \tag{15}$$

which means

$$\mathscr{D} - \mathscr{C} = \mathrm{e}^{\mathrm{i}\psi} P^{-(m-1)} (\mathscr{D} + \mathscr{C}) P. \tag{16}$$

By the fact that all diagonal elements of $\mathscr{C}$ equal zero and by (16), one has

$$\mathscr{D} = \mathrm{e}^{\mathrm{i}\psi} P^{-(m-1)} \mathscr{D} P = \mathrm{e}^{\mathrm{i}\psi} \mathscr{D},$$

which implies $\mathrm{e}^{\mathrm{i}\psi} = 1$. So, by Definition 6 and (15), we know that $\mathscr{A}$ and $|\mathscr{A}|$ are diagonal similar tensors. Thus, from [22, Theorem 2.3], it holds that $\sigma(\mathscr{A}) = \sigma(|\mathscr{A}|)$. □

6 Final remarks

Odd-bipartite and even-bipartite tensors are defined in this paper. Using this, we study the relation between the largest H-eigenvalue of a Z-tensor with nonnegative diagonal elements and the largest H-eigenvalue of the Z-tensor's absolute tensor. Sufficient and necessary conditions for the equality of these largest H-eigenvalues are given when the Z-tensor has even order. For the odd order case, sufficient conditions are presented. Examples are given to verify the authenticity of the conclusions. On the other side, relation between spectral sets of an even order symmetric Z-tensor with nonnegative diagonal entries and its absolute tensor are studied.

In this paper, we only study the case of H-eigenvalues of Z-tensors. Do Z-eigenvalues of Z-tensors also hold in such case? This may be an interesting work in the future.

Acknowledgements The authors were thankful to the anonymous referees for the meaningful suggestions to improve the paper. The first author's work was supported in part by the National Natural Science Foundation of China (Grant No. 11171180) and the second author's work was supported by the Hong Kong Research Grant Council (Grant Nos. PolyU 502111, 501212, 501913, 15302114).

References

1. Cartwright D, Sturmfels B. The number of eigenvalues of a tensor. Linear Algebra Appl, 2013, 438: 942–952
2. Chang K C, Pearson K, Zhang T. Perron Frobenius Theorem for nonnegative tensors. Commun Math Sci, 2008, 6: 507–520
3. Chang K C, Pearson K, Zhang T. On eigenvalue problems of real symmetric tensors. J Math Anal Appl, 2009, 350: 416–422
4. Chen H, Qi L. Positive definiteness and semi-definiteness of even order symmetric Cauchy tensors. J Ind Manag Optim, 2015, 11: 1263–1274
5. Chen Z, Qi L. Circulant tensors with applications to spectral hypergraph theory and stochastic process. J Ind Manag Optim, 2016, 12: 1227–1247
6. Cooper J, Dutle A. Spectra of uniform hypergraphs. Linear Algebra Appl, 2012, 436: 3268–3292
7. Ding W, Qi L, Wei Y. M-Tensors and nonsingular M-tensors. Linear Algebra Appl, 2013, 439: 3264–3278
8. Friedland S, Gaubert S, Han L. Perron-Frobenius theorem for nonnegative multilinear forms and extensions. Linear Algebra Appl, 2013, 438: 738–749
9. Hu S, Qi L. The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform. Discrete Appl Math, 2014, 169: 140–151
10. Hu S, Qi L, Xie J. The largest Laplacian and signless Laplacian H-eigenvalues of a uniform hypergraph. Linear Algebra Appl, 2015, 469: 1–27
11. Li G, Qi L, Yu G. Semismoothness of the maximum eigenvalue function of a symmetric tensor and its application. Linear Algebra Appl, 2013, 438(2): 813–833
12. Lim L -H. Singular values and eigenvalues of tensors: a variational approach. Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP '05), 2005, 1: 129–132
13. Mantica C A, Molinari L G. Weakly Z-symmetric manifolds. Acta Math Hungar, 2012, 135(1): 80–96
14. Ng M, Qi L, Zhou G. Finding the largest eigenvalue of a nonnegative tensor. SIAM J Matrix Anal Appl, 2009, 31: 1090–1099
15. Oeding L, Ottaviani G. Eigenvectors of tensors and algorithms for Waring decomposition. J Symbolic Comput, 2013, 54: 9–35
16. Pearson K, Zhang T. On spectral hypergraph theory of the adjacency tensor. Graphs Combin, 2014, 30: 1233–1248
17. Qi L. Eigenvalue of a real supersymmetric tensor. J Symbolic Comput, 2005, 40: 1302–1324
18. Qi L. Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl, 2013, 439: 228–238
19. Qi L. H^{+}-eigenvalues of Laplacian and signless Laplacian tensors. Commun Math Sci, 2014, 12: 1045–1064
20. Qi L, Dai H, Han D. Conditions for strong ellipticity and M-eigenvalues. Front Math China, 2009, 4: 349–364
21. Qi L, Wang Y, Wu E X. D-eigenvalues of diffusion kurtosis tensor. J Comput Appl Math, 2008, 221: 150–157
22. Shao J. A general product of tensors with applications. Linear Algebra Appl, 2013, 439: 2350–2366

23. Shao J, Shan H, Wu B. Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs. Linear Multilinear Algebra (ahead-of-print), 2015, 1–14
24. Yang Y, Yang Q. Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J Matrix Anal Appl, 2010, 31: 2517–2530
25. Yang Y, Yang Q. On some properties of nonnegative weakly irreducible tensors. arXiv: 1111.0713, 2011
26. Zhang L, Qi L, Zhou G. M-tensors and some applications. SIAM J Matrix Anal Appl, 2014, 35: 437–452