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CIRCULANT TENSORS WITH APPLICATIONS TO SPECTRAL HYPERGRAPH THEORY AND STOCHASTIC PROCESS

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Abstract. Circulant tensors naturally arise from stochastic process and spectral hypergraph theory. The joint moments of stochastic processes are symmetric circulant tensors. The adjacency, Laplacian and signless Laplacian tensors of circulant hypergraphs are also symmetric circulant tensors. The adjacency, Laplacian and signless Laplacian tensors of directed circulant hypergraphs are circulant tensors, but they are not symmetric in general. In this paper, we study spectral properties of circulant tensors and their applications in spectral hypergraph theory and stochastic process. We show that in certain cases, the largest H-eigenvalue of a circulant tensor can be explicitly identified. In particular, the largest H-eigenvalue of a nonnegative circulant tensor can be explicitly identified. This confirms the results in circulant hypergraphs and directed circulant hypergraphs. We prove that an even order circulant B_0 tensor is always positive semi-definite. This shows that the Laplacian tensor and the signless Laplacian tensor of a directed circulant even-uniform hypergraph are positive semi-definite. If a stochastic process is mth order stationary, where m is even, then its mth order moment, which is a circulant tensor, must be positive semi-definite. In this paper, we give various conditions for an even order circulant tensor to be positive semi-definite.

1. **Introduction.** Circulant matrices are Topelitz matrices. They form an important class of matrices in linear algebra and its applications [5, 9, 29]. As a natural extension of circulant matrices, circulant tensors naturally arise from stochastic process and spectral hypergraph theory.

Denote $[n] := \{1, \cdots, n\}$. A real mth order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{j_1 \cdots j_m})$ is a multi-array of real entries $a_{j_1 \cdots j_m}$, where $j_l \in [n]$ for $l \in [m]$. Let

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$\mathcal{A} = (a_{j_1\cdots j_m})$ be a real mth order n-dimensional tensor. If for $j_l \in [n-1], l \in [m]$, we have

$$a_{j_1\cdots j_m} = a_{j_1+1\cdots j_m+1},$$

then we say that $\mathcal{A}$ is an mth order **Toeplitz tensor**. If for $j_l, k_l \in [n], k_l = j_l + 1 \mod(n)$, $l \in [m]$, we have

$$a_{j_1\cdots j_m} = a_{k_1\cdots k_m}, \tag{1}$$

then we say that $\mathcal{A}$ is an mth order **circulant tensor**. Clearly, a circulant tensor is a Toeplitz tensor. By the definition, all the diagonal entries of a Toeplitz tensor are the same. Thus, we may say the **diagonal entry** of a Toeplitz or circulant tensor. In fact, if $\mathcal{A} = (a_{j_1\cdots j_m})$ is a Toeplitz tensor, we have that for $j_l \in [n], l \in [m]$,

$$a_{j_1\cdots j_m} = a_{j_1+k\cdots j_m+k}, \quad \forall\ 0 \le k \le \min\{n-j_1, \cdots, n-j_m\}.$$

When $m = 3$, the definition of Toeplitz tensors is consistent with Definition 3.1 of [1]. Tensors which are circulant with respect to two modes were studied in [27]. Note that the circulant tensors considered here are circulant with respect to all the modes.

We denote by $T_{m,n}$ the set of all real mth order n-dimensional tensors. Then $T_{m,n}$ is a linear space of dimension n^m. Denote the set of all real mth order n-dimensional circulant tensors by $C_{m,n}$. Then $C_{m,n}$ is a linear subspace of $T_{m,n}$, with dimension n^{m-1}. Let $\mathcal{A} = (a_{j_1\cdots j_m}) \in T_{m,n}$. If the entries $a_{j_1\cdots j_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a **symmetric tensor**. Denote the set of all real symmetric mth order n-dimensional tensors by $S_{m,n}$. Then $S_{m,n}$ is a linear subspace of $T_{m,n}$.

Let $\mathcal{A} = (a_{j_1\ldots j_m}) \in T_{m,n}$ and $\mathbf{x} \in \Re^n$. Then $\mathcal{A}\mathbf{x}^m$ is a homogeneous polynomial of degree m, defined by

$$\mathcal{A}\mathbf{x}^m = \sum_{j_1,\ldots,j_m=1}^{n} a_{j_1\ldots j_m} x_{j_1} \ldots x_{j_m}.$$

Assume that m is even. If $\mathcal{A}\mathbf{x}^m \ge 0$ for all $\mathbf{x} \in \Re^n$, then we say that $\mathcal{A}$ is positive semi-definite. If $\mathcal{A}\mathbf{x}^m > 0$ for all $\mathbf{x} \in \Re^n, \mathbf{x} \ne \mathbf{0}$, then we say that $\mathcal{A}$ is positive definite. Clearly, if m is odd, there is no nontrivial positive semi-definite tensors. The definition of positive semi-definite tensors was first introduced in [21] for symmetric tensors. Here we extend that definition to any tensors in $T_{m,n}$. To the best of our knowledge, positive semi-definite tensors and their corresponding homogeneous polynomials have applications in automatical control [21], magnetic resonance imaging [3, 10, 25, 26] and spectral hypergraph theory [11, 16, 22].

In this paper, we study spectral properties of circulant tensors and their applications in spectral hypergraph theory and stochastic process. In the next section, we study the applications of circulant tensors in stochastic process and spectral hypergraph theory. In particular, we study what are the concerns of the properties of circulant tensors in these applications. If a stochastic process is mth order stationary, where m is even, then its mth order moment, which is a circulant tensor, must be positive semi-definite. Hence, in the following three sections, we give various conditions for an even order circulant tensor to be positive semi-definite.

It is well-known that a circulant matrix is generated from the first row vector of that circulant matrix [5, 9, 29]. We may also generate a circulant tensor in this way. In Section 3, we define the **root tensor** $\mathcal{A}_1 \in T_{m-1,n}$ and the **associated tensor** $\bar{\mathcal{A}}_1 \in T_{m-1,n}$ for a circulant tensor $\mathcal{A} \in C_{m,n}$. We show that $\mathcal{A}$ is generated from

$\mathcal{A}_1$. It is also well-known that the eigenvalues and eigenvectors of a circulant matrix can be written explicitly [5, 9, 29]. After reviewing the definitions of eigenvalues and H-eigenvalues of a tensor in $T_{m,n}$, we show that for any circulant tensors $\mathcal{A} \in C_{m,n}$ with any $m \geq 2$, including circulant matrices in $C_{2,n}$, the same n independent vectors are their eigenvectors. For a circulant tensor $\mathcal{A} \in C_{m,n}$, we introduce a one variable polynomial $f_{\mathcal{A}}(t)$ as its **associated polynomial**. Using $f_{\mathcal{A}}(t)$, we may find the n eigenvalues $\lambda_k(\mathcal{A})$ for $k = 0, \cdots, n-1$, corresponding to these n eigenvectors. We call these n eigenvalues the native eigenvalues of that circulant tensor $\mathcal{A}$. In particular, the first native eigenvalue $\lambda_0(\mathcal{A})$, which is equal to the sum of all the entries of the root tensor, is an H-eigenvalue of $\mathcal{A}$. We show that when the associated tensor is a nonnegative tensor, $\lambda_0(\mathcal{A})$ is the largest H-eigenvalue of $\mathcal{A}$. This confirms the results in circulant hypergraphs and directed circulant hypergraphs.

In Section 4, we study positive semi-definiteness of an even order circulant tensor. Recently, it was proved in [24] that an even order symmetric B_0 tensor is positive semi-definite, and an even order symmetric B tensor is positive definite. In this section, for any tensor $\mathcal{A} \in T_{m,n}$, we define a symmetric tensor $\mathcal{B} \in S_{m,n}$ as its **symmetrization**, and denote it $sym(\mathcal{A})$. An even order tensor is positive semi-definite or positive definite if and only if its symmetrization is positive semi-definite or positive definite, respectively. We show that the symmetrization of a circulant B_0 tensor is still a circulant B_0 tensor, and the symmetrization of a circulant B tensor is still a circulant B tensor. This implies that an even order circulant B_0 tensor is always positive semi-definite, and an even order circulant B tensor is always positive definite. Thus, the Laplacian tensor and the signless Laplacian tensor of a directed circulant even-uniform hypergraph are positive semi-definite. Some other sufficient conditions for positive semi-definiteness of an even order circulant tensor are also given in that section.

In Section 5, we study positive semi-definiteness of even order circulant tensors with special root tensors. When the root tensor $\mathcal{A}_1$ is a diagonal tensor, we show that in this case, the n native eigenvalues are indeed all the eigenvalues of that circulant tensor $\mathcal{A}$, with some adequate multiplicities and more eigenvectors. We give all such eigenvectors explicitly. Then we present some conditions for an even order circulant tensor with a diagonal root tensor to be positive semi-definite. When the root tensor $\mathcal{A}_1$ itself is a circulant tensor, we call $\mathcal{A}$ a **doubly circulant** tensor. We show that when m is even and $\mathcal{A}_1$ is a doubly circulant tensor itself, if the root tensor of $\mathcal{A}_1$ is positive semi-definite, then $\mathcal{A}$ is also positive semi-definite. An algorithm for determining positive semi-definiteness of an even order circulant tensor with a diagonal root tensor, and its numerical experiments are also presented.

Throughout this paper, we assume that $m, n \geq 2$. We use small letters $x, u, v, \alpha, \cdots$, for scalers, small bold letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. We reserve the letter i for the imaginary unit. Denote $\mathbf{1}_j \in \Re^n$ as the jth unit vector for $j \in [n]$, $\mathbf{0}$ the zero vector in $\Re^n$, $\mathbf{1}$ the all 1 vector in $\Re^n$, and $\hat{\mathbf{1}}$ the alternative sign vector $(1, -1, 1, -1, \cdots)^\top \in \Re^n$. We call a tensor in $T_{m,n}$ the **identity tensor** of $T_{m,n}$, and denote it $\mathcal{I}$ if all of its diagonal entries are 1 and all of its off-diagonal entries are 0.

2. Applications in stochastic process and hypergraphs.

In this section, we study stochastic process, circulant hypergraphs and directed circulant hypergraphs. We show that circulant tensors naturally arise from these applications. We study what are the concerns on the properties of circulant tensors in these applications.

2.1. **Stochastic process.** For a vector-valued random variable $\mathbf{x} = (x_1, \dots, x_n)$, the joint moment of $\mathbf{x}$ is defined as the expected value of their product:

$$\mathrm{Mom}(x_1, \cdots, x_n) = E\{x_1 x_2 \cdots x_n\}.$$

The mth order moment of the stochastic vector $\mathbf{x} = (x_1, \dots, x_n)$ is a mth order n-dimensional tensor, defined by

$$M_m(\mathbf{x}) = [\mathrm{Mom}(x_{i_1}, \cdots, x_{i_m})]_{i_1,\cdots,i_m=1}^n .$$

By definition, we have: (i) $M_m(\mathbf{x})$ is symmetric; (ii) when $m = 2$, $M_2(\mathbf{x})$ is the covariance matrix of the stochastic vector $\mathbf{x}$ with mean $\mathbf{0}$; (iii) if $\mathbf{y} = A^\top \mathbf{x}$ with $A \in \Re^{n\times N}$, then $M_m(\mathbf{y}) = M_m(\mathbf{x})A^m$, where the product is defined in Section 3.

On the other hand, a discrete stochastic process $\mathbf{x} = \{x_k, k = 1, 2, \cdots\}$ is called mth order stationary if for any points $t_1, \cdots, t_m \in \mathbb{Z}_+$, the joint distribution of

$$\{x_{t_1}, \cdots, x_{t_m}\}$$

is the same as the joint distribution of

$$\{x_{t_1+1}, \cdots, x_{t_m+1}\}.$$

A stochastic process is stationary if it is mth order stationary for any positive integer m. It is well-known that a Markov chain is a stationary process if the initial state is chosen according to the stationary distribution. We can see that the mth order moment of a mth order stationary stochastic process $\mathbf{x}$, $M_m(\mathbf{x})$, is a mth order Toeplitz tensor with infinite dimension. In practice, it may be difficult to handle this case. Instead, a stochastic process $\mathbf{x} = \{x_k, k = 1, 2, \cdots\}$ can be approximated by a stochastic process with period n, $\mathbf{x}^n = \{x_k^n, k = 1, 2, \cdots\}$, where $x_k^n = x_j^n$ if $k = j \bmod(n)$. For example, $\mathbf{x}^1 = \{x_1, x_1, x_1, x_1, \cdots\}$ and $\mathbf{x}^2 = \{x_1, x_2, x_1, x_2, \cdots\}$. We can see that the mth order moment of $\mathbf{x}^n$ can be expressed by a mth order n-dimensional tensor $M_m(\mathbf{x}^n)$ since

$$\mathrm{Mom}(x_{i_1}^n, \cdots, x_{i_m}^n) = \mathrm{Mom}(x_{j_1}^n, \cdots, x_{j_m}^n),$$

where $i_k = j_k \bmod(n)$ for $k \in [m]$. If the stochastic process $\mathbf{x}$ is mth order stationary, the mth order moment of the approximation with period n, $M_m(\mathbf{x}^n)$, is a circulant tensor of order m and dimension n.

Given a stochastic process $\mathbf{x}^n$ with period n, by Theorem 7.1 of Chapter 9 [28], one can derive that $\mathbf{x}^n$ is the second order stationary if and only if $M_2(\mathbf{x}^n)$ is positive semi-definite. In general, $M_m(\mathbf{x}^n)$ is positive semi-definite when the order m is even.

Proposition 1. *For a stochastic process $\mathbf{x}^n$ with period n, $M_m(\mathbf{x}^n)$ is positive semi-definite when m is even.*

Proof. For any $\alpha \in \Re^n$, we have

$$\begin{aligned} M_m(\mathbf{x}^n)\alpha^m &= \sum_{i_1,\cdots,i_m=1}^n \alpha_{i_1}\cdots\alpha_{i_m} \mathrm{Mom}(x_{i_1}^n, \cdots, x_{i_m}^n) \\ &= \mathrm{Mom}\left(\sum_{i_1=1}^n \alpha_{i_1} x_{i_1}^n, \cdots, \sum_{i_m=1}^n \alpha_{i_m} x_{i_m}^n\right) \\ &= E\left\{\left(\sum_{i=1}^n \alpha_i x_i^n\right)^m\right\}. \end{aligned}$$

Then, $M_m(\mathbf{x}^n)\alpha^m \geq 0$ since m is even, which means $M_m(\mathbf{x}^n)$ is positive semi-definite. $\square$

This shows that positive semi-definiteness of curculant tensors is important. In this paper, we will study conditions of positive semi-definiteness of circulant tensors.

2.2. **Circulant hypergraphs.** In the recent years, a number of papers appeared in spectral hypergraph theory via tensors [4, 11, 12, 13, 14, 16, 19, 22, 23, 31, 30].

A hypergraph G is a pair (V, E), where $V = [n]$ is the set of vertices and E is a set of subsets of V. The elements of E are called edges. An edge $e \in E$ has the form $e = (j_1, \cdots, j_m)$, where $j_l \in V$ for $l \in [m]$ and $j_l \neq j_k$ if $l \neq k$. The order of $j_1, \cdots, j_m$ is irrelevant for an edge. Given an integer $m \geq 2$, a hypergraph G is said to be m-uniform if $|e| = m$ for all $e \in E$, where $|e|$ denotes number of vertices in the edge e. The degree of a vertex $j \in V$ is defined as $d(j) = |E(j)|$, where $E(j) = \{e \in E : j \in e\}$. If for all $j \in V$, the degrees $d(j)$ have the same value d, then G is called a regular hypergraph, or a d-regular hypergraph to stress its degree d.

An m-uniform hypergraph $G = (V, E)$ with $V = [n]$ is called a **circulant hypergraph** if G has the following property: if $e = (j_1, \cdots, j_m) \in E$, $k_l = j_l + 1 \mod(n), l \in [m]$, then $\bar{e} = (k_1, \cdots, k_m) \in E$. Clearly, a circulant hypergraph is a regular hypergraph.

For an m-uniform hypergraph $G = (V, E)$ with $V = [n]$, the adjacency tensor $\mathcal{A} = \mathcal{A}(G)$ is a tensor in $S_{m,n}$, defined by $\mathcal{A} = (a_{j_1 \cdots j_m})$,

$$a_{j_1 \cdots j_m} = \frac{1}{(m-1)!} \begin{cases} 1 & \text{if } (j_1, \cdots, j_m) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The degree tensor $\mathcal{D} = \mathcal{D}(G)$ of G, is a diagonal tensor in $S_{m,n}$, with its jth diagonal entry as $d(j)$. The Laplacian tensor and the signless Laplacian tensor of G are defined by $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ and $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$, which were initially introduced in [22], and studied further in [12, 14, 23]. The adjacency tensor, the Laplacian tensor and the signless Laplacian tensors of a uniform hypergraph are symmetric. The adjacency tensor and the signless Laplacian tensor are nonnegative. The Laplacian tensor and the signless Laplacian tensor of an even-uniform hypergraph are positive semi-definite [22]. It is known [22] that the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of a uniform hypergraph always have H-eigenvalues. The smallest H-eigenvalue of the Laplacian tensor is zero with an H-eigenvector $\mathbf{1}$. The largest H-eigenvalues of the adjacency tensor and the signless Laplacian tensor of a d-regular hypergraph are d and $2d$ respectively [22].

Clearly, the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of a circulant hypergraph are symmetric circulant tensors.

2.3. **Directed circulant hypergraphs.** Directed hypergraphs have found applications in imaging processing [6], optical network communications [17], computer science and combinatorial optimization [7]. However, unlike spectral theory of undirected hypergraphs, it is almost blank for spectral theory of directed hypergraphs.

A directed hypergraph G is a pair (V, A), where $V = [n]$ is the set of vertices and A is a set of ordered subsets of V. The elements of A are called arcs. An arc $e \in A$ has the form $e = (j_1, \cdots, j_m)$, where $j_l \in V$ for $l \in [m]$ and $j_l \neq j_k$ if $l \neq k$. The order of $j_2, \cdots, j_m$ is irrelevant. But the order of j_1 is special. The vertex j_1 is called the tail of the arc e. It must be in the first position of the arc. The other vertices $j_2, \cdots, j_m$ are called the heads of the arc e. Similar to m-uniform hypergraphs, we have m-uniform directed hypergraphs. The degree of a vertex $j \in V$ is defined as $d(j) = |A(j)|$, where $A(j) = \{e \in A : j \text{ is a tail of } e\}$. If for all $j \in V$, the degrees $d(j)$ have the same value d, then G is called a directed regular hypergraph, or a directed d-regular hypergraph.

Similarly, an m-uniform directed hypergraph $G = (V, A)$ with $V = [n]$ is called a **directed circulant hypergraph** if G has the following property: if $e = (j_1, \cdots, j_m) \in A$, $k_l = j_l + 1 \bmod(n), l \in [m]$, then $\bar{e} = (k_1, \cdots, k_m) \in A$. Clearly, a directed circulant hypergraph is a regular directed hypergraph.

For an m-uniform directed hypergraph $G = (V, A)$ with $V = [n]$, the adjacency tensor $\mathcal{A} = \mathcal{A}(G)$ is a tensor in $T_{m,n}$, defined by $\mathcal{A} = (a_{j_1 \cdots j_m})$,

$$a_{j_1 \cdots j_m} = \frac{1}{(m-1)!} \begin{cases} 1 & \text{if } (j_1, \cdots, j_m) \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then, the degree tensor $\mathcal{D} = \mathcal{D}(G)$ of G, is a diagonal tensor in $T_{m,n}$, with its jth diagonal entry as $d(j)$. The Laplacian tensor and the signless Laplacian tensor of G are defined by $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ and $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$.

The adjacency tensor, the Laplacian tensor and the signless Laplacian tensors of a uniform directed hypergraph are not symmetric in general. The adjacency tensor and the signless Laplacian tensor are still nonnegative. In general, we do not know if the Laplacian tensor and the signless Laplacian tensor of an even-uniform directed hypergraph are positive semi-definite or not. We may still show that the smallest H-eigenvalue of the Laplacian tensor of an m-uniform directed hypergraph is zero with an H-eigenvector $\mathbf{1}$, and the largest H-eigenvalues of the adjacency tensor and the signless Laplacian tensor of a directed d-regular hypergraph are d and $2d$ respectively.

Clearly, the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of a directed circulant hypergraph are circulant tensors. In general, they are not symmetric.

3. Eigenvalues of a circulant tensor.

It is well-known that the other row vectors of a circulant matrix are rotated from the first row vector of that circulant matrix [5, 9, 29]. We may also regard a circulant tensor in this way. In order to do this, we introduce **row tensors** for a tensor $\mathcal{A} = (a_{j_1 \cdots j_m}) \in T_{m,n}$. Let $\mathcal{A}_k = (a^{(k)}_{j_1 \cdots j_{m-1}}) \in T_{m-1,n}$ be defined by $a^{(k)}_{j_1 \cdots j_{m-1}} \equiv a_{k j_1 \cdots j_{m-1}}$. We call $\mathcal{A}_k$ the kth row tensor of $\mathcal{A}$ for $k \in [n]$. Let $\mathcal{A}$ be a circulant tensor. Then we see that the row tensors $\mathcal{A}_k$ for $k = 2, \cdots, n$, are generated from $\mathcal{A}_1 = (\alpha_{j_1 \cdots j_{m-1}})$, where $\alpha_{j_1 \cdots j_{m-1}} \equiv a^{(1)}_{j_1 \cdots j_{m-1}}$. We call $\mathcal{A}_1$ the **root tensor** of $\mathcal{A}$. We see that $c_0 = \alpha_{1 \cdots 1}$ is the diagonal entry of $\mathcal{A}$. The off-diagonal entries of $\mathcal{A}$ are generated by the other entries of $\mathcal{A}_1$. Thus, we define $\bar{\mathcal{A}}_1 = (\bar{\alpha}_{j_1 \cdots j_{m-1}}) \in T_{m-1,n}$ by $\bar{\alpha}_{1 \cdots 1} = 0$ and $\bar{\alpha}_{j_1 \cdots j_{m-1}} = \alpha_{j_1 \cdots j_{m-1}}$ if $(j_1, \cdots, j_{m-1}) \neq (1, \cdots, 1)$, and call $\bar{\mathcal{A}}_1$ the **associated tensor** of $\mathcal{A}$.

We may further quantify this generating operation. Let $\mathcal{A} = (a_{j_1 \cdots j_m}) \in T_{m,n}$ and $Q = (q_{jk}) \in T_{2,n}$. Then as in [21], $\mathcal{B} = (b_{k_1 \cdots k_m}) \equiv \mathcal{A} Q^m$ is defined by

$$b_{k_1 \cdots k_m} = \sum_{j_1, \cdots, j_m = 1}^{n} a_{j_1 \cdots j_m} q_{j_1 k_1} \cdots q_{j_m k_m},$$

for $k_1, \cdots, k_m \in [n]$. Now we denote $P = (p_{jk}) \in T_{2,n}$ as a permutation matrix with $p_{jj+1} = 1$ for $j \in [n-1]$, $p_{n1} = 1$ and $p_{jk} = 0$ otherwise, i.e.,

$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ 0 & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2}$$

Then, from the definition of circulant tensors, we have the following proposition.

Proposition 2. *Suppose that $\mathcal{A} \in C_{m,n}$ and P is defined by (2). Then for $k \in [n]$, we have*

$$\mathcal{A}_{k+1} = \mathcal{A}_k P^{m-1},$$

where $\mathcal{A}_{n+1} \equiv \mathcal{A}_1$.

We may also use the definition of circulant tensors to prove the following proposition. As the proof is simple, we omit the proof here.

Proposition 3. *Suppose that $\mathcal{A} \in T_{m,n}$ and P is defined by (2). Then the following three statements are equivalent.*

(a). $\mathcal{A} \in C_{m,n}$.
(b). $\mathcal{A}P^m = \mathcal{A}$.
(c). For any $C \in C_{2,n}$, $\mathcal{A}C^m \in C_{m,n}$.

We may denote a circulant matrix $C \in C_{2,n}$ as

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}. \tag{3}$$

It is well-known [5, 9, 29] that the eigenvectors of $\mathcal{C}$ are given by

$$\mathbf{v}_k = \left(1, \omega_k, \omega_k^2, \cdots, \omega_k^{n-1}\right)^\top, \tag{4}$$

where $\omega_k = e^{\frac{2\pi i k}{n}}$ for $k+1 \in [n]$, with corresponding eigenvalues $\lambda_k = f_C(\omega_k)$, where f_C is the associated polynomial of C, defined by

$$f_C(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}.$$

We may also extend this result to circulant tensors. Note that $\mathbf{v}_0 = \mathbf{1}$ is a real vector.

For $\mathcal{A} = (a_{j_1 \cdots j_m}) \in T_{m,n}$ and $\mathbf{x} = (x_1, \cdots, x_n)^\top \in C^n$, let $\mathcal{A}\mathbf{x}^{m-1}$ be a vector in C^n whose jth component is defined as

$$(\mathcal{A}\mathbf{x}^{m-1})_j = \sum_{j_2, \ldots, j_m = 1}^{n} a_{jj_2 \ldots j_m} x_{j_2} \ldots x_{j_m},$$

and let $\mathbf{x}^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^\top$. If $\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$ for some $\lambda \in C$ and $\mathbf{x} \in C^n \setminus \{\mathbf{0}\}$, then λ is called an eigenvalue of $\mathcal{A}$ and $\mathbf{x}$ is called an eigenvector of $\mathcal{A}$, associated with λ. If $\mathbf{x}$ is real, then λ is also real. In this case, they are called an H-eigenvalue and an H-eigenvector respectively. The largest modulus of the eigenvalues of $\mathcal{A}$ is called the **spectral radius** of $\mathcal{A}$, and denoted as $\rho(\mathcal{A})$. The definition of eigenvalues was first given in [21] for symmetric tensors. It was extended to tensors in $T_{m,n}$ in [2].

Suppose that $\mathcal{A} \in C_{m,n}$. Let its root tensor be $\mathcal{A}_1 = (\alpha_{j_1\cdots j_{m-1}})$. Define the **associated polynomial** $f_{\mathcal{A}}$ by

$$f_{\mathcal{A}}(t) = \sum_{j_1,\cdots,j_{m-1}=1}^{n} \alpha_{j_1\cdots j_{m-1}} t^{j_1+\cdots+j_{m-1}-m+1}. \tag{5}$$

Theorem 3.1. *Suppose that $\mathcal{A} \in C_{m,n}$, its root tensor is $\mathcal{A}_1 = (\alpha_{j_1\cdots j_{m-1}})$, and its associated tensor is $\bar{\mathcal{A}}_1 = (\bar{\alpha}_{j_1\cdots j_{m-1}})$. Denote the diagonal entry of $\mathcal{A}$ by $c_0 = a_{1\ldots 1} = \alpha_{1\ldots 1}$. Then any eigenvalue λ of $\mathcal{A}$ satisfies the following inequality:*

$$|\lambda - c_0| \le \sum_{j_1,\cdots,j_{m-1}=1}^{n} \left|\bar{\alpha}_{j_1\cdots j_{m-1}}\right|. \tag{6}$$

Furthermore, the vectors $\mathbf{v}_k$, defined by (4), are eigenvectors of $\mathcal{A}$, with corresponding eigenvalues $\lambda_k = \lambda_k(\mathcal{A}) = f_{\mathcal{A}}(\omega_k)$, where $f_{\mathcal{A}}$ is the associated polynomial of $\mathcal{A}$, defined by (5). In particular, $\mathcal{A}$ always has an H-eigenvalue

$$\lambda_0 = \lambda_0(\mathcal{A}) = \sum_{j_1,\cdots,j_{m-1}=1}^{n} \alpha_{j_1\cdots j_{m-1}}, \tag{7}$$

with an H-eigenvector $\mathbf{1}$, and when n is even,

$$\lambda_{\frac{n}{2}} = \lambda_{\frac{n}{2}}(\mathcal{A}) = \sum_{j_1,\cdots,j_{m-1}=1}^{n} \alpha_{j_1\cdots j_{m-1}}(-1)^{j_1+\cdots+j_{m-1}-m+1} \tag{8}$$

is also an H-eigenvalue of $\mathcal{A}$ with an H-eigenvector $\hat{\mathbf{1}}$.

Proof. By the definition of circulant tensors and Theorem 6(a) of [21], all the eigenvalues of $\mathcal{A}$ satisfy (6). Let $\mathcal{A}_j$ be the jth row tensor of $\mathcal{A}$ for $j \in [n]$. Let P be defined by (2) and $k+1 \in [n]$. It is easy to verify that $P\mathbf{v}_k = \omega_k \mathbf{v}_k$. To prove that $(\mathbf{v}_k, \lambda_k)$ is an eigenpair of $\mathcal{A}$, it suffices to prove that for $j \in [n]$,

$$\mathcal{A}_j \mathbf{v}_k^{m-1} = \lambda_k \omega_k^{(j-1)(m-1)}. \tag{9}$$

We prove (9) by induction. By the definition of the associate polynomial, we see that (9) holds for $j = 1$. Assume that (9) holds for $j-1$. By Proposition 2, we have

$$\begin{aligned} \mathcal{A}_j \mathbf{v}_k^{m-1} &= \mathcal{A}_{j-1} P^{m-1} \mathbf{v}_k^{m-1} \\ &= \mathcal{A}_{j-1} (P\mathbf{v}_k)^{m-1} \\ &= \mathcal{A}_{j-1} (\omega_k \mathbf{v}_k)^{m-1} \\ &= \omega_k^{m-1} \mathcal{A}_{j-1} \mathbf{v}_k^{m-1} \\ &= \omega_k^{m-1} \lambda_k \omega_k^{(j-2)(m-1)} \\ &= \lambda_k \omega_k^{(j-1)(m-1)}. \end{aligned}$$

This proves (9). The other conclusions follow from this by the definition of H-eigenvalues and H-eigenvectors. The proof is completed. ∎

However, unlike a circulant matrix, these n pairs of eigenvalues and eigenvectors are not the only eigenpairs of a circulant tensor when $m \ge 3$. We may see this from the following example.

Example 1. A circulant tensor $\mathcal{A} = (a_{jkl}) \in C_{3,3}$ is generated from the following root tensor

$$\mathcal{A}_1 = \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & b \end{pmatrix}, \tag{10}$$

where $a = 5.91395$, $b = 2.47255$, $c = 2.92646$, $d = 8.49514$. By Theorem 3.1, we see that it has eigenvalues $\lambda_0 = 39.1013$, $\lambda_1 = 14.8057 + 1.1793i$ and $\lambda_2 = 14.8057 - 1.1793i$. Using the polynomial system solver **Nsolve** available in Mathematica, provided by Wolfram Research Inc., Version 8.0, 2010, we may verify that these three eigenvalues are indeed eigenvalues of $\mathcal{A}$. However, we found that $\mathcal{A}$ has three more eigenvalues $\lambda_3 = 4.92535$, $\lambda_4 = -2.08688 + 13.6795i$ and $\lambda_5 = -2.08688 - 13.6795i$.

Thus, for a circulant tensor $\mathcal{A}$, we call the n eigenvalues $\lambda_k(\mathcal{A})$ for $k+1 \in [n]$, provided by Theorem 3.1, the **native eigenvalues** of $\mathcal{A}$, call $\lambda_0(\mathcal{A})$ the **first native eigenvalue** of $\mathcal{A}$, and call $\lambda_{\frac{n}{2}}(\mathcal{A})$ the **alternative native eigenvalue** of $\mathcal{A}$ when n is even.

We now show that the first native eigenvalue $\lambda_0(\mathcal{A})$ plays a special role in certain cases.

Theorem 3.2. *Suppose that $\mathcal{A} \in C_{m,n}$, and its associated tensor is $\bar{\mathcal{A}}_1 = (\bar{\alpha}_{j_1\cdots j_{m-1}})$. If $\bar{\mathcal{A}}_1$ is a nonnegative tensor, then the first native eigenvalue $\lambda_0(\mathcal{A})$ is the largest H-eigenvalue of $\mathcal{A}$. If $\bar{\mathcal{A}}_1$ is a non-positive tensor, then the first native eigenvalue $\lambda_0(\mathcal{A})$ is the smallest H-eigenvalue of $\mathcal{A}$.*

Proof. By Theorem 3.1, we have

$$\lambda_0(\mathcal{A}) = c_0 + \sum_{j_1,\cdots,j_{m-1}=1}^{n} \bar{\alpha}_{j_1\cdots j_{m-1}}.$$

By this and (6), the conclusions hold. ∎

We may apply this theorem to the adjacency, Laplacian and signless Laplacian tensors of a circulant hypergraph or a directed circulant hypergraph. Then we see that the smallest H-eigenvalue of the Laplacian tensor is zero, the largest H-eigenvalue of the adjacency tensor is d, the largest H-eigenvalue of the signless Laplacian tensor is $2d$, where d is the common degree of the circulant hypergraph or the directed circulant hypergraph. These confirm the results in Section 2.

When n is even, the alternative native eigenvalue $\lambda_{\frac{n}{2}}(\mathcal{A})$ also plays a special role in certain cases. In order to study the role of the alternative native eigenvalue, we introduce alternative and negatively alternative tensors. We call a tensor $\mathcal{B} = (b_{j_1\cdots j_m}) \in T_{m,n}$ an **alternative tensor**, if $b_{j_1\cdots j_m}(-1)^{\sum_{k=1}^m j_k - m} \geq 0$. We call $\mathcal{B}$ **negatively alternative** if $-\mathcal{B}$ is alternative.

Then, by definition, we have the following proposition.

Proposition 4. *Suppose $\mathcal{B} \in T_{m,n}$ and let $\mathcal{B}_k$ be the kth row tensor of $\mathcal{B}$ for $k \in [n]$. Then $\mathcal{B} \in T_{m,n}$ is alternative if and only if $\mathcal{B}_k$ is alternative when k is odd and $\mathcal{B}_k$ is negatively alternative when k is even. In particular, $\mathcal{B}_1$ is alternative if $\mathcal{B}$ is alternative.*

Proof. By definition, we have for $k \in [n]$,

$$b_{kj_1\cdots j_{m-1}}(-1)^{\sum_{l=1}^{m-1} j_l + k - m} = b_{kj_1\cdots j_{m-1}}(-1)^{\sum_{l=1}^{m-1} j_l - m + 1}(-1)^{k-1} \geq 0.$$

It means that when k is odd,

$$b^{(k)}_{j_1\cdots j_{m-1}}(-1)^{\sum_{l=1}^{m-1} j_l-m+1} \geq 0$$

and when k is even,

$$b^{(k)}_{j_1\cdots j_{m-1}}(-1)^{\sum_{l=1}^{m-1} j_l-m+1} \leq 0.$$

So the proof is completed. $\square$

However, when $\mathcal{A}$ is circulant, $\mathcal{A}$ may be not alternative even if $\mathcal{A}_1$ is alternative. A simple counter-example can be given as follows.

Example 2. A circulant tensor $\mathcal{A} = (a_{jk}) \in C_{3,2}$ is given by

$$\mathcal{A}_1 = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}.$$

We can see that $\mathcal{A}_1$ and $\mathcal{A}_2$ are alternative but by Proposition 4, $\mathcal{A}$ is not alternative.

On the other hand, when m and n are even, we can see that a circulant tensor is alternative if and only if its root tensor is alternative.

Proposition 5. *Suppose $\mathcal{A} \in C_{m,n}$, where m and n are even. Then, $\mathcal{A}$ is (negatively) alternative if and only if its root tensor $\mathcal{A}_1$ is (negatively) alternative.*

Proof. By Proposition 4, we only prove that $\mathcal{A}$ is alternative if its root tensor $\mathcal{A}_1$ is alternative. Let $\mathcal{A}_k$ be the kth row tensor of $\mathcal{A}$ for $k \in [n]$. We first show that $\mathcal{A}_2$ is negatively alternative since $\mathcal{A}_1$ is alternative. For any $j_1, \cdots, j_{m-1} \in [n]$, let s be the number of the indexes that are equal to 1. Without loss of generality, we assume $j_1 = \cdots = j_s = 1$. By Proposition 2, we have

$$\begin{aligned}
& a^{(2)}_{j_1\cdots j_{m-1}}(-1)^{\sum_{l=1}^{m-1} j_l-m+1} \\
= \; & a^{(2)}_{j_1\cdots j_{m-1}}(-1)^{\sum_{l=s+1}^{m-1}(j_l-1)} \\
= \; & a^{(1)}_{n\cdots n j_{s+1}-1\cdots j_{m-1}-1}(-1)^{\sum_{l=s+1}^{m-1}(j_l-1)} \\
= \; & a^{(1)}_{n\cdots n j_{s+1}-1\cdots j_{m-1}-1}(-1)^{ns+\sum_{l=s+1}^{m-1}(j_l-1)-m+1}(-1)^{m-1-ns} \\
\leq \; & 0.
\end{aligned}$$

The last inequality holds because $\mathcal{A}_1$ is alternative and $m-1-ns$ is odd for any $s \in [m-1] \cup \{0\}$ since m and n are even. By induction, one can obtain that $\mathcal{A}_k$ is alternative when k is odd and $\mathcal{A}_k$ is negatively alternative when k is even, which means that $\mathcal{A}$ is alternative by Proposition 4. $\square$

Theorem 3.3. *Let n be even. Suppose that $\mathcal{A} \in C_{m,n}$, and its associated tensor is $\bar{A}_1 = (\bar{\alpha}_{j_1\cdots j_{m-1}})$. If $\bar{A}_1$ is an alternative tensor, then the alternative native eigenvalue $\lambda_{\frac{n}{2}}(\mathcal{A})$ is the largest H-eigenvalue of $\mathcal{A}$. If $\bar{A}_1$ is a negatively alternative tensor, then the alternative native eigenvalue $\lambda_{\frac{n}{2}}(\mathcal{A})$ is the smallest H-eigenvalue of $\mathcal{A}$.*

Proof. Let n be even. By Theorem 3.1, we have

$$\lambda_{\frac{n}{2}}(\mathcal{A}) = c_0 + \sum_{j_1\cdots,j_{m-1}=1}^{n} \bar{\alpha}_{j_1\cdots j_{m-1}}(-1)^{\sum_{k=1}^{m-1} j_k-m+1}.$$

By this and (6), the conclusions hold. $\square$

Note that the native eigenvalues other than $\lambda_0(\mathcal{A})$ and $\lambda_{\frac{n}{2}}(\mathcal{A})$ are in general not H-eigenvalues.

4. **Positive semi-definiteness of even order circulant tensors.** Let $j_l \in [n]$ for $l \in [m]$. Define the generalized Kronecker symbol [21, 24] by

$$\delta_{j_1\cdots j_m} = \begin{cases} 1 & \text{if } j_1 = \cdots = j_m, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\mathcal{A} = (a_{j_1\cdots j_m}) \in T_{m,n}$. We say that $\mathcal{A}$ is a B_0 tensor if for all $j \in [n]$

$$\sum_{j_2,\cdots,j_m=1}^{n} a_{jj_2\cdots j_m} \geq 0 \tag{11}$$

and

$$\frac{1}{n^{m-1}} \sum_{j_2,\cdots,j_m=1}^{n} a_{jj_2\cdots j_m} \geq a_{jk_2\cdots k_m}, \text{ if } \delta_{jk_2\cdots k_m} = 0. \tag{12}$$

If strict inequalities hold in (11) and (12), then $\mathcal{A}$ is called a B tensor [24]. The definitions of B and B_0 tensors are generalizations of the definition of B matrix [20]. It was proved in [24] that an even order symmetric B tensor is positive definite and an even order symmetric B_0 tensor is positive semi-definite. We may apply this result to even order symmetric circulant B_0 or B tensors. What we wish to show is that an even order circulant B tensor is positive definite and an even order circulant B_0 tensor is positive semi-definite, i.e., we do not require the tensor to be symmetric here. In this way, we may apply our result to directed circulant hypergraphs. The tool for realizing this is symmetrization.

By the definition of circulant tensors, it is easy to see that for $\mathcal{A} = (a_{j_1\cdots j_m}) \in C_{m,n}$, $\mathcal{A}$ is a circulant B_0 tensor if and only if

$$\sum_{j_1,\cdots,j_m=1}^{n} a_{j_1\cdots j_m} \geq 0 \tag{13}$$

and

$$\frac{1}{n^{m}} \sum_{j_1,\cdots,j_m=1}^{n} a_{j_1\cdots j_m} \geq \max\{a_{k_1\cdots k_m} : \delta_{k_1\cdots k_m} = 0\}. \tag{14}$$

If strict inequalities hold in (13) and (14), then $\mathcal{A}$ is a circulant B tensor.

It was established in [21] that an even order real symmetric tensor has always H-eigenvalues, and it is positive semi-definite (positive definite) if and only if all of its H-eigenvalues are nonnegative (positive). This is not true in general for a non-symmetric tensor. In order to use the first native eigenvalue or the alternative eigenvalue of a nonsymmetric circulant tensor to check its positive semi-definiteness, we may also use symmetrization.

We now link a general tensor $\mathcal{A} \in T_{m,n}$ to a symmetric tensor $\mathcal{B} \in S_{m,n}$.

Let $\mathcal{A} \in T_{m,n}$. Then there is a unique symmetric tensor $\mathcal{B} \in S_{m,n}$ such that for all $\mathbf{x} \in \Re^n$, $\mathcal{A}\mathbf{x}^m = \mathcal{B}\mathbf{x}^m$. We call $\mathcal{B}$ the **symmetrization** of $\mathcal{A}$, and denote it $sym(\mathcal{A})$. Thus, when m is even, a tensor $\mathcal{A} \in T_{m,n}$ is positive semi-definite (positive definite) if and only if all of the H-eigenvalues of $sym(\mathcal{A})$ are nonnegative (positive).

We call an index set $(k_1, \cdots, k_m)$ a permutation of another index set $(j_1, \cdots, j_m)$ if $(k_1, \cdots, k_m)$ is a rearrangement of $(j_1, \cdots, j_m)$, denote this operation by σ, and denote $\sigma(j_1, \cdots, j_m) = (k_1, \cdots, k_m)$. Denote the set of all distinct permutations of an index set $(j_1, \cdots, j_m)$, by $\Sigma(j_1, \cdots, j_m)$. Note that $|\Sigma(j_1, \cdots, j_m)|$, the cardinality of $\Sigma(j_1, \cdots, j_m)$, is variant for different index sets. For example, if $j_1 = \cdots = j_m$, then $|\Sigma(j_1, \cdots, j_m)| = 1$; but if all of $j_1, \cdots, j_m$ are distinct, $|\Sigma(j_1, \cdots, j_m)| = m!$.

Let $\mathcal{A} = (a_{j_1\cdots j_m}) \in T_{m,n}$ and $sym(\mathcal{A}) = \mathcal{B} = (b_{j_1\cdots j_m})$. Then it is not difficult to see that

$$b_{j_1\cdots j_m} = \frac{\sum_{\sigma\in\Sigma(j_1,\cdots,j_m)} a_{\sigma(j_1,\cdots,j_m)}}{|\Sigma(j_1,\cdots,j_m)|}. \tag{15}$$

For any $A \in T_{m,n}$, we use $\mathcal{D}(\mathcal{A})$ to denote a diagonal tensor in $T_{m,n}$, whose diagonal entries are the same as those of $\mathcal{A}$.

With this preparation, we are now ready to prove the following theorem.

Theorem 4.1. *Let $\mathcal{A} = (a_{j_1\cdots j_m}) \in T_{m,n}$. Then we have the following conclusions:*

(a). $\mathcal{D}(\mathcal{A}) = \mathcal{D}(sym(\mathcal{A}))$.

(b). If $\mathcal{A} - \mathcal{D}(\mathcal{A})$ are nonnegative (or non-positive or alternative or negatively alternative, respectively), then $sym(\mathcal{A}) - \mathcal{D}(sym(\mathcal{A}))$ are also nonnegative (or non-positive or alternative or negatively alternative, respectively).

(c). The symmetrization of a Toeplitz tensor is still a Toeplitz tensor. The symmetrization of a circulant tensor is still a circulant tensor.

(d). The symmetrization of a circulant B_0 tensor is still a circulant B_0 tensor. The symmetrization of a circulant B tensor is still a circulant B tensor.

(e). Suppose that $\mathcal{A} \in C_{m,n}$. Then we have

$$\lambda_0(\mathcal{A}) = \lambda_0(sym(\mathcal{A})).$$

If the associated tensor of a circulant tensor is nonnegative (or non-positive), then the associated tensor of the symmetrization of a circulant tensor is also nonnegative (or non-positive).

(f). Suppose $\mathcal{A} \in C_{m,n}$, where m and n are even. Then, we have

$$\lambda_{\frac{n}{2}}(\mathcal{A}) = \lambda_{\frac{n}{2}}(sym(\mathcal{A})).$$

Proof. We have (a) and (b) from (15) directly.

(c). Let $\mathcal{A} = (a_{j_1\cdots j_m}) \in T_{m,n}$ be a Toeplitz tensor, and $sym(\mathcal{A}) = \mathcal{B} = (b_{j_1\cdots j_m})$. By (15), for $j_l \in [n-1], l \in [m]$,

$$\begin{aligned} b_{j_1\cdots j_m} &= \frac{\sum_{\sigma\in\Sigma(j_1,\cdots,j_m)} a_{\sigma(j_1,\cdots,j_m)}}{|\Sigma(j_1,\cdots,j_m)|} \\ &= \frac{\sum_{\sigma\in\Sigma(j_1+1,\cdots,j_m+1)} a_{\sigma(j_1+1,\cdots,j_m+1)}}{|\Sigma(j_1+1,\cdots,j_m+1)|} \\ &= b_{j_1+1\cdots j_m+1}. \end{aligned}$$

Thus, $sym(\mathcal{A}) = \mathcal{B}$ is a Toeplitz tensor. When $\mathcal{A}$ is a circulant tensor, we may prove that $sym(\mathcal{A})$ is a circulant tensor similarly.

(d). Let $\mathcal{A} = (a_{j_1\cdots j_m}) \in C_{m,n}$ and $sym(\mathcal{A}) = \mathcal{B} = (b_{j_1\cdots j_m})$. By (c), $\mathcal{B} \in C_{m,n}$. Suppose now that $\mathcal{A}$ ia B_0 tensor. By (15) and (13), we have

$$\sum_{j_1,\cdots,j_m=1}^{n} b_{j_1\cdots j_m} = \sum_{j_1,\cdots,j_m=1}^{n} a_{j_1\cdots j_m} \geq 0.$$

By (15) and (14), we have

$$\begin{aligned} \frac{1}{n^m}\sum_{j_1,\cdots,j_m=1}^{n} b_{j_1\cdots j_m} &\geq \frac{1}{n^m}\sum_{j_1,\cdots,j_m=1}^{n} a_{j_1\cdots j_m} \geq \max\{a_{k_1\cdots k_m} : \delta_{k_1\cdots k_m} = 0\} \\ &\geq \max\{b_{k_1\cdots k_m} : \delta_{k_1\cdots k_m} = 0\}. \end{aligned}$$

Thus, $\mathcal{B}$ is also a B_0 tensor. Similarly, if $\mathcal{A}$ is a B tensor, then $\mathcal{B}$ is also a B tensor.

(e). Let $\mathcal{A} \in C_{m,n}$. The equality $\lambda_0(\mathcal{A}) = \lambda_0(sym(\mathcal{A}))$ holds because

$$\lambda_0(\mathcal{A}) = \frac{1}{n}\mathcal{A}\mathbf{v}_0^m = \frac{1}{n}sym(\mathcal{A})\mathbf{v}_0^m = \lambda_0(sym(\mathcal{A})).$$

The last conclusion follows from (b).

(f). For $k+1 \in [n]$, let ω_k and $\mathbf{v}_k$ be defined in (4). By Theorem 3.1, one can obtain

$$\lambda_k(\mathcal{A})\mathbf{v}_k^\top \mathbf{v}_k^{[m-1]} = \mathcal{A}\mathbf{v}_k^m = sym(\mathcal{A})\mathbf{v}_k^m = \lambda_k(sym(\mathcal{A}))\mathbf{v}_k^\top \mathbf{v}_k^{[m-1]}.$$

By simple computation, we have

$$\mathbf{v}_k^\top \mathbf{v}_k^{[m-1]} = \sum_{j=1}^{n} \omega_k^{m(j-1)} = \begin{cases} \frac{1-\omega_k^{nm}}{1-\omega_k^m} = 0 & \text{if } \omega_k^m \neq 1, \\ m & \text{if } \omega_k^m = 1. \end{cases}$$

In particular, since m and n are even, we have $\omega_{\frac{n}{2}}^m = (-1)^m = 1$. It follows that $\lambda_{\frac{n}{2}}(\mathcal{A}) = \lambda_{\frac{n}{2}}(sym(\mathcal{A}))$. $\square$

In fact, from the proof of Theorem 4.1, we can see that $\lambda_k(\mathcal{A}) = \lambda_k(sym(\mathcal{A}))$ if $\omega_k^m = 1$. And the equality $\lambda_0(\mathcal{A}) = \lambda_0(sym(\mathcal{A}))$ holds since $\omega_0 = 1$. Note that when m is odd, the equality $\lambda_{\frac{n}{2}}(\mathcal{A}) = \lambda_{\frac{n}{2}}(sym(\mathcal{A}))$ may not hold. See Example 2. By computation, $sym(\mathcal{A}) \in C_{3,2}$ is generated by the root tensor

$$sym(\mathcal{A})_1 = \begin{pmatrix} 1 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}.$$

We can see that $\lambda_1(\mathcal{A}) = 6$ and $\lambda_1(sym(\mathcal{A})) = 2/3$. On the other hand, we can also see that $\lambda_0(\mathcal{A}) = \lambda_0(sym(\mathcal{A})) = 2$ and $\lambda_1(\mathcal{A})$ is the largest H-eigenvalue of $\mathcal{A}$ since $\mathcal{A}_1$ is alternative.

We now have the following corollaries.

Corollary 1. *An even order circulant B_0 tensor is positive semi-definite. An even order circulant B tensor is positive definite.*

Proof. Suppose that $\mathcal{A}$ is an even order circulant B_0 tensor. Then by (d) of Theorem 4.1, $\mathcal{B} = \mathrm{sym}(\mathcal{A})$ is also an even order circulant B_0 tensor. Since $\mathcal{B}$ is symmetric, by [24], it is positive semi-definite. Since $\mathcal{A}$ is positive semi-definite if and only if $\mathrm{sym}(\mathcal{A})$ is positive semi-definite. The other conclusion holds similarly. $\square$

Note that an even order B_0 tensor may not be positive semi-definite. Let

$$A = \begin{pmatrix} 10 & 10 \\ 1 & 1 \end{pmatrix}.$$

Then A is a B_0 tensor. Let $\mathbf{x} = (1, -9)^\top$. Then $\mathbf{x}^\top A\mathbf{x} = -8$. Thus, A is not positive semi-definite.

In the next corollary, we stress that we may use (13) and (14) instead of (11) and (12) to check an even order circulant tensor is positive semi-definite or not. The conditions (13) and (14) contain less number of inequalities than (11) and (12).

Corollary 2. *Suppose that $\mathcal{A} = (a_{j_1 \cdots j_m}) \in C_{m,n}$ and m is even. If (13) and (14) hold, then $\mathcal{A}$ is positive semi-definite. If strict inequalities hold in (13) and (14), then $\mathcal{A}$ is positive definite.*

We may apply these two corollaries to directed circulant hypergraphs.

Corollary 3. *The Laplacian tensor and the signless Laplacian tensor of a directed circulant even-uniform hypergraph are positive semi-definite.*

As positive semi-definiteness of the Laplacian tensor and the signless Laplacian tensor of an even-uniform hypergraph plays an important role in spectral hypergraph theory [11, 12, 13, 14, 16, 22, 31, 30], The above result will be useful in the further research for directed circulant hypergraphs.

We may have some other corollaries of Theorem 4.1 as follows.

Corollary 4. *Suppose that m is even. If the associated tensor of a circulant tensor $\mathcal{A}$ is non-positive, then $\mathcal{A}$ is positive semi-definite if and only if $\lambda_0(\mathcal{A})$ is nonnegative.*

Corollary 5. *Suppose that m and n are even, $\mathcal{A} \in C_{m,n}$, and its associated tensor $\bar{\mathcal{A}}_1$ is negatively alternative. Then $\mathcal{A}$ is positive semi-definite if and only if $\lambda_{\frac{n}{2}}(\mathcal{A}) \geq 0$.*

Proof. By definition, we can see that the associate tensor $\bar{\mathcal{A}}_1$ is the root tensor of $\mathcal{A}-\mathcal{D}(\mathcal{A})$. By Proposition 5, one can derive that $\mathcal{A}-\mathcal{D}(\mathcal{A})$ is negatively alternative since m and n are even. By Theorem 4.1 (b), it follows that $sym(\mathcal{A}) - \mathcal{D}(sym(\mathcal{A}))$ is also negatively alternative. Again, by Proposition 5, $\overline{sym(\mathcal{A})}_1$ is also negatively alternative. By Theorem 3.3, in this case, $\lambda_{\frac{n}{2}}(sym(\mathcal{A}))$ is the smallest H-eigenvalue of $sym(\mathcal{A})$. By Theorem 4.1 (e), we have $\lambda_{\frac{n}{2}}(\mathcal{A}) = \lambda_{\frac{n}{2}}(sym(\mathcal{A}))$. The conclusion follows now. □

Corollary 6. *Suppose that m is even. Suppose that $\mathcal{A} \in C_{m,n}$ is positive semi-definite, and its diagonal entry is c_0. Then $c_0 \geq 0$ and $\lambda_0(\mathcal{A}) \geq 0$. If furthermore that n is even, then $\lambda_{\frac{n}{2}}(\mathcal{A}) \geq 0$.*

We may further establish a sufficient condition for positive semi-definiteness of an even order circulant tensor.

Theorem 4.2. *Suppose that m is even, $\mathcal{A} = (a_{j_1\cdots j_m}) \in C_{m,n}$, the diagonal entry of $\mathcal{A}$ is c_0, and the associated tensor of $\mathcal{A}$ is $\bar{\mathcal{A}}_1 = (\bar{\alpha}_{j_1\cdots j_{m-1}})$. If $\mathcal{A}$ is diagonally dominated, i.e.,*

$$c_0 \geq \sum_{j_1,\cdots,j_{m-1}=1}^{n} |\bar{\alpha}_{j_1\cdots j_{m-1}}|, \tag{16}$$

then $\mathcal{A}$ is positive semi-definite. If strict inequality holds in (16), then $\mathcal{A}$ is positive definite.

Proof. Let $\mathcal{A} - \mathcal{D}(\mathcal{A}) = (b_{j_1\cdots j_m}) \in C_{m,n}$. Then $\mathcal{A} \in C_{m,n}$ is diagonally dominated if and only if

$$c_0 \geq \frac{1}{n} \sum_{j_1,\cdots,j_m=1}^{n} |b_{j_1\cdots j_m}|.$$

On the other hand, suppose that (16) holds. Let $sym(\mathcal{A}) - \mathcal{D}(sym(\mathcal{A})) = (c_{j_1\cdots j_m}) \in C_{m,n}$. By the definition of symmetrization, it follows that

$$\begin{array}{rcl} c_0 & \geq & \frac{1}{n}\sum_{j_1,\cdots,j_m=1}^{n} |b_{j_1\cdots j_m}| \\ & = & \frac{1}{n}\sum_{j_1\leq\cdots\leq j_m} \sum_{\sigma\in\Sigma(j_1,\cdots,j_m)} |b_{\sigma(j_1,\cdots,j_m)}| \\ & \geq & \frac{1}{n}\sum_{j_1\leq\cdots\leq j_m} \left|\sum_{\sigma\in\Sigma(j_1,\cdots,j_m)} b_{\sigma(j_1,\cdots,j_m)}\right| \\ & = & \frac{1}{n}\sum_{j_1\leq\cdots\leq j_m} |\Sigma(j_1,\cdots,j_m)|\,|c_{j_1\cdots j_m}| \\ & = & \frac{1}{n}\sum_{j_1\leq\cdots\leq j_m} \sum_{\sigma\in\Sigma(j_1,\cdots,j_m)} |c_{\sigma(j_1,\cdots,j_m)}| \\ & = & \frac{1}{n}\sum_{j_1,\cdots,j_m=1}^{n} |c_{j_1\cdots j_m}|, \end{array}$$

which means that $sym(\mathcal{A}) \in C_{m,n}$ is also a diagonally dominated tensor. By Theorem 3 of [24], we can derive that all the H-eigenvalues of $sym(\mathcal{A})$ are nonnegative. So $\mathcal{A}$ is positive semi-definite. Similarly, if strict inequality holds in (16), we may prove that $\mathcal{A}$ is positive definite. $\square$

Note that Corollary 2 does not imply Theorem 4.2, and Theorem 4.2 does not imply Corollary 2. Thus, they are two different sufficient conditions for positive semi-definiteness of even order circulant tensors.

5. **Circulant tensors with special root tensors.**

In this section, we consider conditions for positive semi-definiteness of even order circulant tensors with special root tensors, including diagonal root tensors and circulant root tensors.

5.1. **Circulant tensors with diagonal root tensors.**

Suppose that $\mathcal{A} \in C_{m,n}$ and $\mathcal{A}_1$ is its root tensor. Assume that $\mathcal{A}_1 = (\alpha_{j_1 \cdots j_{m-1}})$ is a diagonal tensor, with $\alpha_{j_1 \cdots j_{m-1}} = c_{j-1}$ if $j_1 = \cdots = j_{m-1} = j \in [n]$, and $\alpha_{j_1 \cdots j_{m-1}} = 0$ otherwise. In this case, we may give all the eigenvalues and eigenvectors (up to some scaling constants) explicitly. Such a circulant tensor may be one of the simple cases of circulant tensors. We study its properties such that we can understand more about circulant tensors.

Theorem 5.1. *Let circulant matrix C be defined by (3). With the above assumptions, the n native eigenvalues λ_k of $\mathcal{A}$ are all possible eigenvalues of $\mathcal{A}$. They are exactly the n eigenvalues of the circulant matrix C. For $k+1 \in [n]$, each eigenvalue λ_k has the following eigenvectors $\mathbf{y}_{kl} = (1, \eta_{kl}, \eta_{kl}^2, \cdots, \eta_{kl}^{n-1})^\top$, where $\eta_{kl} = e^{\frac{2\pi kli}{n(m-1)}}$ for $l+1 \in [m-1]$.*

Proof. Let $\mathbf{y} = (y_1, \cdots, y_n)^\top \in C^n \setminus \{\mathbf{0}\}$ and λ be an eigenpair of $\mathcal{A}$. Define $c_{j-n} = c_j$ for $j \in [n]$. Then for $j \in [n]$, we have

$$\lambda y_j^{m-1} = \left(\mathcal{A}\mathbf{y}^{m-1}\right)_j = \sum_{l=1}^{n} c_{l-j} y_l^{m-1}. \tag{17}$$

Let $\mathbf{x} = (y_1^{m-1}, y_2^{m-1}, \cdots, y_n^{m-1})^\top$. Then we see that (17) is equivalent to $\lambda\mathbf{x} = C\mathbf{x}$, i.e., $(\lambda, \mathbf{x})$ form an eigenpair of circulant matrix C. Now the conclusion can be derived easily. $\square$

It is easy to see that $\mathcal{A}$, the circulant tensor with a diagonal root tensor discussed above, is symmetric if and only if $c_j = 0$ for $j \in [n-1]$. Thus, in general, such a circulant tensor is not symmetric.

Now we discuss positive semi-definiteness of a circulant tensor with a diagonal root tensor. First, by direct derivation, we have the following result.

Proposition 6. *Let $\mathcal{A} \in C_{m,n}$ have a diagonal root tensor as described above. Then for any $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \Re^n$,*

$$\mathcal{A}\mathbf{x}^m = \mathbf{x}^\top C\mathbf{x}^{[m-1]} \equiv \sum_{j,l=1}^{n} c_{l-j} x_j x_l^{m-1} = c_0 \sum_{l=1}^{n} x_l^m + \sum_{\substack{j,l=1 \\ j \neq l}}^{n} c_{l-j} x_j x_l^{m-1}, \tag{18}$$

where C is the circulant matrix defined by (3).

Example 3. Let $m = 4$ and $n = 2$. Let $c_0 = c_1 = 1$. Then by Proposition 6,

$$\mathcal{A}\mathbf{x}^4 = x_1^4 + x_1^3 x_2 + x_1 x_2^3 + x_2^4 = (x_1 + x_2)^2 (x_1^2 - x_1 x_2 + x_2^2) \geq 0$$

for any $\mathbf{x} \in \Re^2$. Thus, $\mathcal{A}$ is positive semi-definite.

By (7) and (8), we have

$$\lambda_0(\mathcal{A}) = \sum_{j=0}^{n-1} c_j, \tag{19}$$

and when n is even,

$$\lambda_{\frac{n}{2}}(\mathcal{A}) = \sum_{j=0}^{n-1} c_j (-1)^{j(m-1)}.$$

In particular, when m is also even, we have

$$\lambda_{\frac{n}{2}}(\mathcal{A}) = \sum_{j=0}^{n-1} c_j (-1)^j. \tag{20}$$

Let $\mathbf{c} = (c_1, \cdots, c_{n-1})^\top \in \Re^{n-1}$. Let $k \leq \frac{n}{2}$. We say that $\mathbf{c}$ is **k-alternative** if $n = 2pk$ for some integer p, $c_{(2q-1)k} \geq 0$ and $c_{2qk} \leq 0$ for $q \in [p]$ and $c_j = 0$ otherwise. When $n = 2pk$, let $\hat{\mathbf{1}}^{(k)}$ be a vector in $\Re^n$ such that $\hat{\mathbf{1}}^{(k)}_j = 1$ for $(2q-2)k+1 \leq j \leq (2q-1)k$ and $\hat{\mathbf{1}}^{(k)}_j = -1$ for $(2q-1)k+1 \leq j \leq 2qk$, for $q \in [p]$.

In the following, we give some necessary conditions, sufficient conditions, necessary and sufficient conditions for an even order circulant tensor with a diagonal root tensor to be positive semi-definite.

Theorem 5.2. *Let $\mathcal{A} \in C_{m,n}$ have a diagonal root tensor as described at the beginning of this section. Suppose that m is even. Then, we have the following conclusions:*

(a). If $\mathcal{A}$ is positive semi-definite, then $c_0 \geq 0$ and $\lambda_0(\mathcal{A}) \geq 0$. If furthermore n is even, then $\lambda_{\frac{n}{2}}(\mathcal{A}) \geq 0$.

(b). If

$$c_0 \geq \sum_{j=1}^{n-1} |c_j|, \tag{21}$$

then $\mathcal{A}$ is positive semi-definite.

(c). If $\mathbf{c}$ is non-positive, then $\mathcal{A}$ is positive semi-definite if and only if (21) holds.

(d). If $n = 2pk$ for some positive integers p and k, and $\mathbf{c}$ is k-alternative, then $\mathcal{A}$ is positive semi-definite if and only if (21) holds.

Proof. (a). This follows from Corollary 6.

(b). This follows from Theorem 4.2.

(c). If $\mathbf{c}$ is non-positive, then the associated tensor of $\mathcal{A}$ is non-positive. By Corollary 4, $\mathcal{A}$ is positive semi-definite if and only if

$$\lambda_0(\mathcal{A}) = \sum_{j=0}^{n} c_j \geq 0.$$

Since $\mathbf{c}$ is non-positive and $c_0 \geq 0$, the above inequality holds if and only if (21) holds. This proves (c).

(d). Suppose that $n = 2pk$ for some positive integers p and k, and $\mathbf{c}$ is k-alternative. Then (21) holds in this case. By (b), $\mathcal{A}$ is positive semi-definite. On the other hand, if (21) does not hold, Let $\mathbf{x} = \hat{\mathbf{1}}^{(k)}$ in (18). We have $\mathcal{A}\mathbf{x}^m < 0$, i.e., $\mathcal{A}$ is not positive semi-definite. This proves (d). The theorem is proved. ∎

Are there some other cases such that (21) is also a sufficient and necessary condition such that $\mathcal{A}$ is positive semi-definite?

Suppose that m is even. Can we give all the H-eigenvalues of $sym(\mathcal{A})$ explicitly? If so, we may determine $\mathcal{A}$ is positive semi-definite or not. Otherwise, can we construct an algorithm to find the global optimal value of one of the following two minimization problems when m is even? The two minimization problems are as follows:

$$\begin{aligned} \min \quad & c_0 \sum_{l=1}^n x_l^m + \sum_{\substack{j,l=1 \\ j \neq l}}^n c_{l-j} x_j x_l^{m-1} \\ \text{subject to} \quad & \sum_{j=1}^n x_j^2 = 1, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \min \quad & c_0 \sum_{l=1}^n x_l^m + \sum_{\substack{j,l=1 \\ j \neq l}}^n c_{l-j} x_j x_l^{m-1} \\ \text{subject to} \quad & \sum_{j=1}^n x_j^m = 1. \end{aligned} \tag{23}$$

By Proposition 6, $\mathcal{A}$ is positive semi-definite if and only if the global optimal value of (22) or (23) is nonnegative. In Subsection 5.3, we will give an algorithm to determine positive semi-definiteness of an even order circulant tensor with a diagonal root tensor.

5.2. **Doubly circulant tensors.** Let $\mathcal{A} \in C_{m,n}$. If its root tensor $\mathcal{A}_1$ itself is a circulant tensor, by Propositions 2 and 3, we see that all the row tensors of $\mathcal{A}$ are duplicates of $\mathcal{A}_1$, i.e., $\mathcal{A}_k = \mathcal{A}_1$ for $k \in [n]$. We call such a circulant tensor $\mathcal{A}$ a **doubly circulant tensor**.

Let $\mathcal{A}$ be an even order doubly circulant tensor. Suppose that $\mathcal{A}_{11} \in T_{m-2,n}$ is the root tensor of $\mathcal{A}_1$. A natural question is that if there is a relation between $\mathcal{A}_{11}$ and $\mathcal{A}$ in terms of the positive semi-definiteness, i.e., if $\mathcal{A}_{11}$ is positive semi-definite, is $\mathcal{A}$ also positive semi-definite? And if $\mathcal{A}$ is positive semi-definite, is $\mathcal{A}_{11}$ also positive semi-definite? Unfortunately, the answers to these two questions are both "no". See the following example.

Example 4. Let $\mathcal{A}_{11} = diag\{d_1, d_2\}$. Then, for any $\mathbf{x} \in \Re^2$, we have

$$\begin{aligned} \mathcal{A}\mathbf{x}^4 &= (x_1 + x_2)\mathcal{A}_1\mathbf{x}^3 \\ &= (x_1 + x_2)[d_1(x_1^3 + x_2^3) + d_2(x_1^2 x_2 + x_2^2 x_1)] \\ &= (x_1 + x_2)^2[d_1 x_1^2 + (d_2 - d_1)x_1 x_2 + d_1 x_2^2]. \end{aligned}$$

Case 1. $d_1 = 1$, $d_2 = 5$. $\mathcal{A}_{11}$ is positive semi-definite. However, $\mathcal{A}$ is not positive semi-definite since $\mathcal{A}\mathbf{x}^4 < 0$ for $\mathbf{x} = (1, -2)^\top$.
Case 2. $d_1 = 1$, $d_2 = -0.5$. $\mathcal{A}$ is positive semi-definite since $\mathcal{A}\mathbf{x}^4 = (x_1 + x_2)^2[x_1^2 - 1.5x_1x_2 + x_2^2] \geq 0$. However, $\mathcal{A}_{11}$ is not positive semi-definite since $d_2 < 0$.

However, we may answer this question positively if $\mathcal{A}_1$ is also doubly circulant.

Proposition 7. *If $\mathcal{A} \in T_{m,n}$ is a doubly circulant tensor and $\mathcal{A}_1$ is its root tensor, then for any $\mathbf{x} \in \Re^n$, we have*

$$\mathcal{A}\mathbf{x}^m = \sum_{k=1}^{n} x_k \mathcal{A}_1 \mathbf{x}^{m-1}.$$

On the other hand, if m is even and $\mathcal{A}_1$ is doubly circulant, then we have the following conclusions:

(a). $\mathcal{A}$ is doubly circulant.

(b). If $\mathcal{A}_{11}$ is positive semi-definite, then $\mathcal{A}$ is positive semi-definite.

(c). If $\mathcal{A}$ is positive semi-definite, then for any $\mathbf{x} \in \Re^n$ satisfying $\sum_{k=1}^n x_k \neq 0$, we have

$$\mathcal{A}_{11}\mathbf{x}^{m-2} \geq 0,$$

where $\mathcal{A}_{11}$ is the root tensor of $\mathcal{A}_1$.

Proof. If $\mathcal{A}$ is doubly circulant, then we have $\mathcal{A}_k = \mathcal{A}_1$ for $k \in [n]$. It follows that for any $\mathbf{x} \in \Re^n$, one can obtain

$$\mathcal{A}\mathbf{x}^m = \sum_{k=1}^{n} x_k \mathcal{A}_1 \mathbf{x}^{m-1}.$$

On the other hand, if $\mathcal{A}_1$ is doubly circulant, then we have $\mathcal{A}$ is doubly circulant by definition and

$$\mathcal{A}\mathbf{x}^m = \sum_{k=1}^{n} x_k \mathcal{A}_1 \mathbf{x}^{m-1} = \left(\sum_{k=1}^{n} x_k\right)^2 \mathcal{A}_{11}\mathbf{x}^{m-2},$$

where $\mathcal{A}_{11}$ is the root tensor of $\mathcal{A}_1$. The conclusions (a)-(c) follow immediately. □

5.3. **An algorithm and numerical tests.** In Subsection 5.1, we show that a circulant tensor with a diagonal root tensor is positive semi-definite if and only if the global optimal value of (22) or (23) is nonnegative. In this subsection, we present an algorithm to solve the minimization problem (22). Here, m is even, and the norm $\|\cdot\|$ in this section is the 2-norm.

Suppose $\mathcal{A} \in C_{m,n}$. The minimization problem

$$\begin{aligned} \min \quad & \mathcal{A}\mathbf{x}^m \\ \text{subject to} \quad & \|\mathbf{x}\| = 1, \end{aligned} \tag{24}$$

can be equivalent to be written as

$$\begin{aligned} \min \quad & \mathcal{A}\mathbf{x}^1 \cdots \mathbf{x}^m \\ \text{subject to} \quad & \sum_{j=1}^{m} A_j \mathbf{x}^j = \mathbf{0} \\ & \|\mathbf{x}^k\| = 1, \quad k \in [m], \end{aligned} \tag{25}$$

where

$$A_1 = \begin{pmatrix} I_m \\ \vdots \\ \vdots \\ -I_m \end{pmatrix}, \quad A_2 = \begin{pmatrix} -I_m \\ I_m \\ \vdots \\ \vdots \end{pmatrix}, \quad \cdots, \quad A_m = \begin{pmatrix} \vdots \\ \vdots \\ -I_m \\ I_m \end{pmatrix}.$$

Denote $f(\mathbf{x}^1, \cdots, \mathbf{x}^m) := \mathcal{A}\mathbf{x}^1 \cdots \mathbf{x}^m$. Then the augmented Lagrangian function of (25) $L_\beta(\mathbf{x}^1, \cdots, \mathbf{x}^m, \lambda)$ is defined as

$$L_\beta(\mathbf{x}^1, \cdots, \mathbf{x}^m, \lambda) = f(\mathbf{x}^1, \cdots, \mathbf{x}^m) - \lambda^\top \left(\sum_{j=1}^m A_j \mathbf{x}^j \right) + \frac{\beta}{2} \left\| \sum_{j=1}^m A_j \mathbf{x}^j \right\|^2,$$

with the given constant $\beta > 0$.

We use the alternating direction method of multipliers to solve (25).

Algorithm 1. Alternating direction method of multipliers for circulant tensors
Step 0. Given $\epsilon > 0$, $\mathbf{w}^0 = [(\mathbf{x}^1)^0, \cdots, (\mathbf{x}^m)^0, \lambda^0] \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^{mn}$ and set $k = 0$.
Step 1. Generate $\mathbf{w}^{k+1}$ from $\mathbf{w}^k$, i.e., for $j \in [m]$

$$(\mathbf{x}^j)^{k+1} = \underset{\mathbf{x}^\top \mathbf{x}=1}{\arg\min}\ L_\beta[(\mathbf{x}^1)^{k+1}, \cdots, (\mathbf{x}^{j-1})^{k+1}, \mathbf{x}, (\mathbf{x}^{j+1})^k, \cdots, (\mathbf{x}^m)^k, \lambda^k] \tag{26}$$

and

$$\lambda^{k+1} = \lambda^k - \beta \sum_{j=1}^m A_j (\mathbf{x}^j)^{k+1}.$$

Step 2. If $\|\mathbf{w}^{k+1} - \mathbf{w}^k\| < \epsilon$, stop. Otherwise, $k := k+1$, go to Step 1.

Note that the subproblem (26) is exactly equivalent to a convex quadratic programming on the unit ball, i.e.,

$$\begin{array}{ll} \min & \mathbf{x}^\top \mathbf{x} + b^\top \mathbf{x} \\ \text{subject to} & \|\mathbf{x}\| = 1, \end{array}$$

with a given vector b. It is well known that it has a closed form solution. So this algorithm is easily implemented.

Under certain condition, the convergence of the algorithm has also been proved, see [8, 15, 18]. Though the sequence generated from the algorithm may converge to a KKT point, the following numerical results show that the iterative sequence converges to the global minimal solution with a high probability if we choose the initial point randomly. Note that all the diagonal elements of the root tensor are generated randomly in $[-10, 10]$.

Example 5. A circulant tensor $\mathcal{A} \in C_{4,3}$ is generated from a diagonal root tensor with $c_0 = -4.75046$, $c_1 = 3.58365$ and $c_2 = 8.252$.

Example 6. A circulant tensor $\mathcal{A} \in C_{4,4}$ is generated from a diagonal root tensor with $c_0 = 3.30134$, $c_1 = -9.68746$, $c_2 = 2.31954$ and $c_3 = 7.60276$.

In the implementation of Algorithm 1, we set the parameters $\beta = 1.2$ and $\epsilon = 10^{-6}$. And the initial point is generated randomly. All codes were written by Matlab R2012b and all the numerical experiments were done on a laptop with Intel Core i5-2430M CPU 2.4GHz and 1.58GB memory. The numerical results are reported in Table 1. In the table, $\bar{k}$, $\bar{t}$ and $\bar{\lambda}$ denote the average number of iteration, average time and average value derived after 100 experiments. λ^* means the global minimal solution derived by the polynomial system solver **Nsolve** available in Mathematica, provided by Wolfram Research Inc., Version 8.0, 2010. The frequency of success is also recorded. If $\|\lambda - \lambda^*\| \leq 10^{-5}$, we say that the algorithm can find the global minimal solution of (22) successfully.

Table 1. Numerical results for Example 5 and Example 6

	$\bar{k}$	$\bar{t}$	$\bar{\lambda}$	λ^*	Frequency of success
Example 5	62.73	0.35607	-6.39448	-6.39448	100%
Example 6	92.49	0.52795	-1.79658	-1.79658	100%

From Table 1, we can see that the alternative direction method of multiplies can be efficient for solving the minimization problem (22) in some cases. We also test some problems with larger scale. However, it may be hard to verify the value derived by the algorithm since the solver **Nsolve** could not work for larger scale problems.

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