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Perturbation bounds of tensor eigenvalue and singular value problems with even order

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The main purpose of this paper is to investigate the perturbation bounds of the tensor eigenvalue and singular value problems with even order. We extend classical definitions from matrices to tensors, such as, λ-tensor and the tensor polynomial eigenvalue problem. We design a method for obtaining a mode-symmetric embedding from a general tensor. For a given tensor, if the tensor is mode-symmetric, then we derive perturbation bounds on an algebraic simple eigenvalue and Z-eigenvalue. Otherwise, based on symmetric or mode-symmetric embedding, perturbation bounds of an algebraic simple singular value are presented. For a given tensor tuple, if all tensors in this tuple are mode-symmetric, based on the definition of a λ-tensor, we estimate perturbation bounds of an algebraic simple polynomial eigenvalue. In particular, we focus on tensor generalized eigenvalue problems and tensor quadratic eigenvalue problems.

Keywords: algebraic simple; mode-symmetry; mode-symmetric embedding; mode-k tensor polynomial eigenvalue; tensor generalized eigenvalue; tensor quadratic eigenvalue; tensor generalized singular value; nonsingular tensor

AMS Subject Classifications: 15A18; 15A69; 65F15; 65F10

1. Introductions

Qi [1] defined two kinds of eigenvalue and investigated relative results similar to the matrix eigenvalue. Independently, Lim [2] proposed another definition of eigenvalue, eigenvectors, singular value and singular vectors for tensors based on a constrained variational approach, much like the Rayleigh quotient for symmetric matrix eigenvalue (see [3, Chapter 8]).

Chang et al. [4,5] introduced the eigenvalue and defined generalized tensor eigenproblems. To our best knowledge, Kolda and Mayo [6], Cui et al. [7] proposed two algorithms for solving generalized tensor eigenproblems, and they pointed out that the generalized eigenvalue framework unifying definitions of tensor eigenvalue, such as, eigenvalue and H-eigenvalue [1,2], E-eigenvalue, Z-eigenvalue [1] and D-eigenvalue [8]. Ding and Wei

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[9] focused on the properties and perturbations of the spectra of regular tensor pairs and extended results from matrices or matrix pairs to tensor pairs.

Li and Ng [10,11] extended the well-known column sum bound of the spectral radius for nonnegative matrices to the tensor case, and also derived an upper bound of the spectral radius for a nonnegative tensor via the largest eigenvalue of a symmetric tensor.

Throughout this paper, we assume that $m,\ n\ (\geq 2)$ are positive integers and m is even. We use small letters $x, u, v, \ldots,$ for scalars, small bold letters $\mathbf{x}, \mathbf{u}, \mathbf{v}, \ldots,$ for vectors, capital letters $A, B, C, \ldots,$ for matrices and calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots,$ for tensors. Denote $[n]$ by $\{1, 2, \ldots, n\}$. Denote $\langle n\rangle$ by $\{n_1, n_2, \ldots, n_m\}$.

The set $T_{m,n}$ consists of all order m dimension n tensors and each element of $\mathcal{A} \in T_{m,n}$ is real, that is, $\mathcal{A}_{i_1 i_2 \ldots i_m} \in \mathbb{R}$ where $i_k \in [n]$ with $k \in [m]$, and the set $T_{m,\langle n\rangle}$ consists of all order m tensors of size $n_1 \times n_2 \times \cdots \times n_m$ and each element of $\mathcal{A} \in T_{m,\langle n\rangle}$ is real, that is, $\mathcal{A}_{i_1 i_2 \ldots i_m} \in \mathbb{R}$ where $i_k \in [n_k]$ with $k \in [m]$. For a vector $\mathbf{x} \in \mathbb{C}^n$, $\|\mathbf{x}\|^2 = \mathbf{x}^*\mathbf{x}$ where '$*$' represents conjugate transposition, $|\mathbf{x}|_m^m$ means $x_1^m + x_2^m + \cdots + x_n^m$. In particular, when $\mathbf{x} \in \mathbb{R}^n$, the vector m-norm for $|\mathbf{x}|_m^m$ is $\|\mathbf{x}\|_m^m = |x_1|^m + |x_2|^m + \cdots + |x_n|^m$. $\mathbf{0}$ means the zero vector in $\mathbb{C}^n$.

$\mathcal{A} \in T_{m,n}$ is *nonnegative* (see [4]), if all elements are nonnegative, and we denote nonnegative tensors by $NT_{m,n}$. $\mathcal{D} \in T_{m,n}$ is *diagonal* (see [1]), if all off-diagonal entries are zero. In particular, if the diagonal entries of $\mathcal{D}$ are 1, then $\mathcal{D}$ is called the *identity tensor* (see [1]) and denote it by $\mathcal{I}$.

The rest of our paper is organized as follows. Section 2 introduces some definitions, such as mode-k determinant, the polynomial tensor eigenvalue problem, from matrices to tensors, derives a method for obtaining mode-symmetric embedding from $\mathcal{A} \in T_{m,\langle n\rangle}$, and covers a classical result about the perturbation of a simple eigenvalue of $A \in \mathbb{C}^{n\times n}$. In Section 3, for a mode-symmetric tensor, we derive some perturbation bounds of an algebraic simple eigenvalue and Z-eigenvalue, based on symmetric or mode-symmetric embedding, we explore the perturbation bounds of an algebraic simple singular value of $\mathcal{A} \in T_{m,\langle n\rangle}$. In Section 4, for a given λ-tensor, we derive the first-order perturbation of an algebraic simple polynomial eigenvalue and obtain the coefficient of the first-order perturbation term. In particular, we consider the tensor generalized eigenvalue problem, and present a new perturbation bound of an algebraic simple eigenvalue. We show ill-condition tensors for computing Z-eigenvalue or singular value via random numerical examples. We conclude our paper in Section 6.

2. Preliminaries

In this section, we present several definitions generalized from matrices to tensors, we state some remarks and properties associated with these definitions. We shall recall a lemma about the perturbation of a simple eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$.

2.1. *Definitions*

The mode-k product (see [12,13]) of a tensor $\mathcal{A} \in T_{m,n}$ by a matrix $B \in \mathbb{R}^{n\times n}$, denoted by $\mathcal{A} \times_k B$ is a tensor $\mathcal{C} \in T_{m,n}$,

$$\mathcal{C}_{i_1 \ldots i_{k-1} j i_{k+1} \ldots i_m} = \sum_{i_k=1}^{n} \mathcal{A}_{i_1 \ldots i_{k-1} i_k i_{k+1} \ldots i_m} b_{j i_k}, \quad k \in [m].$$

In particular, the mode-k multiplication of a tensor $\mathcal{A} \in T_{m,n}$ by a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted by $\mathcal{A}\bar{\times}_k\mathbf{x}$. When we set $\mathcal{C} = \mathcal{A}\bar{\times}_k\mathbf{x}$, by elementwise, we have

$$\mathcal{C}_{i_1\ldots i_{k-1}i_{k+1}\ldots i_m} = \sum_{i_k=1}^{n} \mathcal{A}_{i_1\ldots i_{k-1}i_k i_{k+1}\ldots i_m} x_{i_k}.$$

Let m vectors $\mathbf{x}_k \in \mathbb{R}^n$, $\mathcal{A}\bar{\times}_1\mathbf{x}_1 \ldots \bar{\times}_m\mathbf{x}_m$ is easy to define. If these m vectors are also the same vectors, denoted by $\mathbf{x}$, then $\mathcal{A}\bar{\times}_1\mathbf{x} \ldots \bar{\times}_m\mathbf{x}$ can be simplified as $\mathcal{A}\mathbf{x}^m$. The mode-k product of a tensor $\mathcal{A} \in T_{m,\langle n\rangle}$ by a matrix $B \in \mathbb{R}^{p\times n_k}$ is easy to define.

The Frobenius norm of a tensor $\mathcal{A} \in T_{m,\langle n\rangle}$ (see [13,14]) is the square root of the sum of the squares of all its elements, i.e.

$$\|\mathcal{A}\|_F = \sqrt{\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} \mathcal{A}^2_{i_1 i_2\ldots i_m}},$$

which is a generalization of the well-known Frobenius-norm of a matrix $A \in \mathbb{C}^{m\times n}$.

Analogous to the reducible matrices (see [15, Chapter 2]), $\mathcal{A} \in T_{m,n}$ is called *reducible* (see [4]), if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$\mathcal{A}_{i_1 i_2\ldots i_m} = 0, \text{ for all } i_1 \in I \text{ and } i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ *irreducible*.

$\mathcal{A}$ is called symmetric (see [1,2]) if $\mathcal{A}_{i_1 i_2\ldots i_m}$ is invariant by any permutation π, that is, $\mathcal{A}_{i_1 i_2\ldots i_m} = \mathcal{A}_{\pi(i_1,i_2,\ldots,i_m)}$, where all $i_k \in [n]$ with $k \in [m]$. We denote all symmetric tensors by $ST_{m,n}$.

Symmetric tensor is a special case of the following definition of 'mode-symmetric'. We shall provide a more general definition about mode-symmetric of a tensor.

Definition 2.1 *Let $\mathcal{A} \in T_{m,n}$. $\mathcal{A}$ is called mode-symmetric, if its entries satisfy the following formulae*

$$\mathcal{A}_{i_1 i_2\ldots i_m} = \mathcal{A}_{i_2 i_3\ldots i_m i_1} = \cdots = \mathcal{A}_{i_m i_1\ldots i_{m-1}},$$

where $i_k \in [n]$ with $k \in [m]$. We denote all mode-symmetric tensors by $MST_{m,n}$.

The following definition generalizes that of Lim [2], and when $\mathbf{x}^*\mathbf{x}$ is represented by $\mathbf{x}^\top\mathbf{x}$, this definition extends that of Qi [1].

Definition 2.2 *Let $\mathcal{A} \in T_{m,n}$. For any $k \in [m]$, if there exists unit $\mathbf{x}_k \in \mathbb{C}^n$ and $\lambda_k \in \mathbb{C}$ such that*

$$\mathcal{A}\bar{\times}_1\mathbf{x}_k \ldots \bar{\times}_{k-1}\mathbf{x}_k\bar{\times}_{k+1}\mathbf{x}_k \ldots \bar{\times}_m\mathbf{x}_k = \lambda_k\mathbf{x}_k, \tag{2.1}$$

then the pair $(\lambda_k, \mathbf{x}_k)$ is called a mode-k E-eigenpair of $\mathcal{A}$.

If $\mathbf{x}_k \in \mathbb{R}^n$ and $\lambda_k \in \mathbb{R}$, then the pair $(\lambda_k, \mathbf{x}_k)$ is called a mode-k Z-eigenpair of $\mathcal{A}$. Moreover, the mode-k E-spectrum and Z-spectrum of $\mathcal{A}$ are defined as

$$E_k(\mathcal{A}) = \{\lambda | \lambda \textit{ is a mode-k E-eigenvalue of } \mathcal{A}\},$$
$$Z_k(\mathcal{A}) = \{\lambda | \lambda \textit{ is a mode-k Z-eigenvalue of } \mathcal{A}\}.$$

Given $\mathcal{A} \in T_{m,n}$, a mode-k eigenpair, which is a little bit different from Lim [2], is defined as follows.

Definition 2.3 *Let $\mathcal{A} \in T_{m,n}$. For any $k \in [m]$, if there exists nonzero $\mathbf{x}_k \in \mathbb{C}^n$ and $\lambda_k \in \mathbb{C}$ such that*

$$\mathcal{A}\bar{\times}_1\mathbf{x}_k \dots \bar{\times}_{k-1}\mathbf{x}_k\bar{\times}_{k+1}\mathbf{x}_k \dots \bar{\times}_m\mathbf{x}_k = \lambda_k\mathbf{x}_k^{[m-1]}, \tag{2.2}$$

where $\mathbf{x}^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1})^\top$, then the pair $(\lambda_k, \mathbf{x}_k)$ is called a mode-k eigenpair of $\mathcal{A}$.

If $\mathbf{x}_k \in \mathbb{R}^n$ and $\lambda_k \in \mathbb{R}$, then the pair $(\lambda_k, \mathbf{x}_k)$ is called a mode-k H-eigenpair of $\mathcal{A}$. Moreover, the mode-k spectrum $\sigma_k(\mathcal{A})$ of $\mathcal{A}$ is defined as

$$\sigma_k(\mathcal{A}) = \{\lambda | \lambda \textit{ is a mode-k eigenvalue of } \mathcal{A}\}.$$

The mode-k spectral radius $\rho_k(\mathcal{A})$ is $\max\{|\lambda| \mid \lambda \in \sigma_k(\mathcal{A})\}$. While the spectral radius $\rho(\mathcal{A})$ of a tensor $\mathcal{A}$ is denoted by $\rho(\mathcal{A}) = \max_{1\le k\le m} \rho_k(\mathcal{A})$.

Mode-k eigenvectors are generalized by left and right eigenvectors of a matrix $A \in \mathbb{R}^{n\times n}$. For $k \in [m]$, some properties of the mode-k eigenpairs of $\mathcal{A}$ are presented in the following:

(a) Given a tensor $\mathcal{A} \in T_{m,n}$ and a vector $\mathbf{x} \in \mathbb{C}^n$, the following equalities

$$\mathcal{A}\bar{\times}_2\mathbf{x}\bar{\times}_3\mathbf{x}\dots\bar{\times}_m\mathbf{x} = \mathcal{A}\bar{\times}_1\mathbf{x}\bar{\times}_3\mathbf{x}\dots\bar{\times}_m\mathbf{x} = \cdots = \mathcal{A}\bar{\times}_1\mathbf{x}\bar{\times}_2\mathbf{x}\dots\bar{\times}_{m-1}\mathbf{x}$$

do not hold. However, when $\mathcal{A}$ is symmetric or mode-symmetric, above equalities hold.

(b) Generally, $\sigma_k(\mathcal{A})$ ($k \in [m]$) are different sets. Furthermore, $\rho_k(\mathcal{A}) \neq \rho_l(\mathcal{A})$, where $k \neq l \in [m]$.

(c) Suppose that $\mathcal{A}$ is symmetric (mode-symmetric), if $(\lambda, \mathbf{x})$ is a mode-k eigenpair of $\mathcal{A}$, then, $(\lambda, \mathbf{x})$ is also other mode-l eigenpairs of $\mathcal{A}$, where $k \neq l \in [m]$. Furthermore, $\sigma_k(\mathcal{A})$ ($k \in [m]$) are the same sets, denoted by $\sigma(\mathcal{A})$.

Now, we state the reason why (a) exists for symmetric or mode-symmetric tensor. Without loss of generality, let $m = 4$. We have

$$\begin{aligned}
(\mathcal{A}\bar{\times}_2\mathbf{x}\bar{\times}_3\mathbf{x}\bar{\times}_4\mathbf{x})_i &= \sum_{jkl=1}^{n} \mathcal{A}_{ijkl}x_jx_kx_l,\\
(\mathcal{A}\bar{\times}_1\mathbf{x}\bar{\times}_3\mathbf{x}\bar{\times}_4\mathbf{x})_j &= \sum_{ikl=1}^{n} \mathcal{A}_{ijkl}x_ix_kx_l = \sum_{kli=1}^{n} \mathcal{A}_{jkli}x_kx_lx_i,\\
(\mathcal{A}\bar{\times}_1\mathbf{x}\bar{\times}_2\mathbf{x}\bar{\times}_4\mathbf{x})_k &= \sum_{ijl=1}^{n} \mathcal{A}_{ijkl}x_ix_kx_l = \sum_{lij=1}^{n} \mathcal{A}_{klij}x_kx_ix_j,\\
(\mathcal{A}\bar{\times}_1\mathbf{x}\bar{\times}_2\mathbf{x}\bar{\times}_3\mathbf{x})_l &= \sum_{ijk=1}^{n} \mathcal{A}_{ijkl}x_ix_jx_k = \sum_{ijk=1}^{n} \mathcal{A}_{lijk}x_ix_jx_k.
\end{aligned}$$

According to Definitions 2.1 and 2.3, we have

$$\mathcal{A}_{ijkl} = \mathcal{A}_{jkli} = \mathcal{A}_{klij} = \mathcal{A}_{lijk}, \quad i, j, k, l \in [n].$$

Then, (a) holds for symmetric or mode-symmetric tensors.

For a given tensor $\mathcal{A} \in T_{m,n}$, if a pair $(\lambda, \mathbf{x})$ is the mode-1 eigenpair of $\mathcal{A}$, then (2.2) in Definition 2.3 can be simplified as $\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$, with $\mathcal{A}\mathbf{x}^{m-1} := \mathcal{A}\bar{\times}_2\mathbf{x}\bar{\times}_3\mathbf{x}\ldots\bar{\times}_m\mathbf{x}$. For a symmetric tensor, Qi [1] derived some properties of a mode-1 eigenvalue. And for a generic tensor, Chang et al. [16] described mode-1 spectrum and the mode-1 spectral radius.

Qi [1] defined the symmetric hyper-determinant of a super-symmetric tensor $\mathcal{A}$. The following definition generalizes from Hu et al. [17, Definition 1.2] and we name it as mode-k determinant of $\mathcal{A}$ with $k \in [m]$, where $\mathcal{A} \in T_{m,n}$.

Definition 2.4 *Suppose that $\mathcal{A} \in T_{m,n}$. For $k \in [m]$, mode-k determinant of $\mathcal{A}$, denoted by* $\mathrm{Det}_k(\mathcal{A})$*, is defined as the resultant of polynomial system*

$$\mathcal{A}\bar{\times}_1\mathbf{x}\ldots\bar{\times}_{k-1}\mathbf{x}\bar{\times}_{k+1}\mathbf{x}\ldots\bar{\times}_m\mathbf{x} = \mathbf{0}.$$

When $\mathrm{Det}_k(\mathcal{A}) \neq 0$*, then $\mathcal{A}$ is called mode-k nonsingular.*

When $k = 1$, Hu et al. [17, Corollary 6.5] derived that

$$\mathrm{Det}_1(\mathcal{A}) = \prod_{\lambda_i \in \sigma_1(\mathcal{A})} \lambda_i.$$

According to Definitions 2.3 and 2.4, we can derive a more general result:

$$\mathrm{Det}_k(\mathcal{A}) = \prod_{\lambda_i \in \sigma_k(\mathcal{A})} \lambda_i.$$

For the set $T_{m,n}$, Chang et al. [4] considered the tensor generalized eigenvalue problem, and a more general case than the tensor generalized eigenvalue problem is the tensor polynomial eigenvalue problem. For given tensors $\mathcal{A}_0, \ldots, \mathcal{A}_l \in T_{m,n}$ and we define the λ-tensor $P_l(\lambda)$ as

$$P_l(\lambda) = \lambda^l\mathcal{A}_l + \lambda^{l-1}\mathcal{A}_{l-1} + \cdots + \lambda\mathcal{A}_1 + \mathcal{A}_0.$$

Hence, we state the definition of the polynomial tensor eigenvalue problem.

Definition 2.5 *For any $k \in [m]$, if there exists nonzero vector $\mathbf{x}_k \in \mathbb{C}^n$ and $\lambda_k \in \mathbb{C}$ such that*

$$P_l(\lambda_k)\bar{\times}_1\mathbf{x}_k\ldots\bar{\times}_{k-1}\mathbf{x}_k\bar{\times}_{k+1}\mathbf{x}_k\ldots\bar{\times}_m\mathbf{x}_k = \mathbf{0},$$

then the pair $(\lambda_k, \mathbf{x}_k)$ is called a mode-k polynomial eigenpair of $P_l(\lambda)$.

If $\mathbf{x}_k \in \mathbb{R}^n$ and $\lambda_k \in \mathbb{R}$, then the pair $(\lambda_k, \mathbf{x}_k)$ is called a mode-k polynomial H-eigenpair of $P_l(\mathcal{A})$. Meanwhile, we denote the set of $P_l(\lambda)$'s mode-k polynomial eigenvalue by

$$\begin{aligned}\Lambda_k(P_l(\lambda)) &= \{\lambda \mid \mathrm{Det}_k(P_l(\lambda)) = 0\}\\ &= \{\lambda \mid \lambda \textit{ is a mode-k polynomial eigenvalue of } P_l(\lambda)\}.\end{aligned}$$

Ding et al. [18] defined a regular tensor pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}, \mathcal{B} \in T_{m,n}$. In general, according to Definition 2.4 and λ-tensor $P_l(\lambda)$, mode-k regular about a tensor $(l+1)$-tuple is defined as follows.

Definition 2.6 *For a given λ-tensor $P_l(\lambda)$, a tensor $(l+1)$-tuple $(\mathcal{A}_l, \ldots, \mathcal{A}_1, \mathcal{A}_0)$ is mode-k singular if for all λ, $\mathrm{Det}_k(P_l(\lambda)) \equiv 0$ holds, where all tensors in this tuple belong to $T_{m,n}$, or if all tensors in this tuple belong to $T_{m,\langle n\rangle}$, where these exist two different indices $k, l \in [m]$ such that $n_k \neq n_l$. Otherwise the $(l+1)$-tuple $(\mathcal{A}_l, \ldots, \mathcal{A}_1, \mathcal{A}_0)$ is said to be mode-k regular, where all tensors in the tuple belong to $T_{m,n}$.*

In this paper, we only consider the tensor polynomial eigenvalue problem where associated tensor $(l+1)$-tuple $(\mathcal{A}_l, \ldots, \mathcal{A}_1, \mathcal{A}_0)$ is regular. Meanwhile, for a given mode-k regular tensor tuple, according to Definition 2.6, we know that there exists a $\widehat{\lambda}$ such that $\mathrm{Det}_k(P_l(\widehat{\lambda})) \neq 0$. Then, we can choice another $(l+1)$-tuple $(\widetilde{\mathcal{A}}_l, \ldots, \widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_0)$ such that $\widetilde{\mathcal{A}}_l = \sum_{i=0}^{l} \widehat{\lambda}^l \mathcal{A}_l$ and there is a one-to-one map between $\Lambda_k(P_l(\lambda))$ and $\Lambda_k(\widetilde{P}_l(\lambda))$, where $\mathrm{Det}_k(\widetilde{\mathcal{A}}_l) \neq 0$ and $\widetilde{P}_l(\lambda) = \lambda^l \widetilde{\mathcal{A}}_l + \lambda^{l-1}\widetilde{\mathcal{A}}_{l-1} + \cdots + \lambda\widetilde{\mathcal{A}}_1 + \widetilde{\mathcal{A}}_0$.

Furthermore, we can also suppose that $\mathrm{Det}_k(\mathcal{A}_l) \neq 0$, that is, $\mathcal{A}_l$ is nonsingular. For a given $P_l(\lambda)$, when $\mathcal{A}_l$ is nonsingular, we will show that, for all $k \in [m]$, $\Lambda_k(P_l(\lambda))$ are finite subsets of $\mathbb{C}$.

Meanwhile, for a given tensor $(l+1)$-tuple $(\mathcal{A}_l, \ldots, \mathcal{A}_1, \mathcal{A}_0)$, we can also define a (α, β)-tensor. Let $P_l(\alpha, \beta) = \alpha^l \mathcal{A}_l + \alpha^{l-1}\beta\mathcal{A}_{l-1} + \cdots + \alpha\beta^{l-1}\mathcal{A}_1 + \beta^l \mathcal{A}_0$. It is obvious that $P_l(\alpha, \beta)$ is a homogeneous polynomial on α and β. The relationship between $P_l(\lambda)$ and $P_l(\alpha, \beta)$ is listed below. If $\beta \neq 0$, then $P_l(\alpha, \beta) = \beta^l P_l(\alpha/\beta)$; and if $\alpha \neq 0$, then $P_l(\alpha, \beta) = \alpha^l \widetilde{P}_l(\beta/\alpha)$, where $\widetilde{P}_l(\beta/\alpha) = \mathcal{A}_l + t\mathcal{A}_{l-1} + \cdots + t^{l-1}\mathcal{A}_1 + t^l \mathcal{A}_0$ and $t = \beta/\alpha$.

For a given λ-tensor $P_l(\lambda)$, when the pair $(\lambda_k, \mathbf{x}_k)$ is a mode-k polynomial eigenpair of $P_l(\lambda)$, then, we can choose a pair (α_k, β_k) such that

$$P_l(\alpha_k, \beta_k)\bar{\times}_1\mathbf{x}_k \ldots \bar{\times}_{k-1}\mathbf{x}_k\bar{\times}_{k+1}\mathbf{x}_k \ldots \bar{\times}_m\mathbf{x}_k = \mathbf{0}$$

and

$$\lambda_k = \begin{cases} \alpha_k/\beta_k, & \beta_k \neq 0, \\ \infty, & \beta_k = 0, \end{cases}$$

with $(\alpha_k, \beta_k) \neq (0, 0)$.

For a given tensor $\mathcal{A} \in T_{m,\langle n\rangle}$, let $\mathbf{x}_k \in \mathbb{R}^{n_k}$ be nonzero vectors and $\|\mathbf{x}_k\| = 1$ with $k \in [m]$. If $(\sigma, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is a solution of this following nonlinear equations

$$\begin{cases} \mathcal{A}\bar{\times}_2\mathbf{x}_2\bar{\times}_3\mathbf{x}_3 \ldots \bar{\times}_m\mathbf{x}_m = \sigma\mathbf{x}_1, \\ \mathcal{A}\bar{\times}_1\mathbf{x}_1\bar{\times}_3\mathbf{x}_3 \ldots \bar{\times}_m\mathbf{x}_m = \sigma\mathbf{x}_2, \\ \qquad\vdots \\ \mathcal{A}\bar{\times}_1\mathbf{x}_1\bar{\times}_2\mathbf{x}_2 \ldots \bar{\times}_{m-1}\mathbf{x}_{m-1} = \sigma\mathbf{x}_m, \end{cases} \tag{2.3}$$

then, the unit vector $\mathbf{x}_k$ and σ are called the mode-k singular vector, $k \in [m]$, and singular value of $\mathcal{A}$, respectively (see [2]).

Meanwhile, for a given tensor $\mathcal{A} \in T_{m,\langle n\rangle}$, its singular value and associated mode-k singular vectors can be generalized the in following form.

Definition 2.7 *Let $B_k \in \mathbb{R}^{n_k \times n_k}$ be positive definite matrices and $\mathbf{x}_k \in \mathbb{R}^{n_k}$ be nonzero vectors where $\mathbf{x}_k^\top B_k\mathbf{x}_{k=1}$ with $k \in [m]$. If $(\sigma, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is a solution of this following* *nonlinear equations*

$$\begin{cases} \mathcal{A}\bar{\times}_2\mathbf{x}_2\bar{\times}_3\mathbf{x}_3\ldots\bar{\times}_m\mathbf{x}_m = \sigma B_1\mathbf{x}_1, \\ \mathcal{A}\bar{\times}_1\mathbf{x}_1\bar{\times}_3\mathbf{x}_3\ldots\bar{\times}_m\mathbf{x}_m = \sigma B_2\mathbf{x}_2, \\ \qquad\vdots \\ \mathcal{A}\bar{\times}_1\mathbf{x}_1\bar{\times}_2\mathbf{x}_2\ldots\bar{\times}_{m-1}\mathbf{x}_{m-1} = \sigma B_m\mathbf{x}_m, \end{cases} \tag{2.4}$$

then, $\mathbf{x}_k$ *and* σ *are called the restricted mode-*k *singular vector,* $k \in [m]$*, and restricted singular value of* $\mathcal{A}$*, respectively.*

When all matrices B_k are the identity matrix, Formulae (2.4) reduces to Formulae (2.3).

2.2. *Some remarks*

As we know, many scholars extended definitions from matrices to tensors. However, for these definitions, there are some differences between matrices and tensors given in following remark.

Remark 2.1 For all mode-k E-spectrum of $\mathcal{A}$ and all mode-k tensor polynomial eigenvalue of $P_l(\lambda)$, some statements are given.

(1) When $\mathcal{A} \in T_{m,n}$, $E_k(\mathcal{A})$ ($k \in [m]$) are the different sets. However, either $\mathcal{A}$ is symmetric or mode-symmetric, $E_k(\mathcal{A})$ ($k \in [m]$) are the same sets, denoted by $E(\mathcal{A})$. Similar to the case of $Z_k(\mathcal{A})$.
(2) When $\mathcal{A} \in T_{m,n}$, $\mathrm{Det}_k(\mathcal{A}) \neq \mathrm{Det}_l(\mathcal{A})$ with $k \neq l \in [m]$. Either $\mathcal{A}$ is symmetric or mode-symmetric, for all $k \in [m]$, $\mathrm{Det}_k(\mathcal{A})$ are the same number, denoted by $\mathrm{Det}(\mathcal{A})$.
(3) If all tensors in $P_l(\lambda)$ are symmetric or mode-symmetric, $\Lambda_k(P_l(\lambda))$ ($k \in [m]$) are the same sets, denoted by $\Lambda(P_l(\lambda))$. If there exists a tensor in $P_l(\lambda)$ is not symmetric or mode-symmetric, then the result does not hold.
(4) Hereinafter, when we refer to an Z- (or E- or polynomial) eigenpair $(\lambda, \mathbf{x})$, it means that $(\lambda, \mathbf{x})$ is a mode-1 Z- (or E- or polynomial) eigenpair.

We know that the matrix eigenvalue problem and the generalized matrix eigenvalue problem are two special cases of the polynomial matrix eigenvalue problem (see [3]). For the polynomial tensor eigenvalue problem, similar statements are in following.

Remark 2.2 Let $P_l(\lambda) = \lambda^l\mathcal{A}_l + \lambda^{l-1}\mathcal{A}_{l-1} + \cdots + \lambda\mathcal{A}_1 + \mathcal{A}_0$ with $\mathcal{A}_i \in T_{m,n}$ $(i = 0 : l-1)$. The following three special cases of Definition 2.5 should be emphasized.

(a) When $l = 1$, $\mathcal{A}_1 = \mathcal{I}$ and $\mathcal{A}_0$ is symmetric, Definition 2.5 is a generalization of Lim [2] and Qi [1] derived some properties of a mode-1 eigenpair, i.e. the tensor eigenvalue problem.
(b) When $l = 1$ and $\mathcal{A}_1$ is not the identity tensor, Chang et al. [4] considered Definition 2.5, and further developed by Zhang [19] with $k = 1$. We call a pair $(\lambda_k, \mathbf{x}_k)$, satisfying Definition 2.5, is a mode-k generalized eigenpair of a 2-tuple $(\mathcal{A}_1, \mathcal{A}_0)$, i.e. the tensor generalized eigenvalue problem.

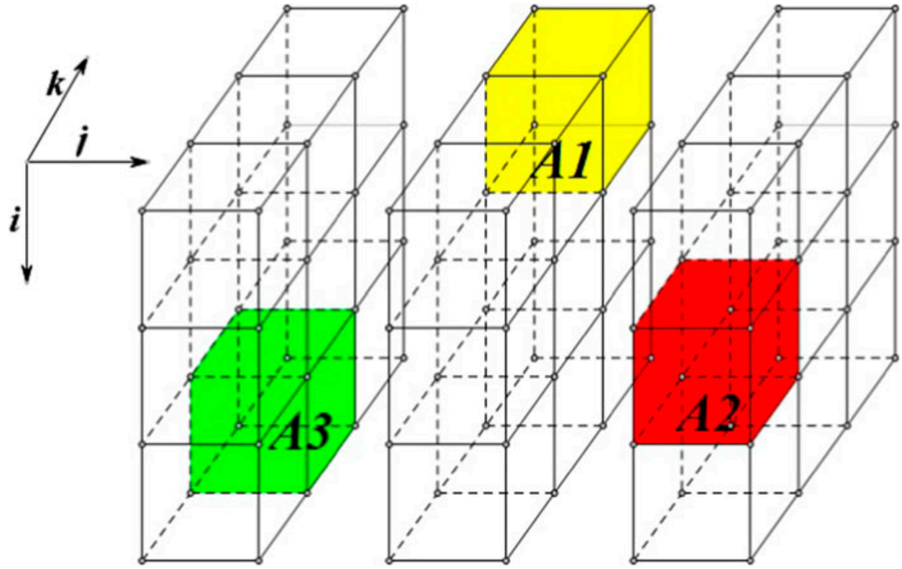

Figure 1. Intuitive performance of $\mathbf{msym}(\mathcal{A})$: A1 for the first part, the second part represented by A2; and A3 for the third part. Not labelled part means all zero elements of $\mathbf{msym}(\mathcal{A})$.

(c) When $l = 2$, We call a pair (λ_k, x_k), satisfying Definition 2.5, is a mode-k quadratic eigenpair of a 3-tuple $(\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0)$, i.e. the tensor quadratic eigenvalue problem.

Some properties of the mode-k tensor matrix product are given as follows.

Lemma 2.1 ([14, Property 2 and 3], [13]) *Given a tensor $\mathcal{A} \in T_{m,n}$ and the matrices $F \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$. For different integers k and l, one has*

$$(\mathcal{A} \times_k F) \times_l G = (\mathcal{A} \times_l G) \times_k F = \mathcal{A} \times_k F \times_l G, \ (\mathcal{A} \times_k F) \times_k G = \mathcal{A} \times_k (G \cdot F),$$

where '$\cdot$' means the multiplication of two matrices.

Suppose that $A \in \mathbb{C}^{n\times n}$. Wilkinson [20], Demmel [21] and Stewart and Sun [22] concentrated on computing the condition number of a *simple* eigenvalue, respectively.

Lemma 2.2 ([21, Theorem 4.4], [22, Theorem 2.3]) *Let λ be a simple eigenvalue of A with right eigenvector $\mathbf{x}$ and left eigenvector $\mathbf{y}$, normalized so that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. Let $\lambda + \delta\lambda$ be the corresponding eigenvalue of $A + \delta A$. Then*

$$\delta\lambda = \frac{\mathbf{y}^*\delta A\mathbf{x}}{\mathbf{y}^*\mathbf{x}} + O(\|\delta A\|_2^2), \ \text{or}$$
$$|\delta\lambda| \le \frac{\|\delta A\|_2}{\mathbf{y}^*\mathbf{x}} + O(\|\delta A\|_2^2) = \sec\Theta(\mathbf{y}, \mathbf{x})\|\delta A\|_2 + O(\|\delta A\|_2^2),$$

where $\Theta(\mathbf{y},\mathbf{x})$ *is the acute angle between* $\mathbf{y}$ *and* $\mathbf{x}$ *and* $\|A\|_2$ *is the largest singular value of the matrix* A*. In other words,* $\sec\Theta(\mathbf{y},\mathbf{x}) = 1/|\mathbf{y}^*\mathbf{x}|$ *is the condition number of the eigenvalue* λ*.*

2.3. *Symmetric and mode-symmetric embeddings*

Assume that $A \in \mathbb{R}^{m\times n}$, well known relationship exists between the singular value decomposition of A and the Schur decomposition of its symmetric embedding $\mathbf{sym}(A) = ([\mathbf{0}\ A;\ A^\top\ \mathbf{0}])$ (see [3, Chapter 8.6]). For a general tensor $\mathcal{A} \in T_{m,\langle n\rangle}$, Ragnarsson et al. [23] derived a method for obtaining a symmetric embedding $\mathbf{sym}(\mathcal{A})$ from $\mathcal{A}$, where $\mathbf{sym}(\mathcal{A}) \in ST_{m,\hat{n}}$ with $\hat{n} = n_1 + n_2 + \cdots + n_m$.

In the rest of this subsection, we consider how to obtain a mode-symmetric embedding $\mathbf{msym}(\mathcal{A})$ from $\mathcal{A}$, where $\mathbf{msym}(\mathcal{A}) \in MST_{m,\hat{n}}$ with $\hat{n} = n_1 + n_2 + \cdots + n_m$.

Let $K_t = n_1 + n_2 + \cdots + n_t$, $t \in [m]$, and $\mathbf{msym}(\mathcal{A}) \in T_{m,K_m}$. Then all entries of $\mathbf{msym}(\mathcal{A})$ satisfy

$$\mathbf{msym}(\mathcal{A})_{i_1 i_2 \dots i_m} = \begin{cases} \mathcal{A}_{i_1 i_2 \dots i_m} & 1 \le i_1 \le K_1,\ K_1+1 \le i_2 \le K_2,\ K_2+1 \le i_3 \le K_3 \\ & K_3+1 \le i_4 \le K_4, \dots, K_{m-1}+1 \le i_m \le K_m, \\ \mathcal{A}_{i_2 \dots i_m i_1} & K_1+1 \le i_2 \le K_2,\ K_2+1 \le i_3 \le K_3,\ K_3+1 \le i_4 \le K_4, \\ & \dots, K_{m-1}+1 \le i_m \le K_m,\ 1 \le i_1 \le K_1, \\ \dots & \dots \\ \mathcal{A}_{i_m i_1 \dots i_{m-1}} & K_{m-1}+1 \le i_m \le K_m,\ 1 \le i_1 \le K_1,\ K_1+1 \le i_2 \le K_2 \\ & K_2+1 \le i_3 \le K_3, \dots, K_{m-2}+1 \le i_{m-1} \le K_{m-1}, \\ 0 & \text{otherwise.} \end{cases}$$

According to Definition 2.1, it is obvious that $\mathbf{msym}(\mathcal{A})$ is mode-symmetric. When $m = 3$, intuitive performance of $\mathbf{msym}(\mathcal{A})$ is given in Figure 1.

Suppose that $\mathbf{x}_k \in \mathbb{R}^{n_k}$ with $k \in [m]$. Let $\mathbf{x} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top, \dots, \mathbf{x}_m^\top)^\top$, then, we have

$$\mathbf{sym}(\mathcal{A})\mathbf{x}^m \ne \mathbf{msym}(\mathcal{A})\mathbf{x}^m = m\mathcal{A}\bar{\times}_1\mathbf{x}_1\bar{\times}_2\mathbf{x}_2\dots\bar{\times}_m\mathbf{x}_m.$$

Meanwhile, according to $\mathcal{A}$ and $\mathbf{x}_k$ with $k \in [m]$, it is easy to derive all entries of $\mathbf{msym}(\mathcal{A})\mathbf{x}^{m-1}$. Here, we do not list them out. The reader can also find these expressions of $\mathbf{sym}(\mathcal{A})\mathbf{x}^m$ and $\mathbf{sym}(\mathcal{A})\mathbf{x}^{m-1}$ in [23,24].

3. Perturbation bounds of Z-eigenvalue and singular value

In this section, we consider the properties of eigenvalue and Z-eigenvalue of a mode-symmetric tensor. We also investigate perturbation bounds of an algebraic simple eigenvalue and Z-eigenvalue of a mode-symmetric tensor. Finally, given a tensor $\mathcal{A} \in T_{m,\langle n\rangle}$, based on symmetric or mode-symmetric embedding from $\mathcal{A}$, perturbation bounds of an algebraic simple singular value are obtained.

3.1. *Properties of eigenvalue and Z-eigenvalue*

In this subsection, we assume that $\mathcal{A} \in ST_{m,n}$. Qi [1] derived some properties of eigenvalue and Z-eigenvalue of a symmetric tensor. Those results also hold with a mode-symmetric tensor. In order to prove Theorems 3.2 and 3.4, we need the following theorem.

THEOREM 3.1 *Assume that $\mathcal{A}$ is a mode-symmetric tensor and m is even. The following conclusions holds for $\mathcal{A}$:*

(a) *$\mathcal{A}$ always has H-eigenvalue. $\mathcal{A}$ is positive definite (positive semidefinite) if and only if all of its H-eigenvalue are positive (nonnegative).*
(b) *$\mathcal{A}$ always has Z-eigenvalue. $\mathcal{A}$ is positive definite (positive semidefinite) if and only if all of its Z-eigenvalue are positive (nonnegative).*

Proof Firstly, we prove part (a). We see (2.2) is the optimality condition of

$$\max\left\{\mathcal{A}\mathbf{x}^m : \sum_{i=1}^n x_i^m = 1, \mathbf{x} \in \mathbb{R}^n\right\} \tag{3.1}$$

and

$$\min\left\{\mathcal{A}\mathbf{x}^m : \sum_{i=1}^n x_i^m = 1, \mathbf{x} \in \mathbb{R}^n\right\}. \tag{3.2}$$

As the feasible set is compact and the objective function is continuous, the global maximizer and minimizer always exist. This shows that (2.2) has real solutions, i.e. $\mathcal{A}$ always has H-eigenvalue. Since $\mathcal{A}$ is positive definite (positive semidefinite) if and only if the optimal value of (3.1) is positive (nonnegative), we draw the second conclusion of (a).

Now, we will prove part (b). The proof of (b) is similar to the proof of (a), as long as we replace by

$$\max\left\{\mathcal{A}\mathbf{x}^m : \sum_{i=1}^n x_i^m = 1, \mathbf{x} \in \mathbb{R}^n\right\}$$

and

$$\min\left\{\mathcal{A}\mathbf{x}^m : \sum_{i=1}^n x_i^m = 1, \mathbf{x} \in \mathbb{R}^n\right\}.$$

□

3.2. *Algebraic simple Z-eigenvalue*

Suppose that $\mathcal{A} \in MST_{m,n}$. Since m is even, then there exists a positive integer h such that $m = 2h$. Denote by $\mathcal{E}$ the tensor $\underbrace{I \otimes I \otimes \cdots \otimes I}_{h}$. $\mathcal{A}$ always has Z-eigenvalue (see [1, Theorem 5]), if $\widetilde{\lambda}$ is an Z-eigenvalue of $\mathcal{A}$, it is known that $\widetilde{\lambda}$ is a root of $\mathrm{Det}(\mathcal{A} - \lambda\mathcal{E}) = 0$ (see [1, Theorem 2]).

It is observed (see [25]) that the complex E-eigenpairs of a tensor form the equivalence class under a multiplicative transformation. That is (see [26]), if $(\lambda, \mathbf{x})$ is an E-eigenpair of $\mathcal{A}$ and $\mathbf{y} = e^{\iota\varphi}\mathbf{x}$ with $\varphi \in \mathbb{R}$, where $\iota = \sqrt{-1}$ then $\mathbf{y}^*\mathbf{y} = \mathbf{x}^*\mathbf{x} = 1$ and

$$\mathcal{A}\mathbf{y}^{m-1} = e^{\iota(m-1)\varphi}\mathcal{A}\mathbf{x}^{m-1} = e^{\iota(m-1)\varphi}\lambda\mathbf{x} = e^{\iota(m-2)\varphi}\lambda\mathbf{y}. \tag{3.3}$$

Therefore, $(e^{\iota(m-2)\varphi}\lambda, e^{\iota\varphi}\mathbf{x})$ is also an E-eigenpair of $\mathcal{A}$ for any $\varphi \in \mathbb{R}$. Then, we can choose φ_* such that $(e^{\iota(m-2)\varphi_*}\lambda, e^{\iota\varphi_*}\mathbf{x})$ is a Z-eigenpair. Hence, without loss of generality, we only consider the perturbation bounds of a Z-eigenvalue of a symmetric or mode-symmetric tensor.

Chang et al. [4] defined the geometric multiplicity of an eigenvalue λ, meanwhile, Hu et al. [17] considered the algebraic multiplicity of an eigenvalue λ. Similarly, we can define the geometric and algebraic multiplicity of an Z-eigenvalue. An algebraic simple Z-eigenvalue is defined as an Z-eigenvalue whose algebraic multiplicity is one. This definition is applicable to a generalized eigenvalue, a polynomial eigenvalue and a singular value. We present the perturbation bound of an algebraic simple Z-eigenvalue λ of $\mathcal{A}+\varepsilon\mathcal{B}$.

Theorem 3.2 *Suppose $\mathcal{A},\mathcal{B}\in MST_{m,n}$ and $\varepsilon\in\mathbb{R}$. Let $(\lambda,\mathbf{x})$ be an algebraic simple Z-eigenpair of $\mathcal{A}$. Then there exists $\varepsilon_0>0$ and an analytic function $\lambda(\varepsilon)$ with $|\varepsilon|\le\varepsilon_0$ such that*

$$\lambda(0)=\lambda,\quad \lambda'(0)=\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}=\mathcal{B}\mathbf{x}^m,$$

with $\mathbf{x}^\top\mathbf{x}=1$. Therefore, $\lambda(\varepsilon)$ is an algebraic simple Z-eigenvalue of $\mathcal{A}+\varepsilon\mathcal{B}$ over $|\varepsilon|\le\varepsilon_0$, and

$$\lambda(\varepsilon)=\lambda+\varepsilon\mathcal{B}\mathbf{x}^m+O(\varepsilon^2).$$

Proof When $\mathcal{A}$ is symmetric, we know that $\mathcal{A}$ always has Z-eigenvalue (see [1, Theorem 5]), that is, $Z(\mathcal{A})$ is the nonempty set. This result also holds when $\mathcal{A}$ is mode-symmetric (see Theorem 3.1).

By the description of Qi [1], the E-characteristic polynomial of $\mathcal{A}+\varepsilon\mathcal{B}$ becomes

$$\varphi_\varepsilon(z)=\mathrm{Det}(z\mathcal{E}-\mathcal{A}-\varepsilon\mathcal{B}).$$

It is obvious that $\varphi_\varepsilon(z)$ is an analytic function associated to ε and z. Define $\mathfrak{D}_r:=\{z\in\mathbb{C}:|z-\lambda|\le r\}$. Let r be arbitrarily small such that $Z(\mathcal{A})\cap\mathfrak{D}_r=\{\lambda\}$. Denote the boundary of $\mathfrak{D}_r$ as $\partial\mathfrak{D}_r$. Then $\min_{z\in\partial\mathfrak{D}_r}|\varphi_0(z)|=\gamma>0$.

Since $\varphi_\varepsilon(z)$ is a continuous function of ε, then there exists $\varepsilon_0>0$, such that for all ε with $|\varepsilon|\le\varepsilon_0$, $\varphi_\varepsilon(z)$ has only one zero point in $\mathfrak{D}_r$ and $\min_{\substack{z\in\partial\mathfrak{D}_r\\|\varepsilon|\le\varepsilon_0}}|\varphi_\varepsilon(z)|>0$.

It follows from the Residue Theorem (see [27]) that the zero point $\lambda(\varepsilon)$ of $\varphi_\varepsilon(z)$ in $\mathfrak{D}_r$ can be represented as $\lambda(\varepsilon)=\frac{1}{2\pi}\oint_{\partial\mathcal{D}_r}\frac{z\varphi_\varepsilon'(z)}{\varphi_\varepsilon(z)}dz$, where $\varphi_\varepsilon'(z)=d\varphi_\varepsilon(z)/dz$.

Noting that $\frac{z\varphi_\varepsilon'(z)}{\varphi_\varepsilon(z)}$ and $\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z\varphi_\varepsilon'(z)}{\varphi_\varepsilon(z)}\right)$ are continuous on $\partial\mathfrak{D}_r$, by the differential and integral order exchange theorem, $\lambda(\varepsilon)$ is an analytic function, if $|\varepsilon|\le\varepsilon_0$. Hence, $\lambda(\varepsilon)$ can be expressed as

$$\lambda(\varepsilon)=\lambda(0)+\lambda'(0)\varepsilon+O(\varepsilon^2),\quad \lambda(0)=\lambda,\ |\varepsilon|\le\varepsilon_0.$$

For an algebraic simple Z-eigenvalue λ of a mode-symmetric tensor $\mathcal{A}$, if there exist two real vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ such that

$$\mathcal{A}\mathbf{x}_i^{m-1}=\lambda\mathbf{x}_i,\quad \mathbf{x}_i^\top\mathbf{x}_i=1,\quad i\in[2],$$

then, it is obvious that if $\mathbf{x}_1=c\mathbf{x}_2$, then c satisfies that $c^2=1$ and $c^{m-2}=1$. In this case, we can see that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the same vectors, otherwise, we see that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the different vectors. Hence, we denote

$$\delta=\min\{\|\mathbf{y}-\mathbf{x}\|:\ \mathbf{y}\text{ and }\mathbf{x}\text{ are the different eigenvectors associated with }\lambda\}.$$

Then, over $\{\mathbf{z}\in\mathbb{C}^n:\|\mathbf{z}-\mathbf{x}\|<\delta\}$, there exists a unique eigenvector $\mathbf{x}$ of $\mathcal{A}$ associated to λ. (For an algebraic simple Z-eigenvalue λ, if its geometric multiplicity is also 1, then set $\delta\le\varepsilon_0$.)

For an algebraic simple Z-eigenvalue $\lambda(\varepsilon)$ of $\mathcal{A}+\varepsilon\mathcal{B}$, since $\lambda(\varepsilon)$ is an analytic function with $|\varepsilon| \leq \varepsilon_0$, then there exists $\tilde{\delta} = \min\{\delta, \varepsilon_0\}$ such that $\|\mathbf{x}(\varepsilon) - \mathbf{x}\| < \tilde{\delta}$ and $\mathbf{x}(\varepsilon)$ is the unique eigenvector of $\mathcal{A} + \varepsilon\mathcal{B}$ corresponding to $\lambda(\varepsilon)$. According to some results about algebraic functions (see [28]), we derive that $\mathbf{x}(\varepsilon)$ is an analytic function, where $|\varepsilon| \leq \tilde{\delta}$ and $\mathbf{x}(0) = \mathbf{x}$.

As $(\mathcal{A}+\varepsilon\mathcal{B})\mathbf{x}(\varepsilon)^{m-1} = \lambda(\varepsilon)\mathbf{x}(\varepsilon)$, by differentiating this equation with respect to ε, and setting $\varepsilon = 0$, we have

$$\mathcal{A}\tilde{\mathbf{x}} + \mathcal{B}\mathbf{x}(0)^{m-1} = \lambda'(0)\mathbf{x}(0) + \lambda(0)\mathbf{x}'(0),$$

where $\mathcal{A}\tilde{\mathbf{x}} = \mathcal{A}(\bar{\times}_2\mathbf{x}'(0)\bar{\times}_3\mathbf{x}(0)\ldots\bar{\times}_m\mathbf{x}(0) + \cdots + \bar{\times}_2\mathbf{x}(0)\ldots\bar{\times}_{m-1}\mathbf{x}(0)\bar{\times}_m\mathbf{x}'(0))$.

Since $(\lambda, \mathbf{x})$ is a mode-k Z-eigenpair and $\|\mathbf{x}\| = 1$, then

$$\lambda'(0) = \mathcal{B}\mathbf{x}(0)^m = \mathcal{B}\mathbf{x}^m.$$

Hence, this theorem is complete proved. □

Gohberg and Koltracht [29] studied condition numbers of maps in finite-dimensional spaces $F : \mathbb{R}^p \to \mathbb{R}^q$. The condition number of F at a point $\mathbf{a} \in D_F$ characterizes the instantaneous rate of change in $F(\mathbf{a})$ with respect to perturbations in $\mathbf{a}$.

Then, another perturbation result of an nonzero algebraic simple Z-eigenvalue $\lambda = \lambda(\mathcal{A})$ is considered, whose associated eigenvector $\mathbf{x}$ is a real nonzero vector with $\|\mathbf{x}\| = 1$. It is well known that the map $F_\mathcal{E} : \varepsilon \to \lambda(\mathcal{A} + \varepsilon\mathcal{B})$ is analytic in a neighbourhood of 0 (see [17]). Therefore, the map $F : \mathcal{A} \to \lambda(\mathcal{A})$ has continuous partial derivatives with respect to each entry at $\mathcal{A}$ and

$$\frac{\partial F}{\partial_{i_1 i_2 \ldots i_m}}(\mathcal{A}) := \lim_{t\to 0} \frac{F(\mathcal{A}+t\mathcal{J}) - F(\mathcal{A})}{t} = x_{i_1}x_{i_2}\ldots x_{i_m},$$

where $\mathcal{J}$ is the zero tensor except for $\mathcal{J}_{i_1 i_2 \ldots i_m} = 1$. Here, we give an example to illustrate the meaning of $\mathcal{J}$. Without loss of generality, let $m = 4$ and $(i_1, i_2, i_3, i_4) = (1, 2, 3, 4)$, then $\mathcal{J}$ can be written as $\mathcal{J} = \mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_3 \circ \mathbf{e}_4$ where $\mathbf{e}_i$ is the ith column of the $n \times n$ identity matrix with $i \in [4]$. Then, all entries of $\mathcal{J}$ are given in following:

$$\mathcal{J}_{j_1 j_2 j_3 j_4} = \mathbf{e}_1(j_1)\mathbf{e}_2(j_2)\mathbf{e}_3(j_3)\mathbf{e}_4(j_4),$$

where $j_k \in [n]$ and $\mathbf{e}_k(j_k)$ is the j_kth element of $\mathbf{e}_k$ with $k \in [4]$. In general, $\mathcal{J} = \mathbf{e}_{i_1} \circ \mathbf{e}_{i_2} \circ \cdots \circ \mathbf{e}_{i_m}$ where $\mathbf{e}_{i_k}$ is the i_kth column of the $n \times n$ identity matrix with $k \in [m]$.

Hence, as a map from $\mathbb{R}^{n^m} \to \mathbb{R}$, F is differentiable at $\mathcal{A}$ and

$$F'(\mathcal{A}) = \left[\frac{\partial F}{\partial_{11\ldots 1}}, \ldots, \frac{\partial F}{\partial_{11\ldots 1n}}, \frac{\partial F}{\partial_{21\ldots 1}}, \ldots, \frac{\partial F}{\partial_{nn\ldots n}}\right].$$

According to a formula by Gohberg and Koltracht [29], for relatively small componentwise perturbations in $\mathcal{A}$, i.e.

$$|\mathcal{E}_{i_1 i_2 \ldots i_m}| \leq \varepsilon |\mathcal{A}_{i_1 i_2 \ldots i_m}|, \quad i_k \in [n],\ k \in [m],$$

where $\mathcal{E} \in MST_{m,n}$ and $\varepsilon > 0$ is arbitrarily small, the sensitivity of $F(\mathcal{A})$ is characterized by the componentwise condition number of F at $\mathcal{A}$ is

$$c(F, \mathcal{A}) = \frac{\|F'(\mathcal{A})D_{\mathcal{A}}\|_\infty}{\|F(\mathcal{A})\|_\infty},$$

where $D_{\mathcal{A}} = \mathrm{diag}(\mathcal{A}_{11\ldots1}, \ldots, \mathcal{A}_{11\ldots1n}, \mathcal{A}_{21\ldots1}, \ldots, \mathcal{A}_{nn\ldots n})$, and $\lambda \neq 0$.

It indicates that

$$c(F, \mathcal{A}) = \frac{\|(\mathcal{A}_{11\ldots1}x_1x_1\ldots x_1, \ldots, \mathcal{A}_{11\ldots1n}x_1\ldots x_1x_n, \mathcal{A}_{21\ldots1}x_2x_1\ldots x_1, \ldots, \mathcal{A}_{nn\ldots n}x_nx_n\ldots x_n)\|_\infty}{|\lambda|},$$

where the infinity norm of the map $F'(\mathcal{A})D_{\mathcal{A}}$ is, in fact, the 1-norm of the row vector that represents it. Thus

$$c(F, \mathcal{A}) = \frac{\sum_{i_1i_2\ldots i_m=1}^{n} |\mathcal{A}_{i_1i_2\ldots i_m}x_{i_1}x_{i_2}\ldots x_{i_m}|}{|\lambda|} = \frac{|\mathcal{A}||\mathbf{x}|^m}{|\lambda|},$$

where an absolute value of a tensor is the corresponding tensor of the absolute values of its entries. Thus, if $\mathcal{E}$ is a perturbation of $\mathcal{A}$, then we have a relative perturbation bound of a nonzero Z-eigenvalue.

THEOREM 3.3 *Suppose that $\mathcal{E} \in MST_{m,n}$ such that $|\mathcal{E}_{i_1i_2\ldots i_m}| \leq \varepsilon|\mathcal{A}_{i_1i_2\ldots i_m}|$, where $i_k \in [n]$ with $k \in [m]$. Then, for an algebraic simple Z-eigenvalue $\lambda \neq 0$, there exists an Z-eigenvalue $\widehat{\lambda}$ of $\mathcal{A} + \mathcal{E}$ such that*

$$\frac{|\widehat{\lambda} - \lambda|}{|\lambda|} \leq c(F, \mathcal{A})\varepsilon + O(\varepsilon^2).$$

For an irreducible and mode-symmetric nonnegative tensor, the perturbation bound of the Z-spectral radius is derived from Theorem 3.3.

COROLLARY 3.1 *Let $\mathcal{A} \in MST_{m,n}$. If $\mathcal{A}$ is nonnegative and irreducible, and suppose that $\mathcal{E} \in MST_{m,n}$ such that $|\mathcal{E}| \leq \varepsilon\mathcal{A}$, $(0 < \varepsilon < 1)$. Let ϱ and ϱ_ε denote, respectively, the Z-spectral radius of $\mathcal{A}$ and $\mathcal{A} + \mathcal{E}$. Then*

$$\frac{|\varrho_\varepsilon - \varrho|}{\varrho} \leq \varepsilon.$$

Proof Since $|\mathcal{E}| \leq \varepsilon\mathcal{A}$, i.e. $|\mathcal{E}_{i_1,\ldots,i_m}| \leq \varepsilon\mathcal{A}_{i_1,\ldots,i_m}$, we can write it as

$$0 \leq \mathcal{A} - \varepsilon\mathcal{A} \leq \mathcal{A} + \mathcal{E} \leq \mathcal{A} + \varepsilon\mathcal{A}.$$

Since Z-spectral radius $\varrho(\cdot)$ of $\mathcal{A}$ is monotone, it follows that $\varrho(\mathcal{A} - \varepsilon\mathcal{A}) \leq \varrho(\mathcal{A} + \mathcal{E}) \leq \varrho(\mathcal{A} + \varepsilon\mathcal{A})$. As $\varrho(\mathcal{A} \pm \varepsilon\mathcal{A}) = (1 \pm \varepsilon)\varrho(\mathcal{A})$, we obtain that

$$(1 - \varepsilon)\varrho \leq \varrho_\varepsilon \leq (1 + \varepsilon)\varrho.$$

As $\varrho > 0$, the last inequality is equivalent to the result. □

Since $\mathcal{A}$ always has Z-eigenvalue, then, according to Theorem 3.2 and formula (3.3), when $(\lambda, \mathbf{x})$ is an algebraic simple E-eigenpair of $\mathcal{A}$, the perturbation of λ can be also considered.

3.3. *Algebraic simple eigenvalue*

An algebraic *simple* eigenvalue is defined as an eigenvalue whose algebraic multiplicity is one. We present the explicit expression of an algebraic simple eigenvalue λ of $\mathcal{A}+\varepsilon\mathcal{B}$, which extends the classical results in [22, Theorem 2.3].

THEOREM 3.4 *Suppose $\mathcal{A}$ and $\mathcal{B} \in MST_{m,n}$, and $\varepsilon \in \mathbb{R}$. Let $(\lambda, \mathbf{x})$ be an algebraic simple eigenpair of $\mathcal{A}$. If $|\mathbf{x}|_m^m \neq 0$, then there exists $\varepsilon_0 > 0$ and an analytic function $\lambda(\varepsilon)$ with $|\varepsilon| \leq \varepsilon_0$ such that*

$$\lambda(0) = \lambda, \quad \lambda'(0) = \left.\frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \frac{\mathcal{B}\mathbf{x}^m}{|\mathbf{x}|_m^m}.$$

Therefore, $\lambda(\varepsilon)$ is an algebraic simple eigenvalue of $\mathcal{A}+\varepsilon\mathcal{B}$ over $|\varepsilon| \leq \varepsilon_0$, and

$$\lambda(\varepsilon) = \lambda + \varepsilon\frac{\mathcal{B}\mathbf{x}^m}{|\mathbf{x}|_m^m} + O(\varepsilon^2).$$

Particularly, if $(\lambda, \mathbf{x})$ is an algebraic simple H-eigenpair of $\mathcal{A}$, normalized x so that $|\mathbf{x}|_m^m = 1$. Then there exists $\varepsilon_0 > 0$ and an analytic function $\lambda(\varepsilon)$ with $|\varepsilon| \leq \varepsilon_0$ such that

$$\lambda(0) = \lambda, \quad \lambda'(0) = \left.\frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \mathcal{B}\mathbf{x}^m.$$

Thus, $\lambda(\varepsilon)$ is an algebraic simple H-eigenvalue of $\mathcal{A}+\varepsilon\mathcal{B}$ over $|\varepsilon| \leq \varepsilon_0$, and

$$\lambda(\varepsilon) = \lambda + \varepsilon\mathcal{B}\mathbf{x}^m + O(\varepsilon^2). \tag{3.4}$$

Remark 3.1 For Theorem 3.4, when $\mathcal{A}$ and $\mathcal{B}$ are irreducible and symmetric nonnegative tensors, and let ε be a positive number, formula (3.4) can be reduced to the result by Li et al. [11, Theorem 5.2].

Another perturbation result of an nonzero algebraic simple H-eigenvalue $\lambda = \lambda(\mathcal{A})$ is considered, whose associated eigenvector $\mathbf{x}$ is a real nonzero vector.

It is well known that the map $F_{\mathcal{E}} : \varepsilon \to \lambda(\mathcal{A}+\varepsilon\mathcal{B})$ is analytic in a neighbourhood of 0 (see [17]). Therefore, the map $F : \mathcal{A} \to \lambda(\mathcal{A})$ has continuous partial derivatives with respect to each entry at $\mathcal{A}$ and

$$\frac{\partial F}{\partial_{i_1 i_2 \ldots i_m}}(\mathcal{A}) := \lim_{t\to 0}\frac{F(\mathcal{A}+t\mathcal{J}) - F(\mathcal{A})}{t} = \frac{x_{i_1}x_{i_2}\ldots x_{i_m}}{\|\mathbf{x}\|_m^m},$$

where $\mathcal{J} = \mathbf{e}_{i_1} \circ \mathbf{e}_{i_2} \circ \cdots \circ \mathbf{e}_{i_m}$ where $\mathbf{e}_{i_k}$ is the i_kth column of the $n \times n$ identity matrix with $k \in [m]$.

Hence, as a map from $\mathbb{R}^{n^m} \to \mathbb{R}$, F is differentiable at $\mathcal{A}$ and

$$F'(\mathcal{A}) = \left[\frac{\partial F}{\partial_{11\ldots 1}}, \ldots, \frac{\partial F}{\partial_{11\ldots 1n}}, \frac{\partial F}{\partial_{21\ldots 1}}, \ldots, \frac{\partial F}{\partial_{nn\ldots n}}\right].$$

Based on a formula by Gohberg and Koltracht [29], for the *componentwise perturbations* in $\mathcal{A}$, i.e.

$$|\mathcal{E}_{i_1 i_2 \ldots i_m}| \leq \varepsilon|\mathcal{A}_{i_1 i_2 \ldots i_m}|, \quad i_k \in [n],\ k \in [m],$$

where $\mathcal{E} \in MST_{m,n}$ and $\varepsilon > 0$ is arbitrarily small, the sensitivity of $F(\mathcal{A})$ is characterized by the componentwise condition number of F at $\mathcal{A}$

$$c(F, \mathcal{A}) = \frac{\|F'(\mathcal{A})D_{\mathcal{A}}\|_\infty}{\|F(\mathcal{A})\|_\infty},$$

where $D_{\mathcal{A}} = \mathrm{diag}(\mathcal{A}_{11\ldots1}, \ldots, \mathcal{A}_{11\ldots1n}, \mathcal{A}_{21\ldots1}, \ldots, \mathcal{A}_{nn\ldots n})$, and $\lambda \neq 0$.

It indicates that

$$c(F, \mathcal{A}) = \frac{\|(\mathcal{A}_{11\ldots1}x_1x_1\ldots x_1, \ldots, \mathcal{A}_{11\ldots1n}x_1\ldots x_1x_n, \mathcal{A}_{21\ldots1}x_2x_1\ldots x_1, \ldots, \mathcal{A}_{nn\ldots n}x_nx_n\ldots x_n)\|_\infty}{|\lambda|},$$

where the infinity norm of the map $F'(\mathcal{A})D_{\mathcal{A}}$ is 1-norm of the row vector that represents it. Thus

$$c(F, \mathcal{A}) = \frac{\sum_{i_1i_2\ldots i_m=1}^{n} |\mathcal{A}_{i_1i_2\ldots i_m}x_{i_1}x_{i_2}\ldots x_{i_m}|}{|\lambda|\|\mathbf{x}\|_m^m} = \frac{|\mathcal{A}||\mathbf{x}|^m}{|\lambda|\|\mathbf{x}\|_m^m},$$

where an *absolute value of a tensor* is the corresponding tensor of the absolute values of its entries. Thus if $\mathcal{E}$ is a perturbation of $\mathcal{A}$, then we have the relative perturbation bound of an algebraic simple H-eigenvalue λ.

THEOREM 3.5 *Suppose that $\mathcal{E} \in MST_{m,n}$ and $|\mathcal{E}_{i_1i_2\ldots i_m}| \leq \varepsilon|\mathcal{A}_{i_1i_2\ldots i_m}|$, where $i_k \in [n]$ with $k \in [m]$. For an algebraic simple H-eigenvalue $\lambda \neq 0$, then there exists an eigenvalue $\widehat{\lambda}$ of $\mathcal{A}+\mathcal{E}$ such that*

$$\frac{|\widehat{\lambda} - \lambda|}{|\lambda|} \leq c(F, \mathcal{A})\varepsilon + O(\varepsilon^2).$$

For an irreducible and mode-symmetric nonnegative tensor, the perturbation bound of the spectral radius, is derived from Theorem 3.5.

COROLLARY 3.2 *Let $\mathcal{A} \in MST_{m,n}$. If $\mathcal{A}$ is nonnegative and irreducible, and suppose that $\mathcal{E} \in MST_{m,n}$ and $|\mathcal{E}| \leq \varepsilon\mathcal{A}$, $(0 < \varepsilon < 1)$. Let ρ and ρ_ε denote the spectral radius of $\mathcal{A}$ and $\mathcal{A}+\mathcal{E}$, respectively. Then*

$$\frac{|\rho_\varepsilon - \rho|}{\rho} \leq \varepsilon.$$

Proof Since $|\mathcal{E}| \leq \varepsilon\mathcal{A}$, i.e. $|\mathcal{E}_{i_1,\ldots,i_m}| \leq \varepsilon\mathcal{A}_{i_1,\ldots,i_m}$, we can write it as

$$0 \leq \mathcal{A} - \varepsilon\mathcal{A} \leq \mathcal{A} + \mathcal{E} \leq \mathcal{A} + \varepsilon\mathcal{A}.$$

Since spectral radius $\rho(\cdot)$ of $\mathcal{A}$ ia monotone [30], it follows that $\rho(\mathcal{A}-\varepsilon\mathcal{A}) \leq \rho(\mathcal{A}+\mathcal{E}) \leq \rho(\mathcal{A}+\varepsilon\mathcal{A})$. As $\rho(\mathcal{A} \pm \varepsilon\mathcal{A}) = (1 \pm \varepsilon)\rho(\mathcal{A})$, we obtain that

$$(1-\varepsilon)\rho \leq \rho_\varepsilon \leq (1+\varepsilon)\rho.$$

As $\rho > 0$, the last inequality is equivalent to the result. □

Remark 3.2 This corollary is a special case of Theorem 3.5, if $\mathcal{A}$ is an irreducible nonnegative tensor, $c(F, \mathcal{A}) = 1$ for the spectral radius. For nonnegative matrices, the perturbation of Perron root has been discussed by Elsner et al. [31].

3.4. *Singular value*

For a given a tensor $\mathcal{A} \in T_{m,\langle n\rangle}$, Ragnarsson et al. [23] explored the singular value σ and mode-k singular vectors $\mathbf{x}_k$ through its symmetric embedding $\mathbf{sym}(\mathcal{A})$, and Chen et al. [24] further developed the connection between tensor singular value and its symmetric embedding eigenvalue. In the rest part, we consider the connection between tensor singular value and its mode-symmetric embedding Z-eigenvalue.

First, through the example when $m = 3$, we derive the process for implementing how to transform tensor singular value to its mode-symmetric embedding eigenvalue.

When $m = 3$, formulae (2.3) can be simplified as

$$\mathcal{A}\bar{\times}_2\mathbf{x}_2\bar{\times}_3\mathbf{x}_3 = \sigma\mathbf{x}_1,\ \mathcal{A}\bar{\times}_1\mathbf{x}_1\bar{\times}_3\mathbf{x}_3 = \sigma\mathbf{x}_2,\ \mathcal{A}\bar{\times}_1\mathbf{x}_1\bar{\times}_2\mathbf{x}_2 = \sigma\mathbf{x}_3. \tag{3.5}$$

Elementwise, we have

$$\sum_{j,k}\mathcal{A}_{ijk}x_{2,j}x_{3,k} = \sigma x_{1,i},\ \sum_{i,k}\mathcal{A}_{ijk}x_{1,i}x_{3,k} = \sigma x_{2,j},\ \sum_{i,j}\mathcal{A}_{ijk}x_{1,i}x_{2,j} = \sigma x_{3,k}, \tag{3.6}$$

where $\mathcal{A} \in T_{3,\langle n\rangle}$, with $i \in [n_1]$, $j \in [n_2]$ and $k \in [n_3]$.

In formulae (3.6), when we change the sum order, then, componentwise, we have

$$\sum_{j,k}\mathcal{A}_{ijk}x_{2,j}x_{3,k} = \sigma x_{1,i},\ \sum_{k,i}\mathcal{A}_{jki}x_{3,k}x_{1,i} = \sigma x_{2,j},\ \sum_{i,j}\mathcal{A}_{kij}x_{1,i}x_{2,j} = \sigma x_{3,k}.$$

Let $\mathbf{x} = \frac{1}{\sqrt{3}}(\mathbf{x}_1^\top, \mathbf{x}_2^\top, \mathbf{x}_3^\top)^\top$, then we have $\|\mathbf{x}\| = 1$. This is because that $\|\mathbf{x}_k\| = 1$, $k \in [3]$. We have $\left\|(\mathbf{x}_1^\top, \mathbf{x}_2^\top, \mathbf{x}_3^\top)^\top\right\| = \sqrt{3}$. Hence, the formulae (3.5) can be transformed as

$$\frac{1}{\sqrt{3}}\mathbf{msym}(\mathcal{A})\bar{\times}_2\mathbf{x}\bar{\times}_3\mathbf{x} = \sigma\mathbf{x},\ \frac{1}{\sqrt{3}}\mathbf{msym}(\mathcal{A})\bar{\times}_1\mathbf{x}\bar{\times}_3\mathbf{x} = \sigma\mathbf{x},$$
$$\frac{1}{\sqrt{3}}\mathbf{msym}(\mathcal{A})\bar{\times}_1\mathbf{x}\bar{\times}_2\mathbf{x} = \sigma\mathbf{x}.$$

Furthermore, when we set $\widetilde{\mathcal{A}} = \frac{1}{\sqrt{3}}\mathbf{msym}(\mathcal{A})$, then, the above formulae is equivalent to

$$\widetilde{\mathcal{A}}\mathbf{x}^2 = \sigma\mathbf{x}, \quad \|\mathbf{x}\|_2 = 1.$$

Generally, for a given tensor $\mathcal{A} \in T_{m,\langle n\rangle}$, through the above description, formulae (2.3) can be transformed the following eigenvalue problem. Suppose $(\sigma, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ is a solution of (2.3), with $\|\mathbf{x}_k\| = 1$, $k \in [m]$. Let $\mathbf{x} = \frac{1}{\sqrt{m}}(\mathbf{x}_1^\top, \mathbf{x}_2^\top, \ldots, \mathbf{x}_m^\top)^\top$, then we have $\|\mathbf{x}\| = 1$. This is because that $\|\mathbf{x}_k\| = 1$, $k \in [m]$, we get $\left\|(\mathbf{x}_1^\top, \mathbf{x}_2^\top, \ldots, \mathbf{x}_m^\top)^\top\right\| = \sqrt{m}$.

Thus, we can derive

$$\begin{cases} \left(\frac{1}{\sqrt{m}}\right)^{m-2}\mathbf{msym}(\mathcal{A})\bar{\times}_2\mathbf{x}\bar{\times}_3\mathbf{x}\ldots\bar{\times}_m\mathbf{x} = \sigma\mathbf{x}, \\ \left(\frac{1}{\sqrt{m}}\right)^{m-2}\mathbf{msym}(\mathcal{A})\bar{\times}_1\mathbf{x}\bar{\times}_3\mathbf{x}\ldots\bar{\times}_m\mathbf{x} = \sigma\mathbf{x}, \\ \qquad\vdots \\ \left(\frac{1}{\sqrt{m}}\right)^{m-2}\mathbf{msym}(\mathcal{A})\bar{\times}_1\mathbf{x}\bar{\times}_2\mathbf{x}\ldots\bar{\times}_{m-1}\mathbf{x} = \sigma\mathbf{x}. \end{cases}$$

Furthermore, let $\widetilde{\mathcal{A}} = \left(\frac{1}{\sqrt{m}}\right)^{m-2} \mathbf{msym}(\mathcal{A})$, then $(\sigma, \mathbf{x})$ is an Z-eigenpair of $\widetilde{\mathcal{A}}$, that is, $(\sigma, \mathbf{x})$ is the solution of nonlinear equations

$$\widetilde{\mathcal{A}}\mathbf{x}^{m-1} = \sigma \mathbf{x}, \quad \|\mathbf{x}\| = 1.$$

Meanwhile, we have that both $\mathbf{msym}(\mathcal{A})$ and $\widetilde{\mathcal{A}}$ are mode-symmetric. The value σ is called an algebraic simple singular value, when σ is an algebraic simple Z-eigenvalue of $\widetilde{\mathcal{A}}$.

According to symmetric or mode-symmetric embedding of $\mathcal{A} \in T_{m,\langle n \rangle}$ and the perturbation bounds about an algebraic simple Z-eigenvalue of a mode-symmetric tensor, it is obvious to derive these three theorems about the perturbation of an algebraic simple singular value. Hence, the proof is omitted.

THEOREM 3.6 *Suppose $\mathcal{A}, \mathcal{B} \in T_{m,\langle n \rangle}$. Let $(\sigma, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ be a solution of (2.3) and σ be an algebraic simple singular value of $\mathcal{A}$ with $\mathbf{x}_k \in \mathbb{R}^{n_k}$, $k \in [m]$. Then there exists $\varepsilon_0 > 0$ and an analytic function $\sigma(\varepsilon)$ with $|\varepsilon| \le \varepsilon_0$ such that*

$$\sigma(0) = \lambda, \quad \sigma'(0) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0} = \mathcal{B}\bar{\times}_1\mathbf{x}_1\bar{\times}_2\mathbf{x}_2\ldots\bar{\times}_m\mathbf{x}_m,$$

with $\mathbf{x}_k^\top \mathbf{x}_k = 1$ $(k \in [m])$. Therefore, $\sigma(\varepsilon)$ is an algebraic simple singular value of $\mathcal{A} + \varepsilon\mathcal{B}$ over $|\varepsilon| \le \varepsilon_0$, and

$$\sigma(\varepsilon) = \sigma + \varepsilon\mathcal{B}\bar{\times}_1\mathbf{x}_1\bar{\times}_2\mathbf{x}_2\ldots\bar{\times}_m\mathbf{x}_m + O(\varepsilon^2).$$

THEOREM 3.7 *Suppose that $\mathcal{E} \in T_{m,\langle n \rangle}$ and $|\mathcal{E}_{i_1i_2\ldots i_m}| \le \varepsilon|\mathcal{A}_{i_1i_2\ldots i_m}|$, where $i_k \in [n_k]$ with $k \in [m]$. Then, for an algebraic simple singular value $\sigma \neq 0$, there exists a singular value $\widehat{\sigma}$ of $\mathcal{A} + \mathcal{E}$ such that*

$$\frac{|\widehat{\sigma} - \sigma|}{|\sigma|} \le c(F, \mathcal{A})\varepsilon + O(\varepsilon^2),$$

where $c(F, \mathcal{A}) = \frac{1}{|\sigma|}|\mathcal{A}|\bar{\times}_1|\mathbf{x}_1|\bar{\times}_2|\mathbf{x}_2|\ldots\bar{\times}_m|\mathbf{x}_m|$ and all $\mathbf{x}_k$ are the mode-k singular vectors associated with σ.

COROLLARY 3.3 *If $\mathcal{A}$ is nonnegative and irreducible, and suppose that $\mathcal{E} \in T_{m,\langle n \rangle}$ and $|\mathcal{E}| \le \varepsilon\mathcal{A}$, $(0 < \varepsilon < 1)$. Let $\bar{\sigma}$ and $\bar{\sigma}_\varepsilon$ denote, respectively, the largest singular value of $\mathcal{A}$ and $\mathcal{A} + \mathcal{E}$. Then*

$$\frac{|\bar{\sigma}_\varepsilon - \bar{\sigma}|}{\bar{\sigma}} \le \varepsilon.$$

4. Perturbation for the case of $P_l(\lambda)$

In this section, we derive perturbation bounds of an algebraic simple mode-k tensor polynomial eigenvalue λ_k in $\Lambda_k(P_l(\lambda))$. However, according to Remark 2.1, all sets $\Lambda_k(P_l(\lambda))$ are different. When we suppose that all tensors in $P_l(\lambda)$ are symmetric or mode-symmetric and $\mathcal{A}_l$ is nonsingular, we can see that all sets $\Lambda_k(P_l(\lambda))$ are the same finite set, denoted by $\Lambda(P_l(\lambda))$, and an algebraic simple mode-k polynomial eigenvalue λ_k can be simplified as an algebraic simple polynomial eigenvalue λ. Hence, we just study some perturbation bounds of λ.

Meanwhile, we denote $\Delta P_l(\lambda)$ by $\lambda^l \Delta\mathcal{A}_l + \lambda^{l-1}\Delta\mathcal{A}_{l-1} + \cdots + \lambda\Delta\mathcal{A}_1 + \Delta\mathcal{A}_0$ with $\Delta\mathcal{A}_i \in T_{m,n}(i = 0 : l-1)$.

4.1. *Tensor generalized eigenvalue problem*

According to Remark 2.2, for a given tensor pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A},\ \mathcal{B} \in T_{m,n}$, the tensor generalized eigenvalue problem is a special case of the polynomial eigenvalue problem with $l = 1$. Since $\mathcal{A}$ and $\mathcal{B}$ are mode-symmetric, then the set of all generalized eigenvalue λ of a pair $(\mathcal{A}, \mathcal{B})$ is $\Lambda(P_1(\lambda))$ with $P_1(\lambda) = \mathcal{A} + \lambda(-\mathcal{B})$. When $\mathcal{B}$ is nonsingular, some perturbation bounds on an algebraic simple generalized eigenvalue are considered. The following theorem states a perturbation of an algebraic generalized eigenvalue λ.

THEOREM 4.1 *If λ is an algebraic simple generalized eigenvalue of the pair $(\mathcal{A}, \mathcal{B})$. Then, there exists an algebraic simple generalized eigenvalue $\widetilde{\lambda}$ of $(\mathcal{A}+\epsilon\mathcal{A}_1, \mathcal{B}+\epsilon\mathcal{B}_1)$ such that*

$$\widetilde{\lambda} = \lambda + O(\epsilon),$$

where $|\epsilon| \le \epsilon_0$ with sufficient small $\epsilon_0 > 0$.

Proof Since two tensors $\mathcal{A}$ and $\mathcal{B}$ are mode-symmetric, we can denote the set of tensor generalized eigenvalue of the pair $(\mathcal{A}, \mathcal{B})$ by

$$\Lambda(\mathcal{A}, \mathcal{B}) = \{\lambda \in \mathbb{C} \mid \mathrm{Det}(\mathcal{A} - \lambda\mathcal{B}) = 0\}.$$

According to the relationship between the determinant and the eigenvalue of a tensor (see [17]), we know that $\mathrm{Det}(\mathcal{A} - \lambda\mathcal{B})$ is a $n(m-1)^{n-1}$th polynomial about λ with leading coefficient $\mathrm{Det}(\mathcal{B})$.

Since $\mathrm{Det}(\mathcal{B}) \ne 0$, we can derive that there exists $\hat{\epsilon} > 0$ such that $\mathrm{Det}(\mathcal{B} + \hat{\epsilon}\mathcal{B}_1) \ne 0$. Hence, the set of all generalized eigenvalue of the pair $(\mathcal{A} + \epsilon\mathcal{A}_1, \mathcal{B} + \epsilon\mathcal{B}_1)$ $(|\epsilon| \le \hat{\epsilon})$ can be denoted by

$$\Lambda_\epsilon(\mathcal{A}, \mathcal{B}) = \{\lambda \in \mathbb{C} \mid \mathrm{Det}((\mathcal{A} - \lambda\mathcal{B}) + \epsilon(\mathcal{A}_1 - \lambda\mathcal{B}_1)) = 0\}.$$

Meanwhile, $\mathrm{Det}((\mathcal{A} - \lambda\mathcal{B}) + \epsilon(\mathcal{A}_1 - \lambda\mathcal{B}_1))$ is also a $n(m-1)^{n-1}$th polynomial about λ with leading coefficient $\mathrm{Det}(\mathcal{B}+\epsilon\mathcal{B}_1)$. $\mathrm{Det}(\mathcal{B}+\epsilon\mathcal{B}_1)$ is a $n(m-1)^{n-1}$th polynomial about ϵ with the constant term $\mathrm{Det}(\mathcal{B})$.

Hence, we know that $\mathrm{Det}(\mathcal{B}+\epsilon\mathcal{B}_1) \not\equiv 0$. According to theorems of algebraic functions of one variable (see [28]), if λ is an algebraic simple generalized eigenvalue, then there exists an algebraic simple generalized eigenvalue $\widetilde{\lambda}$ of the pair $(\mathcal{A} + \epsilon\mathcal{A}_1, \mathcal{B} + \epsilon\mathcal{B}_1)$ and $\epsilon_0 > 0$ such that

$$\widetilde{\lambda} = \lambda + O(\epsilon),$$

where $|\epsilon| \le \min\{\hat{\epsilon}, \epsilon_0\}$. □

In the above theorem, there are two statements we need to emphasize the following issues.

(1) The choice of the tensor pair $(\mathcal{A}_1, \mathcal{B}_1)$ in Theorem 4.1 is not only one. In general, we suppose that $(\mathcal{A}_1, \mathcal{B}_1) = (\mathcal{A}, \mathcal{B})$.

(2) Theorem 4.1 only states the relationship about the first-order perturbation of an algebraic simple generalized eigenvalue, but does not present the coefficient of the first-order perturbation term.

Hence, in the rest of this subsection, we consider how to present an expression of this coefficient.

THEOREM 4.2 *Suppose that $\Delta\mathcal{A}, \Delta\mathcal{B} \in MST_{m,n}$. If $\lambda \neq 0$ is an algebraic simple generalized H-eigenvalue of $(\mathcal{A}, \mathcal{B})$ with associated generalized eigenvector $\mathbf{x} \in \mathbb{R}^n$. Then, there exists an algebraic simple generalized H-eigenvalue $\widehat{\lambda}$ of $(\mathcal{A} + \Delta\mathcal{A}, \mathcal{B} + \Delta\mathcal{B})$ such that*

$$\frac{|\widehat{\lambda} - \lambda|}{|\lambda|} \leq \epsilon \frac{(\alpha + |\lambda|\beta)\|\mathbf{x}\|^m}{|\lambda||\mathcal{B}\mathbf{x}^m|} + O(\epsilon^2),$$

where $\|\Delta\mathcal{A}\|_F \leq \epsilon\alpha$ and $\|\Delta\mathcal{B}\|_F \leq \epsilon\beta$ with $0 < \epsilon < 1$.

Proof Since $\mathrm{Det}(\mathcal{B}) \neq 0$, it is obvious that $\mathcal{B}\mathbf{x}^m \neq 0$ for all nonzero vectors $x \in \mathbb{R}^n$. Let $\lambda \neq 0$ be an algebraic simple generalized H-eigenvalue of $(\mathcal{A}, \mathcal{B})$, with corresponding eigenvector $\mathbf{x}$, then a normwise condition number of λ can be defined as follows,

$$\kappa(\lambda) := \lim_{\epsilon\to 0}\sup\left\{\frac{|\Delta\lambda|}{\epsilon|\lambda|} : (\mathcal{A} + \Delta\mathcal{A})(\mathbf{x} + \Delta\mathbf{x})^{m-1} = (\lambda + \Delta\lambda)(\mathcal{B} + \Delta\mathcal{B})(\mathbf{x} + \Delta\mathbf{x})^{m-1},\right.$$
$$\left.\|\Delta\mathcal{A}\|_F \leq \epsilon\alpha,\ \|\Delta\mathcal{B}\|_F \leq \epsilon\beta\right\}.$$

Next, we prove that the following formula holds,

$$\kappa(\lambda) = \frac{(\alpha + |\lambda|\beta)\|\mathbf{x}\|^m}{|\lambda||\mathcal{B}\mathbf{x}^m|}.$$

The given expression is clearly an upper bound for $\kappa(\lambda)$. We now show that the bound is attained. From the definition of a normwise condition number of λ, we have

$$\Delta\lambda = \frac{\Delta\mathcal{A}\mathbf{x}^m - \lambda\Delta\mathcal{B}\mathbf{x}^m}{\mathcal{B}\mathbf{x}^m} + O(\epsilon^2). \tag{4.1}$$

Let $\mathcal{G} = (\underbrace{\mathbf{x} \circ \mathbf{x} \circ \cdots \circ \mathbf{x}}_{m})/\|\mathbf{x}\|^m$. Then $\|\mathcal{G}\|_F = 1$ and $\mathcal{G}x^m = \|\mathbf{x}\|^m$. Let $\Delta\mathcal{A} = \epsilon\alpha\mathcal{G}$ and $\Delta\mathcal{B} = -\mathrm{sign}(\lambda)\epsilon\alpha\mathcal{G}$. Then $\|\Delta\mathcal{A}\|_F \leq \epsilon\alpha$ and $\|\Delta\mathcal{B}\|_F \leq \epsilon\beta$ and the modulus of the first-order term of (4.1) is $\epsilon\|\mathbf{x}\|^m(\alpha + |\lambda|\beta)/|\mathcal{B}\mathbf{x}^m|$; dividing (4.1) by $\epsilon|\lambda|$ and taking the limit as $\epsilon \to 0$ then gives the desired equality.

From the definition of $\kappa(\lambda)$ we have, for the perturbation system in (4.1),

$$\frac{|\Delta\lambda|}{|\lambda|} \leq \kappa(\lambda)\epsilon + O(\epsilon^2).$$

Hence, the proof is over. □

THEOREM 4.3 *Suppose that $\Delta\mathcal{A}, \Delta\mathcal{B} \in MST_{m,n}$. If $\lambda \neq 0$ is an algebraic simple generalized H-eigenvalue of $(\mathcal{A}, \mathcal{B})$ with corresponding eigenvector $\mathbf{x} \in \mathbb{R}^n$. Then, there exists an algebraic simple generalized H-eigenvalue $\widehat{\lambda}$ of $(\mathcal{A} + \Delta\mathcal{A}, \mathcal{B} + \Delta\mathcal{B})$ such that*

$$\frac{|\widehat{\lambda} - \lambda|}{|\lambda|} \leq \epsilon \frac{(\mathcal{E} + |\lambda|\mathcal{F})|\mathbf{x}|^m}{|\lambda||\mathcal{B}\mathbf{x}^m|} + O(\epsilon^2),$$

where $|\Delta\mathcal{A}| \leq \epsilon\mathcal{E}$ *and* $|\Delta\mathcal{B}| \leq \epsilon\mathcal{F}$ *with nonnegative tensors* $\mathcal{E}, \mathcal{F} \in MST_{m,n}$ *and* $0 < \epsilon < 1$.

Proof Since $\mathrm{Det}(\mathcal{B}) \neq 0$, it is obvious that $\mathcal{B}\mathbf{x}^m \neq 0$ for all nonzero vectors $x \in \mathbb{R}^n$. Let $\lambda \neq 0$ be an algebraic simple generalized H-eigenvalue of $(\mathcal{A}, \mathcal{B})$, with corresponding eigenvector $\mathbf{x}$, then a componentwise condition number for an algebraic simple generalized H-eigenvalue λ analogous to the normwise condition number is defined by

$$\begin{aligned}\mathrm{cond}(\lambda) := \limsup_{\epsilon\to 0}\Bigg\{&\frac{|\Delta\lambda|}{\epsilon|\lambda|} : (\mathcal{A} + \Delta\mathcal{A})(\mathbf{x} + \Delta\mathbf{x})^{m-1} \\ &= (\lambda + \Delta\lambda)(\mathcal{B} + \Delta\mathcal{B})(\mathbf{x} + \Delta\mathbf{x})^{m-1}, |\Delta\mathcal{A}| \leq \epsilon\mathcal{E},\ |\Delta\mathcal{B}| \leq \epsilon\mathcal{F}\Bigg\}.\end{aligned}$$

From this defintion it follows that,

$$\frac{|\Delta\lambda|}{|\lambda|} \leq \mathrm{cond}(\lambda)\epsilon + O(\epsilon^2).$$

In the following, we will derive the expression of $\mathrm{cond}(\lambda)$. First, according to the definition of a componentwise condition number for an algebraic simple generalized H-eigenvalue λ, we have

$$\mathrm{cond}(\lambda) \geq \frac{(\mathcal{E} + |\lambda|\mathcal{F})|\mathbf{x}|^m}{|\lambda||\mathcal{B}\mathbf{x}^m|}.$$

Next, we will show that the expression for the $\mathrm{cond}(\lambda)$ attained when we choose $\Delta\mathcal{A} = \epsilon\mathcal{E} \times_1 D \times_2 D \cdots \times_m D$ and $\Delta\mathcal{B} = -\mathrm{sign}(\lambda)\epsilon\mathcal{F} \times_1 D \times_2 D \cdots \times_m D$, where $D = \mathrm{diag}(\mathrm{sign}(\mathbf{x}))$. Hence, the proof is over. □

Remark 4.1 For all nonzero vectors $\mathbf{x} \in \mathbb{C}^n$, when a pair $(\lambda, \mathbf{x})$ is an algebraic simple generalized eigenpair of the pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B}\mathbf{x}^m \neq 0$, the above two theorems also hold. However, according to Theorem 4.4, it is known that when $\mathcal{B}$ is nonsingular and all nonzero vectors $\mathbf{x} \in \mathbb{C}^n$, the inequality $\mathcal{B}\mathbf{x}^m \neq 0$ holds (also see [17, Theorem 3.1]).

Hence, Theorems 4.2 and 4.3 hold for an algebraic simple generalized eigenpair $(\lambda, \mathbf{x})$ of a tensor pair $(\mathcal{A}, \mathcal{B})$ when $\mathcal{B}$ is nonsingular.

There are many choices for the pair (α, β) and the pair $(\mathcal{E}, \mathcal{F})$. In practice, we always choose $(\alpha, \beta) = (\|\mathcal{A}\|_F, \|\mathcal{B}\|_F)$ and $(\mathcal{E}, \mathcal{F}) = (\mathcal{A}, \mathcal{B})$.

We consider briefly the special case when $\mathcal{A}$ is an irreducible nonnegative tensor and $\mathcal{B}$ is diagonal with positive diagonal entries. Here, the tensor generalized eigenvalue problem is equivalent to the standard eigenvalue problem for $\mathcal{A} \times_k D^{-1}$, where D is diagonal and its diagonal entries is equivalent to the diagonal entries of $\mathcal{B}$. Perron-Frobenius theorem for nonnegative tensors (see [4]) will be used in the process of the corollary.

Corollary 4.1 *Suppose* $\mathcal{A}$ *is an irreducible and nonnegative tensor and* $\mathcal{B}$ *is a diagonal tensor with positive diagonal entries. B is a diagonal matrix and its diagonal entries are equal to the diagonal entries of* $\mathcal{B}$. *Let* λ_k *be the mode-k Perron root of* $\mathcal{A} \times_k B^{-1}$. *Then, the following statements hold.*

(1) *All* λ_k *are equal, denoted by* λ.
(2) *Assume* λ *be simple, and let* $\mathcal{E} = \mathcal{A}$ *and* $\mathcal{F} = \mathcal{B}$. *Then* $\mathrm{cond}(\lambda) = 2$.

(3) *Moreover, if $\lambda+\Delta\lambda$ is the Perron root of the pair $(\mathcal{A}+\Delta\mathcal{A}, \mathcal{B}+\Delta\mathcal{B})$, for $0 \le \epsilon < 1$, then we have*

$$\frac{|\Delta\lambda|}{|\lambda|} \le \frac{2\epsilon}{1-\epsilon}.$$

Proof Parts 1 and 2 are trivial to verify. For the third part, we only prove when $k = 1$. The rest is analogous to the case of $k = 1$. Note that since $|\Delta\mathcal{B}| \le \epsilon\mathcal{B}$, with $\mathcal{B}$ diagonal, and $|\Delta\mathcal{A}| \le \epsilon\mathcal{A}$,

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)\mathcal{A}\times_1 B^{-1} \le \mathcal{A}\times_1 B^{-1} \le \left(\frac{1+\epsilon}{1-\epsilon}\right)\mathcal{A}\times_1 B^{-1}.$$

Since $\rho_1(\cdot)$ is monotone on the nonnegative tensors,

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)\rho_1(\mathcal{A}\times_1 B^{-1}) \le \rho_1(\mathcal{A}\times_1 B^{-1}) \le \left(\frac{1+\epsilon}{1-\epsilon}\right)\rho_1(\mathcal{A}\times_1 B^{-1}).$$

Hence, the third part is proved. □

When $\mathcal{B}$ is the identity tensor and $\Delta\mathcal{B}$ is the zero tensor, part (3) of Corollary 4.1 gives a perturbation bound about the spectral radius of an nonnegative irreducible tensor.

According to [18, Theorem 2.1] and $\mathrm{Det}(\mathcal{B}) \neq 0$, the number of all generalized eigenvalue is $n(m-1)^{n-1}$ and all generalized eigenvalue are finite numbers. Then a generalized eigenvalue λ can be represented as $\lambda = \alpha/\beta$ with $\beta \neq 0$. Hence, a generalized eigenpair $(\lambda, \mathbf{x})$ can also be represented as $(\alpha, \beta, \mathbf{x})$. We can also denote a generalized eigenvalue λ by (α, β) or $\langle\alpha, \beta\rangle$, where $\langle\alpha, \beta\rangle = \tau(\alpha, \beta)$ with $\tau \neq 0$. A property of the pair $\langle\alpha, \beta\rangle$ is given below.

THEOREM 4.4 *Let $\langle\alpha, \beta\rangle$ be an algebraic simple generalized eigenvalue of the pair $(\mathcal{A}, \mathcal{B})$ with corresponding generalized eigenvector $\mathbf{x}$. Then*

$$\langle\alpha, \beta\rangle = \langle\mathcal{A}x^m, \mathcal{B}x^m\rangle.$$

Proof Since $\det(\mathcal{B}) \neq 0$, then λ is a finite number. Let $\lambda = \alpha/\beta$, we obtain $\beta \neq 0$. For the vector $\mathbf{x}$, there exists a Householder matrix P such that $P\mathbf{x} = \|\mathbf{x}\|\mathbf{e}_1$, where $\mathbf{e}_1$ is the first column of the identity matrix I.

According to [18], we have that $\lambda(\mathcal{A}, \mathcal{B}) = \lambda(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$, where $\widehat{\mathcal{A}} = \mathcal{A}\times_1 P\cdots\times_m P$ and $\widehat{\mathcal{B}} = \mathcal{B}\times_1 P\cdots\times_m P$. Then, the pair $(\alpha, \beta, \mathbf{e}_1)$ satisfies $\beta\widehat{\mathcal{A}}\mathbf{e}_1^{m-1} = \alpha\widehat{\mathcal{B}}\mathbf{e}_1^{m-1}$, that is, $\langle\alpha, \beta\rangle = \langle\widehat{\mathcal{A}}\mathbf{e}_1^m, \widehat{\mathcal{B}}\mathbf{e}_1^m\rangle$. Hence, the proof is over. □

Similar to Theorem 4.1, we can derive the perturbation bound of an algebraic simple generalized eigenvalue $\langle\alpha, \beta\rangle$.

THEOREM 4.5 *Suppose $\mathcal{A}, \mathcal{B} \in MST_{m,n}$. Let $\langle\alpha, \beta\rangle$ be an algebraic simple eigenvalue of the regular pair $(\mathcal{A}, \mathcal{B})$ with associated eigenvector $\mathbf{x}$. Let $\langle\widetilde{\alpha}, \widetilde{\beta}\rangle$ be the corresponding eigenvalue of the $O(\epsilon)$ perturbation $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$. Then*

$$\langle\widetilde{\alpha}, \widetilde{\beta}\rangle = \langle\widetilde{\mathcal{A}}\mathbf{x}^m, \widetilde{\mathcal{B}}\mathbf{x}^m\rangle + O(\epsilon^2).$$

Proof According to the implicit function theory, we find that we may take for the eigenvectors corresponding to $\langle\widetilde{\alpha}, \widetilde{\beta}\rangle$ the vectors $\widetilde{\mathbf{x}} = \mathbf{x} + \mathbf{u}$ where $\mathbf{u} = O(\epsilon)$ (see also [32]). By Theorem 4.4,

$$\langle\widetilde{\alpha}, \widetilde{\beta}\rangle = \langle\widetilde{\mathcal{A}}\widetilde{\mathbf{x}}^m, \widetilde{\mathcal{B}}\widetilde{\mathbf{x}}^m\rangle = \langle\widetilde{\mathcal{A}}\mathbf{x}^m + m\mathbf{u}^\top\mathcal{A}\mathbf{x}^{m-1} + O(\epsilon^2), \widetilde{\mathcal{B}}\mathbf{x}^m + m\mathbf{u}^\top\mathcal{B}\mathbf{x}^{m-1} + O(\epsilon^2)\rangle.$$

Since $\det(\mathcal{B}) \neq 0$, then β must be nonzero. Then

$$m\mathbf{u}^\top\mathcal{A}\mathbf{x}^{m-1} = \alpha\frac{m\mathbf{u}^\top\mathcal{B}\mathbf{x}^{m-1}}{\beta}, \quad m\mathbf{u}^\top\mathcal{B}\mathbf{x}^{m-1} = \beta\frac{m\mathbf{u}^\top\mathcal{B}\mathbf{x}^{m-1}}{\beta}.$$

Thus, $(m\mathbf{u}^\top\mathcal{A}\mathbf{x}^{m-1}, m\mathbf{u}^\top\mathcal{B}\mathbf{x}^{m-1})$ is an order ϵ perturbation of $\langle\widetilde{\mathcal{A}}\mathbf{x}^m, \widetilde{\mathcal{B}}\mathbf{x}^m\rangle$ that lies along (α, β). Hence, the proof is over. $\square$

4.2. *Tensor quadratic eigenvalue problem*

In this section, we will consider another special case of the polynomial eigenvalue problem with $l = 2$. Suppose that all tensors in $P_2(\lambda)$ are symmetric or mode-symmetric and $\mathcal{A}_2$ is nonsingular.

Given $P_2(\lambda) = \lambda^2\mathcal{A}_2 + \lambda\mathcal{A}_1 + \mathcal{A}_0$, let $\mu = \lambda^{1/(m-1)}$ and $\mathbf{y} = \mu\mathbf{x}$, then $P_2(\lambda)\mathbf{x}^{m-1} = \mathbf{0}$ can be represented as

$$\begin{aligned} \lambda\mathcal{A}_2\mathbf{y}^{m-1} + \mathcal{A}_1\mathbf{y}^{m-1} + \mathcal{A}_0\mathbf{x}^{m-1} &= \mathbf{0}, \\ \mathbf{y}^{[m-1]} - \lambda\mathbf{x}^{[m-1]} &= \mathbf{0}. \end{aligned}$$

These above equations are equal to the generalized eigenvalue problems as follows.

$$\widetilde{\mathcal{A}}\widetilde{\mathbf{x}}^{m-1} = \lambda(-\widetilde{\mathcal{B}})\widetilde{\mathbf{x}}^{m-1},$$

where $\widetilde{\mathbf{x}} = (\mathbf{y}^\top, \mathbf{x}^\top)^\top$ and the definitions of $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ are in following:

$$\widetilde{\mathcal{A}}_{i_1i_2\ldots i_m} = \begin{cases} \mathcal{A}_{1,i_1i_2\ldots i_m} & i_k = 1, 2, \ldots, n, \quad k \in [m], \\ \mathcal{A}_{0,i_1(i_2-n)\ldots(i_m-n)} & i_1 = 1, 2, \ldots, n; i_k = n+1, n+2, \ldots, 2n, \quad k \in [m] - \{1\}, \\ \mathcal{I}_{(i_1-n)i_2\ldots i_m} & i_1 = n+1, n+2, \ldots, 2n; i_k = 1, 2, \ldots, n, \quad k \in [m] - \{1\}, \\ 0 \quad \text{otherwise}, \end{cases}$$

and

$$\widetilde{\mathcal{B}}_{i_1i_2\ldots i_m} = \begin{cases} \mathcal{A}_{2,i_1i_2\ldots i_m} & i_k = 1, 2, \ldots, n, \quad k \in [m], \\ -\mathcal{I}_{(i_1-n)(i_2-n)\ldots(i_m-n)} & i_k = n+1, n+2, \ldots, 2n, \quad k \in [m], \\ 0 \quad \text{otherwise}, \end{cases}$$

where, for two given sets $\mathbf{X}$ and $\mathbf{Y}$, an element belongs to the set $\mathbf{X} - \mathbf{Y}$ means that this element belongs to $\mathbf{X}$, but does not belong to $\mathbf{Y}$. Since $\mathcal{A}_2$ is nonsingular and $\widetilde{\mathcal{B}}$ is symmetric, then, according to [17, Theorem 4.2], we obtain that $\mathrm{Det}(\widetilde{\mathcal{B}}) = \mathrm{Det}(\mathcal{A}_2)^{(m-1)^n} \neq 0$, hence, the number of all quadratic eigenvalue is $2n(m-1)^{2n-1}$ and tensor quadratic eigenvalue are finite.

For the tensor quadratic eigenvalue problem, the perturbation of an algebraic simple quadratic eigenvalue is derived by the following theorem.

THEOREM 4.6 *If λ is an algebraic simple quadratic eigenvalue of $P_2(\lambda)$, associated eigenvector $\mathbf{x}$. Then, there exists an algebraic simple quadratic eigenvalue $\widetilde{\lambda}$ of $P_2(\lambda) + \Delta P_2(\lambda)$ such that*

$$\widetilde{\lambda} = \lambda + O(\epsilon),$$

where $|\epsilon| \leq \epsilon_0$ with $\epsilon_0 > 0$.

Proof According to the above description, if $(\lambda, \mathbf{x})$ is a tensor quadratic eigenpair of $P_2(\lambda)$, then, there exists a vector $\widetilde{\mathbf{x}} = (\lambda^{1/(m-1)}\mathbf{x}^\top, \mathbf{x}^\top)^\top$ and two m-order $2n$ dimensional tensors $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ such that $\widetilde{\mathcal{A}}\widetilde{\mathbf{x}}^{m-1} = \lambda(-\widetilde{\mathcal{B}})\widetilde{\mathbf{x}}^{m-1}$. If $(\lambda, \mathbf{x})$ is an algebraic simple quadratic eigenpair, then $(\lambda, \widetilde{\mathbf{x}})$ is an algebraic simple generalized eigenpair of the pair $(\widetilde{\mathcal{A}}, -\widetilde{\mathcal{B}})$.

According to Theorem 4.1, for the pair $(\widetilde{\mathcal{A}}, -\widetilde{\mathcal{B}})$, there exists an algebraic simple generalized eigenvalue $\widehat{\lambda}$ of $(\widetilde{\mathcal{A}} + \epsilon\widehat{\widetilde{\mathcal{A}}}, -(\widetilde{\mathcal{B}} + \epsilon\widehat{\widetilde{\mathcal{B}}}))$ such that $\widehat{\lambda} = \lambda + O(\epsilon)$. Meanwhile, for these two tensors $\widehat{\widetilde{\mathcal{A}}}$ and $\widehat{\widetilde{\mathcal{B}}}$, there exist three tensors $\Delta\mathcal{A}_l$ $(l = 0, 1, 2)$ such that $\widehat{\lambda}$ is the algebraic simple quadratic eigenvalue of $P_2(\lambda) + \Delta P_2(\lambda)$. Then, this theorem is proved. □

4.3. *Tensor polynomial eigenvalue problem*

In this section, we now consider the tensor polynomial eigenvalue problem, given in Definition 2.5, with $l \geq 3$. Suppose all tensors in $P_l(\lambda)$ are symmetric or mode-symmetric and $\mathcal{A}_l$ is nonsingular.

Choosing $\mu = \lambda^{1/(m-1)}$, we can transform, for the nonzero $\mathbf{x} \in \mathbb{C}^n$, $P(\lambda)\mathbf{x}^{m-1} = \mathbf{0}$ to the formula given as follows.

$$\begin{cases} \lambda\mathcal{A}_l\mathbf{y}_{l-1}^{m-1} + \cdots + \mathcal{A}_1\mathbf{y}_1^{m-1} + \mathcal{A}_0\mathbf{x}^{m-1} = \mathbf{0}, \\ \mathbf{y}_{l-1} = \mu\mathbf{y}_{l-2}, \\ \cdots\cdots\cdots \\ \mathbf{y}_1 = \mu\mathbf{x}. \end{cases}$$

Meanwhile, we have $\mathbf{y}_k = \mu^k\mathbf{x}$ with $k \in [l-1]$.

Hence, a solution $(\lambda, \mathbf{x})$ of $P(\lambda)\mathbf{x}^{m-1} = \mathbf{0}$ also solves the generalized eigenvalue problem $\widehat{\mathcal{A}}\widehat{\mathbf{x}}^{m-1} = \lambda(-\widehat{\mathcal{B}})\widehat{\mathbf{x}}^{m-1}$, where $\widehat{\mathbf{x}} = (\mathbf{y}_{l-1}^\top, \ldots, \mathbf{y}_1^\top, \mathbf{x}^\top)^\top$, and all nonzero elements of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are given as follows, respectively,

$$\begin{aligned} &\widehat{\mathcal{B}}(1:n, 1:n, \ldots, 1:n) = \mathcal{A}_l, \\ &\widehat{\mathcal{B}}(ni+1:(i+1)n, ni+1:(i+1)n, \ldots, ni+1:(i+1)n) = -\mathcal{I},\ i = 1, \ldots, l-1, \\ &\widehat{\mathcal{A}}(1:n, ni+1:(i+1)n, \ldots, ni+1:(i+1)n) = \mathcal{A}_{l-1-i},\ i = 0, 1, \ldots, l-1, \\ &\widehat{\mathcal{A}}(n(i+1)+1:n(i+2), ni+1:(i+1)n, \ldots, ni+1 \\ &\quad :(i+1)n) = \mathcal{I},\ i = 0, 1, \ldots, l-1. \end{aligned}$$

Since $\mathcal{A}_l$ is nonsingular and $\widehat{\mathcal{B}}$ is symmetric, then, according to [17, Theorem 4.2], we obtain that $\mathrm{Det}(\widehat{\mathcal{B}}) = \mathrm{Det}(\mathcal{A}_l)^{(m-1)^{(l-1)n}} \neq 0$, hence, the number of tensor quadratic eigenvalue is $ln(m-1)^{ln-1}$ and all polynomial eigenvalue are finite.

THEOREM 4.7 *Suppose that all tensors in $P(\lambda)$ and $\Delta P(\lambda)$ are mode-symmetric. If λ is an algebraic simple eigenvalue of $P(\lambda)$. Then, there exists an algebraic simple generalized* *eigenvalue* $\widetilde{\lambda}$ *of* $P(\lambda) + \Delta P(\lambda)$ *such that*

$$\widetilde{\lambda} = \lambda + O(\epsilon),$$

where $|\epsilon| \leq \epsilon_0$ *with* $\epsilon_0 > 0$.

Proof According to the above argument, if $(\lambda, \mathbf{x})$ is a polynomial eigenpair of $P_l(\lambda)$, then, there exists a vector $\widehat{\mathbf{x}} = (\mathbf{y}_{l-1}^\top, \ldots, \mathbf{y}_1^\top, \mathbf{x}^\top)^\top$ and two tensors $\hat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$, where $\widehat{\mathcal{B}}$ is singular and mode-symmetric, such that $\widehat{\mathcal{A}}\widehat{\mathbf{x}}^{m-1} = \lambda(-\widehat{\mathcal{B}})\widehat{\mathbf{x}}^{m-1}$. When $(\lambda, \mathbf{x})$ is also an algebraic simple polynomial eigenpair, then $(\lambda, \widehat{\mathbf{x}})$ is an algebraic simple generalized eigenpair of the tensor pair $(\widehat{\mathcal{A}}, -\widehat{\mathcal{B}})$.

According to Theorem 4.1, for the pair $(\widehat{\mathcal{A}}, -\widehat{\mathcal{B}})$, there exists an algebraic simple generalized eigenvalue $\widetilde{\lambda}$ of $(\widehat{\mathcal{A}} + \epsilon\widetilde{\widehat{\mathcal{A}}}, -(\widehat{\mathcal{B}} + \epsilon\widetilde{\widehat{\mathcal{B}}}))$ such that $\widetilde{\lambda} = \lambda + O(\epsilon)$. Meanwhile, for these two tensors $\widetilde{\widehat{\mathcal{A}}}$ and $\widetilde{\widehat{\mathcal{B}}}$, there exist $l+1$ tensors $\Delta\mathcal{A}_i$ $(i = 0, 1, \ldots, l)$ such that $\widehat{\lambda}$ is the algebraic simple polynomial eigenvalue of $P_l(\lambda) + \Delta P_l(\lambda)$. Then, this theorem is over. □

Furthermore, for the tensor polynomial eigenvalue problem, the perturbations of an algebraic simple polynomial eigenvalue have some more precise results, generalized by Theorems 4.2 and 4.3.

THEOREM 4.8 *If* $\lambda \neq 0$ *is an algebraic simple polynomial H-eigenvalue of* $P_l(\lambda)$, *associated polynomial eigenvector* $\mathbf{x} \in \mathbb{R}^n$. *Then, when* $P_l'(\lambda)\mathbf{x}^m \neq 0$, *there exists an algebraic simple polynomial H-eigenvalue* $\widehat{\lambda}$ *of* $P_l(\lambda) + \Delta P_l(\lambda)$ *such that*

$$\frac{|\widehat{\lambda} - \lambda|}{|\lambda|} \leq \epsilon \frac{\left(\sum_{i=0}^{l} |\lambda|^i \alpha_i\right) \|\mathbf{x}\|^m}{|\lambda||P_l'(\lambda)\mathbf{x}^m|} + O(\epsilon^2),$$

where $\|\Delta\mathcal{A}_i\|_F \leq \epsilon\alpha_i$ *with* $0 < \epsilon < 1$ *and* $P_l'(\lambda) = l\lambda^{l-1}\mathcal{A}_l + (l-1)\lambda^{l-2}\mathcal{A}_{l-1} + \cdots + \mathcal{A}_1$.

Proof Since $\lambda \neq 0$ is an algebraic simple polynomial H-eigenvalue of $P_l(\lambda)$, with corresponding eigenvector $\mathbf{x}$, then a normwise condition number of λ can be defined as follows.

$$\kappa(\lambda) := \limsup_{\epsilon \to 0} \left\{ \frac{|\Delta\lambda|}{\epsilon|\lambda|} : (P_l(\lambda + \Delta\lambda) + \Delta P_l(\lambda + \Delta\lambda))(\mathbf{x} + \Delta\mathbf{x})^{m-1} = \mathbf{0}, \right.$$
$$\left. \|\Delta\mathcal{A}_i\|_F \leq \epsilon\alpha_i \right\}.$$

Next, we prove that, if $P_l'(\lambda)\mathbf{x}^m \neq 0$, the following formula holds,

$$\kappa(\lambda) = \frac{\left(\sum_{i=0}^{l} |\lambda|^i \alpha_i\right) \|\mathbf{x}\|^m}{|\lambda||P_l'(\lambda)\mathbf{x}^m|}.$$

The given expression is clearly an upper bound for $\kappa(\lambda)$. We now show that the bound is attained. From the definition of a normwise condition number of λ, we have

$$\Delta\lambda = \frac{\left(\sum_{i=0}^{l} \lambda^i \Delta\mathcal{A}_i\right) \mathbf{x}^m}{P_l'(\lambda)\mathbf{x}^m} + O(\epsilon^2). \tag{4.2}$$

Let $\mathcal{G} = (\underbrace{\mathbf{x} \circ \mathbf{x} \circ \cdots \circ \mathbf{x}}_{m})/\|\mathbf{x}\|^m$. Then $\|\mathcal{G}\|_F = 1$ and $\mathcal{G}\mathbf{x}^m = \|\mathbf{x}\|^m$. Let $\Delta\mathcal{A}_i = -\mathrm{sign}(\lambda^i)\epsilon\alpha_i\mathcal{G}$. Then $\|\Delta\mathcal{A}_i\|_F \le \epsilon\alpha_i$ and the modulus of the first-order term of (4.2) is $\left(\sum_{i=0}^{l} |\lambda|^i \alpha_i\right) \|\mathbf{x}\|^m/|\lambda||P_l'(\lambda)\mathbf{x}^m|$; dividing (4.2) by $\epsilon|\lambda|$ and taking The limit as $\epsilon \to 0$ then gives the desired equality.

From the definition of $\kappa(\lambda)$ we have, for the perturbation system in (4.2),

$$\frac{|\Delta\lambda|}{|\lambda|} \le \kappa(\lambda)\epsilon + O(\epsilon^2).$$

Hence, the proof is over. □

THEOREM 4.9 *If $\lambda \neq 0$ is an algebraic simple polynomial H-eigenvalue of $P_l(\lambda)$, associated eigenvector $\mathbf{x} \in \mathbb{R}^n$. Then, when $P_l'(\lambda)\mathbf{x}^m \neq 0$, there exists an algebraic simple polynomial H-eigenvalue $\widehat{\lambda}$ of $P_l(\lambda) + \Delta P_l(\lambda)$ such that*

$$\frac{|\widehat{\lambda} - \lambda|}{|\lambda|} \le \epsilon \frac{\left(\sum_{i=0}^{l} |\lambda|^i \mathcal{E}_i\right) |\mathbf{x}|^m}{|\lambda||P_l'(\lambda)\mathbf{x}^m|} + O(\epsilon^2),$$

where $|\Delta\mathcal{A}_i| \le \epsilon\mathcal{E}_i$ with $0 < \epsilon < 1$ and $P_l'(\lambda) = l\lambda^{l-1}\mathcal{A}_l + (l-1)\lambda^{l-2}\mathcal{A}_{l-1} + \cdots + \mathcal{A}_1$.

Proof Since $\lambda \neq 0$ is an algebraic simple polynomial H-eigenvalue of $P_l(\lambda)$, with corresponding eigenvector $\mathbf{x}$, then a componentwise condition number for an algebraic simple generalized eigenvalue λ analogous to the normwise condition number is defined by

$$\begin{aligned}\mathrm{cond}(\lambda) := \lim_{\epsilon\to 0}\sup \Bigg\{ \frac{|\Delta\lambda|}{\epsilon|\lambda|} &: (P_l(\lambda + \Delta\lambda)(\mathbf{x} + \Delta\mathbf{x})^{m-1} \\ &+(\Delta P_l(\lambda + \Delta\lambda)(\mathbf{x} + \Delta\mathbf{x})^{m-1} = \mathbf{0}, |\Delta\mathcal{A}_i| \le \epsilon\mathcal{E}_i \Bigg\}.\end{aligned}$$

From this defintion it follows that,

$$\frac{|\Delta\lambda|}{|\lambda|} \le \mathrm{cond}(\lambda)\epsilon + O(\epsilon^2).$$

In the following, we derive the expression of $\mathrm{cond}(\lambda)$. According to the definition of a componentwise condition number for an algebraic simple eigenvalue λ, when $P_l'(\lambda)\mathbf{x}^m \neq 0$, we have

$$\mathrm{cond}(\lambda) \ge \frac{\left(\sum_{i=0}^{l} |\lambda|^i \mathcal{E}_i\right) |\mathbf{x}|^m}{|\lambda||P_l'(\lambda)\mathbf{x}^m|}.$$

Next, we show that the expression for the $\mathrm{cond}(\lambda)$ attained when we choose $\Delta\mathcal{A}_i = -\mathrm{sign}(\lambda^i)\epsilon\mathcal{E}_i \times_1 D \times_2 D \cdots \times_m D$, where $D = \mathrm{diag}(\mathrm{sign}(\mathbf{x}))$. Hence, the proof is over. □

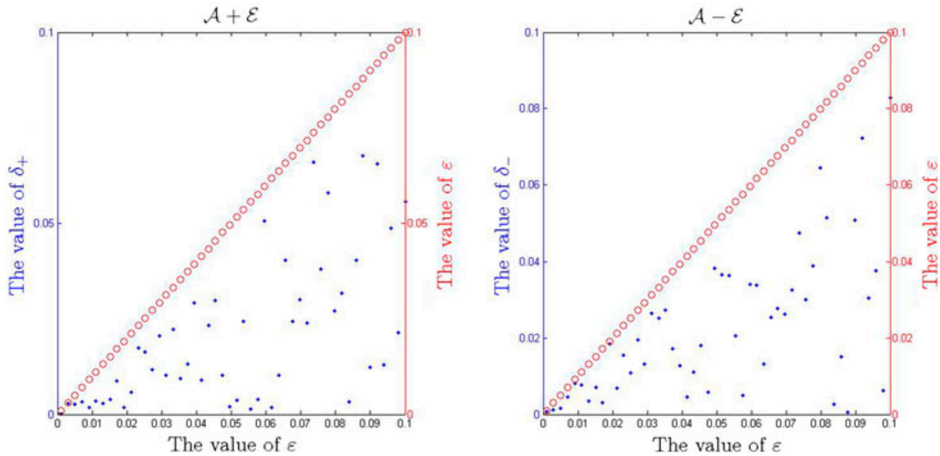

Figure 2. Verification of Corollary 3.1 when $|\mathcal{E}| < \varepsilon\mathcal{A}$ with $\varepsilon = \text{linspace}(0.001, 0.1, 50)$. The blue stars represent the values of δ_+ (δ_-) in each part of this figure (left axis), and the red circles represent the values of ε in each part of this figure (right axis).

Remark 4.2 When the pair $(\lambda, \mathbf{x})$ is not an algebraic simple polynomial H-eigenpair in above two theorems, these two theorems also hold.

5. Numerical examples

The computation of all examples is based on Matlab 2013b and the Matlab Tensor Toolbox [33]. Without loss of generality, we assume that $m = 4$ and $n = 10$, for first two examples in this section and for the third example, let $\langle n\rangle = \{10, 12, 14, 16\}$. We show ill-condition tensors for computing Z-eigenvalue or singular value via three random numerical examples.

Example 5.1 Let $\mathcal{A} \in MST_{m,n}$ be a random nonnegative irreducible tensor with the Z-spectral radius ϱ. The perturbation tensor $\mathcal{E}$ satisfies $|\mathcal{E}| \le \varepsilon\mathcal{A}$, with $0 < \varepsilon < 1$. Denote $\widehat{\varrho}_+$ ($\widehat{\varrho}_-$) by the Z-spectral radius of $\mathcal{A}+\mathcal{E}$ ($\mathcal{A}-\mathcal{E}$) (noting that ϱ, $\widehat{\varrho}_+$, and $\widehat{\varrho}_-$ are computed by the NQZ method [34]). In order to verify Corollary 3.1, we compare ε and $\frac{|\widehat{\varrho}_+-\varrho|}{\varrho}$ $\left(\frac{|\widehat{\varrho}_--\varrho|}{\varrho}\right)$, denoted by δ_+ (δ_-), under two kinds of perturbation tensors: $\mathcal{E} = \text{rand}(1)\varepsilon\mathcal{A}$ and $\mathcal{E} = \varepsilon\mathcal{A}$.

Let $\varepsilon = \text{linspace(a, b, 50)}$ with $0 < a < b < 1$. For these two kinds of perturbation tensors described above, the comparative results are given in Figures 2 and 3, respectively.

Example 5.2 Suppose that $\mathcal{C} \in MST_{m,n}$ is a random nonnegative irreducible tensor with the spectral radius $\lambda_{\max}$ and $\mathcal{B}$ is a diagonal tensor, where its diagonal entries are equal to $\lambda_{\max} + 10$. Let $\mathcal{A} = \mathcal{C}$ where all its diagonal elements are zero. The perturbation tensor pair $(\mathcal{E}, \mathcal{F})$ satisfies $|\mathcal{E}| \le \epsilon\mathcal{A}$ and $|\mathcal{F}| \le \epsilon\mathcal{B}$, with $0 < \epsilon < 1$. Denote λ, $\widehat{\lambda}_+$ and $(\widehat{\lambda}_-)$ by the Perron roots of the tensor pair $(\mathcal{A}, \mathcal{B})$, $(\mathcal{A}+\mathcal{E}, \mathcal{B}+\mathcal{F})$ and $(\mathcal{A}-\mathcal{E}, \mathcal{B}-\mathcal{F})$ (noting that σ, $\widehat{\sigma}_+$ and $\widehat{\sigma}_-$ are computed by the NQZ method [34]). In order to verify Corollary 4.1,

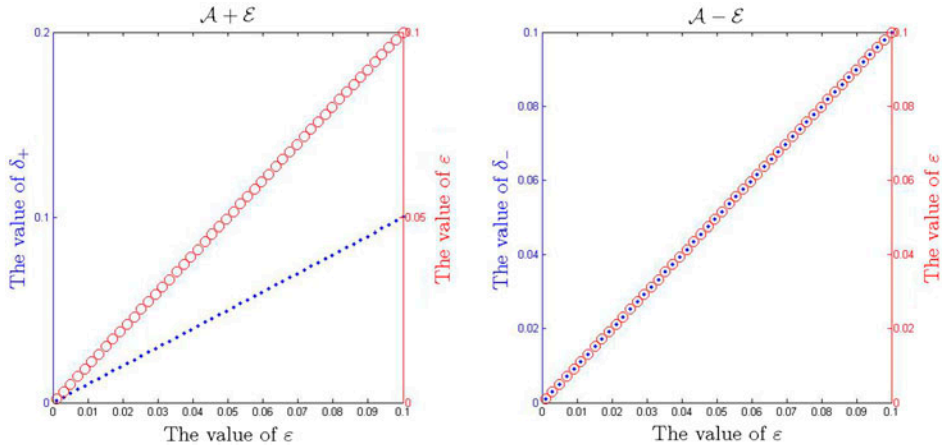

Figure 3. Verification of Corollary 3.1 when $|\mathcal{E}| = \varepsilon\mathcal{A}$ with $\varepsilon = \text{linspace}(0.001, 0.1, 50)$. The blue stars represent the values of δ_+ (δ_-) in each part of this figure (left axis), and the red circles represent the values of ε in each part of this figure (right axis).

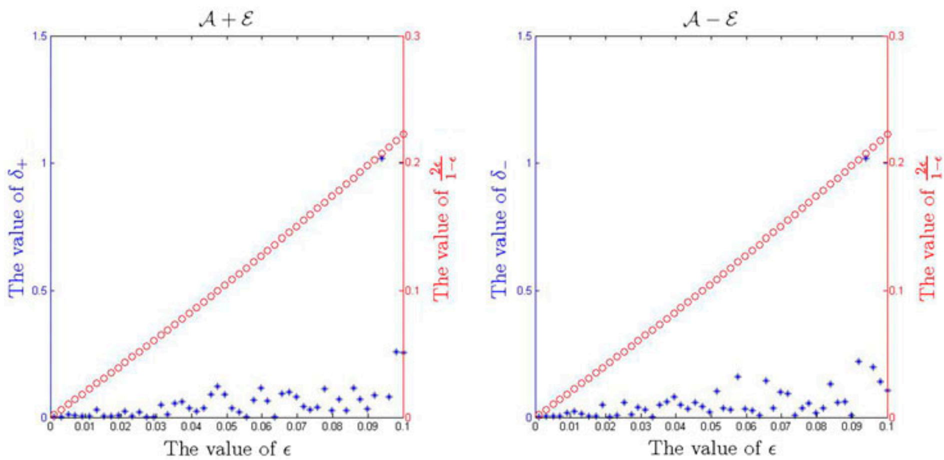

Figure 4. Verification of Corollary 4.1 when $|\mathcal{E}| < \epsilon\mathcal{A}$ and $|\mathcal{F}| < \epsilon\mathcal{B}$ with $\epsilon = \text{linspace}(0.001, 0.1, 50)$. The blue stars represent the values of δ_+ (δ_-) in each part of this figure (left axis), and the red circles represent the values of $\frac{2\epsilon}{1-\epsilon}$ in each part of this figure (right axis).

we compare $\frac{2\epsilon}{1-\epsilon}$ and $\frac{|\widehat{\lambda}_+ - \lambda|}{\lambda}$ $\left(\frac{|\widehat{\lambda}_- - \lambda|}{\lambda}\right)$, denoted by δ_+ (δ_-), under two kinds of perturbation tensor pairs: $(\mathcal{E}, \mathcal{F}) = \text{rand}(1)\epsilon(\mathcal{A}, \mathcal{B})$ and $(\mathcal{E}, \mathcal{F}) = \epsilon(\mathcal{A}, \mathcal{B})$.

Let $\epsilon = \text{linspace(a, b, 50)}$ with $0 < a < b < 1$. For these two kinds of perturbation tensor pairs described above, the comparative results are given in Figures 4 and 5, respectively.

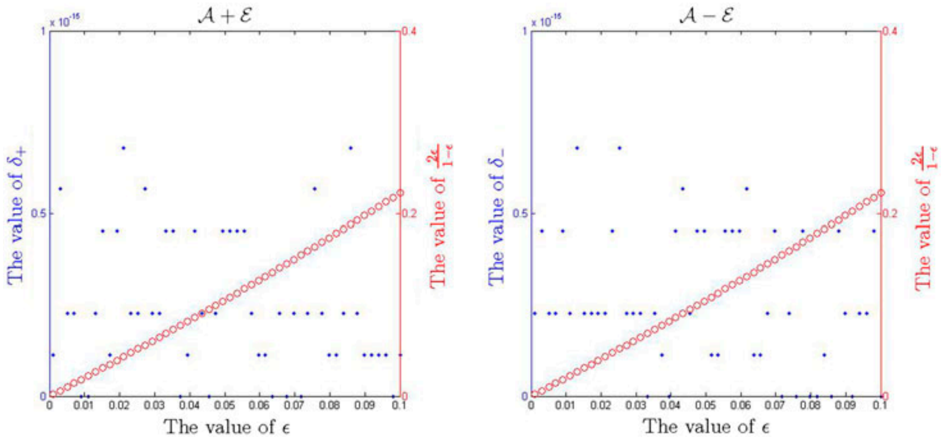

Figure 5. Verification of Corollary 4.1 when $|\mathcal{E}| = \epsilon\mathcal{A}$ and $|\mathcal{F}| = \epsilon\mathcal{B}$ with $\epsilon =$ linspace(0.001, 0.1, 50). The blue stars represent the values of δ_+ (δ_-) in each part of this figure (left axis), and the red circles represent the values of $\frac{2\epsilon}{1-\epsilon}$ in each part of this figure (right axis).

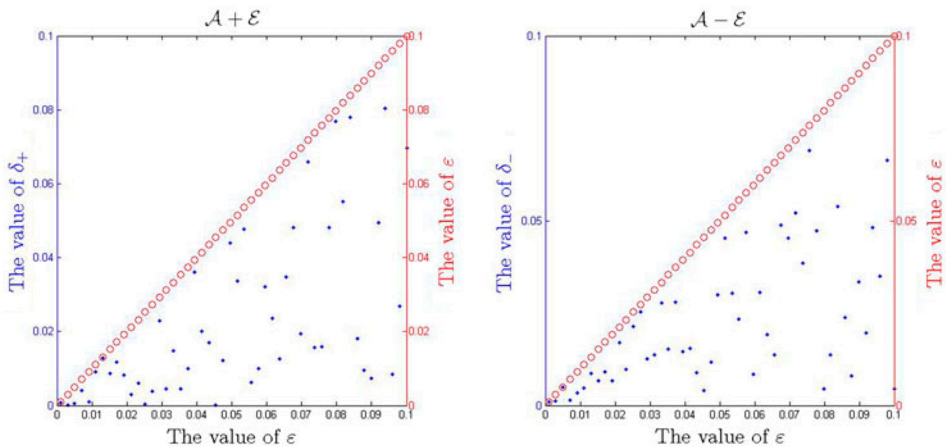

Figure 6. Verification of Corollary 3.3 when $|\mathcal{E}| < \varepsilon\mathcal{A}$ with $\varepsilon =$ linspace(0.001, 0.1, 50). The blue stars represent the values of δ_+ (δ_-) in each part of this figure (left axis), and the red circles represent the values of ε in each part of this figure (right axis).

Example 5.3 Let $\mathcal{A} \in T_{m,\langle n\rangle}$ be a random nonnegative irreducible tensor with the largest singular value σ. The perturbation tensor $\mathcal{E}$ satisfies $|\mathcal{E}| \leq \varepsilon\mathcal{A}$, with $0 < \varepsilon < 1$. Denote $\widehat{\sigma}_+$ ($\widehat{\sigma}_-$) by the Z-spectral radius of $\mathcal{A}+\mathcal{E}$ ($\mathcal{A}-\mathcal{E}$) (Noting that σ, $\widehat{\sigma}_+$, and $\widehat{\sigma}_-$ are computed by the NQZ method [34]). In order to verify Corollary 3.3, we compare ε and $\frac{|\widehat{\sigma}_+-\sigma|}{\sigma}$ ($\frac{|\widehat{\sigma}_--\sigma|}{\sigma}$), denoted by δ_+ (δ_-), under two kinds of perturbation tensors: $\mathcal{E} = \text{rand}(1)\varepsilon\mathcal{A}$ and $\mathcal{E} = \varepsilon\mathcal{A}$.

Let $\varepsilon =$ linspace(a, b, 50) with $0 < a < b < 1$. For these two kinds of perturbation tensors described above, the comparative results are given in Figures 6 and 7, respectively.

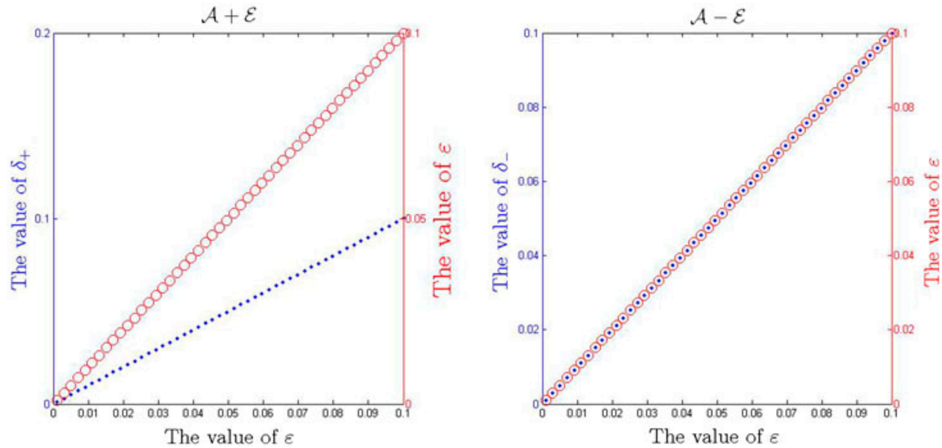

Figure 7. Verification of Corollary 3.3 when $|\mathcal{E}| = \varepsilon\mathcal{A}$ with ε = linspace(0.001, 0.1, 50). The blue stars represent the values of δ_+ (δ_-) in each part of this figure (left axis), and the red circles represent the values of ε in each part of this figure (right axis).

6. Conclusion

In this paper, for a mode-symmetric tensor, the perturbation of an algebraic simple eigenvalue and E-eigenvalue are considered. Based on symmetric or mode-symmetric embedding, for a given tensor $\mathcal{A} \in T_{m,\langle n\rangle}$, we obtain perturbation bounds about a singular value of $\mathcal{A}$. Furthermore, we define the tensor polynomial eigenvalue problem and derive perturbation bounds about an algebraic simple polynomial eigenvalue. In particular, we focus on the tensor generalized eigenvalue problem and tensor quadratic eigenvalue problem.

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Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Qi L. Eigenvalues of a real supersymmetric tensor. J Symbolic Comput. 2005;40:1302–1324. doi:10.1016/j.jsc.2005.05.007.

[2] Lim L. Singular values and eigenvalues of tensors: a variational approach. In: IEEE CAMSAP 2005: First International Workshop on Computational Advances in Multi-Sensor Adaptive Processing. IEEE; 2005. p. 129–132.
[3] Golub GH, Van Loan CF. Matrix computations. 4th ed. Baltimore (MD): Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press; 2013.
[4] Chang KC, Pearson K, Zhang T. Perron–Frobenius theorem for nonnegative tensors. Commun. Math. Sci. 2008;6:507–520. http://projecteuclid.org/euclid.cms/1214949934.
[5] Chang KC, Pearson K, Zhang T. On eigenvalue problems of real symmetric tensors. J. Math. Anal. Appl. 2009;350:416–422. doi:10.1016/j.jmaa.2008.09.067.
[6] Kolda TG, Mayo JR. An adaptive shifted power method for computing generalized tensor eigenpairs. SIAM J. Matrix Anal. Appl. 2014;35:1563–1581. doi:10.1137/140951758.
[7] Cui CF, Dai YH, Nie J. All real eigenvalues of symmetric tensors. SIAM J. Matrix Anal. Appl. 2014;35:1582–1601.
[8] Qi L, Wang Y, Wu EX. D-eigenvalues of diffusion kurtosis tensors. J. Comput. Appl. Math. 2008;221:150–157. doi:10.1016/j.cam.2007.10.012.
[9] Ding W, Wei Y. Generalized tensor eigenvalue problems. SIAM J. Matrix Anal. Appl. 2015; 36:1073–1099.
[10] Li W, Ng MK. The perturbation bound for the spectral radius of a nonnegative tensor. Adv. Numer. Anal. 2014:10 p. Article ID 109525. doi:10.1155/2014/109525.
[11] Li W, Ng MK. Some bounds for the spectral radius of nonnegative tensors. Numer. Math. 2015;130:315–335.
[12] Cichocki A, Zdunek R, Phan AH, Amari Si. Nonnegative matrix and tensor factorizations: applications to exploratory multi-way data analysis and blind source separation. New York (NY): John Wiley & Sons; 2009.
[13] Kolda TG, Bader BW. Tensor decompositions and applications. SIAM Rev. 2009;51:455–500. doi:10.1137/07070111X.
[14] De Lathauwer L, De Moor B, Vandewalle J. A multilinear singular value decomposition. SIAM J. Matrix Anal. Appl. 2000;21:1253–1278. doi:10.1137/S0895479896305696.
[15] Berman A, Plemmons RJ. Nonnegative matrices in the mathematical sciences. Vol. 9, Classics in applied mathematics. Philadelphia (PA): Society for Industrial and Applied Mathematics; 1994. doi:10.1137/1.9781611971262.
[16] Chang KC, Qi L, Zhang T. A survey on the spectral theory of nonnegative tensors. Numer. Linear Algebra Appl. 2013;20:891–912. doi:10.1002/nla.1902.
[17] Hu S, Huang ZH, Ling C, Qi L. On determinants and eigenvalue theory of tensors. J. Symbolic Comput. 2013;50:508–531.
[18] Ding W, Qi L, Wei Y. $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors. Linear Algebra Appl. 2013;439:3264–3278.
[19] Zhang T. Existence of real eigenvalues of real tensors. Nonlinear Anal. 2011;74:2862–2868. doi:10.1016/j.na.2011.01.008.
[20] Wilkinson JH. The algebraic eigenvalue problem. New York (NY): Oxford University Press; 1965.
[21] Demmel JW. Applied numerical linear algebra. Philadelphia (PA): Society for Industrial and Applied Mathematics (SIAM); 1997. doi:10.1137/1.9781611971446.
[22] Stewart GW, Sun JG. Matrix perturbation theory. Boston (MA): Computer Science and Scientific Computing, Academic Press Inc; 1990.
[23] Ragnarsson S, Van Loan CF. Block tensors and symmetric embeddings. Linear Algebra Appl. 2013;438:853–874. doi:10.1016/j.laa.2011.04.014.
[24] Chen Z, Lu L. A tensor singular values and its symmetric embedding eigenvalues. J. Comput. Appl. Math. 2013;250:217–228. doi:10.1016/j.cam.2013.03.014.
[25] Cartwright D, Sturmfels B. The number of eigenvalues of a tensor. Linear Algebra Appl. 2013;438:942–952. doi:10.1016/j.laa.2011.05.040.

[26] Kolda TG, Mayo JR. Shifted power method for computing tensor eigenpairs. SIAM J. Matrix Anal. Appl. 2011;32:1095–1124. doi:10.1137/100801482.
[27] Ahlfors LV. Complex analysis: an introduction of the theory of analytic functions of one complex variable. 2nd ed. New York (NY): McGraw-Hill Book Co.; 1966.
[28] Chevalley C. Introduction to the theory of algebraic functions of one variable. No. VI. Mathematical surveys. New York (NY): American Mathematical Society; 1951.
[29] Gohberg I, Koltracht I. Mixed, componentwise, and structured condition numbers. SIAM J. Matrix Anal. Appl. 1993;14:688–704. doi:10.1137/0614049.
[30] Yang Y, Yang Q. Further results for Perron–Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. 2010;31:2517–2530.
[31] Elsner L, Koltracht I, Neumann M, Xiao D. On accurate computations of the Perron root. SIAM J. Matrix Anal. Appl. 1993;14:456–467. doi:10.1137/0614032.
[32] Zhang T, Golub GH. Rank-one approximation to high order tensors. SIAM J. Matrix Anal. Appl. 2001;23:534–550.
[33] Bader BW, Kolda TG. Matlab Tensor Toolbox Version 2.5. 2012 Jan. Available from: http://www.sandia.gov/tgkolda/TensorToolbox/.
[34] Ng M, Qi L, Zhou G. Finding the largest eigenvalue of a nonnegative tensor. SIAM J. Matrix Anal. Appl. 2009;31:1090–1099. doi:10.1137/09074838X.