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Inheritance properties and sum-of-squares decomposition of Hankel tensors: theory and algorithms

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Abstract In this paper, we show that if a lower-order Hankel tensor is positive semi-definite (or positive definite, or negative semi-definite, or negative definite, or SOS), then its associated higher-order Hankel tensor with the same generating vector, where the higher order is a multiple of the lower order, is also positive semi-definite (or positive definite, or negative semi-definite, or negative definite, or SOS, respectively). Furthermore, in this case, the extremal H-eigenvalues of the higher order tensor are bounded by the extremal H-eigenvalues of the lower order tensor, multiplied with some constants. Based on this inheritance property, we give a concrete sum-of-squares decomposition for each strong Hankel tensor. Then we prove the second inheritance property of Hankel tensors, i.e., a Hankel tensor has no negative (or non-positive, or

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positive, or nonnegative) H-eigenvalues if the associated Hankel matrix of that Hankel tensor has no negative (or non-positive, or positive, or nonnegative, respectively) eigenvalues. In this case, the extremal H-eigenvalues of the Hankel tensor are also bounded by the extremal eigenvalues of the associated Hankel matrix, multiplied with some constants. The third inheritance property of Hankel tensors is raised as a conjecture.

Keywords Hankel tensor · Inheritance property · Positive semi-definite tensor · Sum-of-squares · Convolution

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1 Introduction

Hankel structures are widely employed in data analysis and signal processing. Not only Hankel matrices but also higher-order Hankel tensors arise frequently in many disciplines such as exponential data fitting [3,7,15,16], frequency domain subspace identification [20], multidimensional seismic trace interpolation [22], and so on. Furthermore, the positive semi-definite Hankel matrices are most related to the moment problems, and one can refer to [1,8,19]. In moment problems, some necessary or sufficient conditions for the existence of a desired measure are given as the positive semi-definiteness of a series of Hankel matrices.

The term *tensor* in this paper is used to mean a multi-way array. We call the number of indices as the order of a tensor, that is, the order of a tensor of size $n_1 \times n_2 \times \cdots \times n_m$ is m. Particularly, if the tensor is square, i.e., $n := n_1 = n_2 = \cdots = n_m$, then we call it an mth-order n-dimensional tensor.

An mth-order *Hankel tensor* $\mathscr{H} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_m}$ is a multi-way array whose entries are function values of the sums of indices, i.e.,

$$\mathscr{H}_{i_1,i_2,\ldots,i_m} = h_{i_1+i_2+\cdots+i_m},\ i_k = 0, 1, \ldots, n_k - 1,\ k = 1, 2, \ldots, m,$$

where the vector $\mathbf{h} = (h_i)_{i=0}^{n_1+\cdots+n_m-m}$ is called the *generating vector* of this Hankel tensor $\mathscr{H}$ [7,12,18,24]. The generating vector and the size parameters totally determines the Hankel tensor. Furthermore, if the size parameters change, then the same vector can generate several Hankel tensors with variant orders and sizes. What kinds of common properties will Hankel tensors generated by the same vector but with different orders share? These common properties will be called inheritance properties in this paper.

The multiplication of a tensor $\mathscr{T}$ and a matrix M along kth mode (see [9, Chapter 12.4]) is defined by

$$(\mathscr{T} \times_k M)_{i_1\ldots i_{k-1} j_k i_{k+1}\ldots i_m} := \sum_{i_k=0}^{n-1} \mathscr{T}_{i_1 i_2 \ldots i_m} M_{i_k j_k}.$$

When the matrix degrades into a vector, Qi [17] introduced some simple but useful notations

$$\mathscr{T}\mathbf{x}^m := \mathscr{T} \times_1 \mathbf{x} \times_2 \mathbf{x} \cdots \times_m \mathbf{x},$$
$$\mathscr{T}\mathbf{x}^{m-1} := \mathscr{T} \times_2 \mathbf{x} \cdots \times_m \mathbf{x}.$$

If there is a scalar $\lambda \in \mathbb{C}$ and a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ such that

$$\mathscr{T}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathbf{x}^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^\top$, then we call λ an eigenvalue of $\mathscr{T}$ and $\mathbf{x}$ a corresponding eigenvector. Further when $\mathbf{x}$ is a real eigenvector, we call λ an H-eigenvalue of $\mathscr{T}$. An mth-order n-dimensional square tensor induces a degree-m multivariate polynomial of n variables:

$$p_{\mathscr{T}}(\mathbf{x}) := \mathscr{T}\mathbf{x}^m = \sum_{i_1,\ldots,i_m=0}^{n-1} \mathscr{T}_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m}.$$

Suppose that m is even. If $p_{\mathscr{T}}(\mathbf{x})$ is always nonnegative (positive, non-positive, or negative) for all nonzero real vectors $\mathbf{x}$, then the tensor $\mathscr{T}$ is called a *positive semi-definite* (*positive definite*, *negative semi-definite*, or *negative definite*, respectively) tensor [17]. If $p_{\mathscr{T}}(\mathbf{x})$ can be represented as a sum of squares, then we call the tensor $\mathscr{T}$ an *SOS tensor* [13]. Apparently, an SOS tensor must be positive semi-definite, but the converse is generally not true.

We organize our paper in line with these spectral inheritance properties. However, other spectral inheritance properties, such as tensor decompositions (the Vandermonde and the SOS decompositions) and the convolution formula, will also be introduced as tools for investigating the spectral inheritance properties of Hankel tensors.

It is obvious that the positive definiteness is not well-defined for odd order tensors, since $\mathscr{T}(-\mathbf{x})^m = -\mathscr{T}\mathbf{x}^m$. Qi [17] proved that an even-order tensor is positive semi-definite if and only of it has no negative H-eigenvalues. Thus we shall use the property "no negative H-eigenvalue" instead of the positive semi-definiteness when studying the inheritance properties of odd order Hankel tensors. The basic question about the inheritance of the positive semi-definiteness is:

- If a lower-order Hankel tensor has no negative H-eigenvalues, does a higher-order Hankel tensor with the same generating vector possess no negative H-eigenvalues?

We will consider two situations, i.e.,

- the lower order m is even and the higher order qm is a multiple of m, and
- the lower order is 2,

which provide the basic question with positive answers. Moreover, we guess that it is also true when the lower order m is odd. As cannot prove it or find a counterexample, we leave it as a conjecture.

In fact, Qi [18] showed an inheritance property of Hankel tensors. The generating vector of a Hankel tensor also generates a Hankel matrix, which is called the associated Hankel matrix of that Hankel tensor [18]. It was shown in [18] that if the Hankel tensor is of even order and its associated Hankel matrix is positive semi-definite, then the

Hankel tensor is also positive semi-definite. In [18], a Hankel tensor is called a strong Hankel tensor if its associated Hankel matrix is positive semi-definite. Thus, an even order strong Hankel tensor is positive semi-definite. This actually is the even order case of the second situation. In this paper, we will show that the inheritance property holds in the second situation for odd orders.

The converse of the inheritance properties is not true. A simple case of the converse of the inheritance properties is as follows. Suppose a higher even order Hankel tensor is positive semi-definite. Is the Hankel matrix which shares the same generating vector with the higher even order positive semi-definite Hankel tensor also positive semi-definite? The answer is “no”. Actually, if the answer were “yes”, then all the even order positive semi-definite Hankel tensors would be strong Hankel tensors. In the literature, there are many examples of even order positive semi-definite Hankel tensors which are not strong Hankel tensors:

– the 4th-order 2-dimensional Hankel tensor [18] generated by

$$\left[1, 0, -\tfrac{1}{6}, 0, 1\right]^\top,$$

– the 4th-order 4-dimensional Hankel tensor [6] generated by

$$\left[8, 0, 2, 0, 1, 0, 1, 0, 1, 0, 2, 0, 8\right]^\top,$$

– the 6th-order 3-dimensional Hankel tensor [11] generated by

$$\left[h_0, 0, 0, 0, 0, 0, h_6, 0, 0, 0, 0, 0, h_{12}\right]^\top,$$

where $\sqrt{h_0 h_{12}} \geq \left(560 + 70\sqrt{70}\right) h_6 > 0$.

The following is a summary of the inheritance properties of Hankel tensors, studied in this paper.

The first inheritance property of Hankel tensors is that if a lower-order Hankel tensor is positive semi-definite (positive definite, negative semi-definite, negative definite, or SOS), then its associated higher-order Hankel tensor with the same generating vector, where the higher order is a multiple of the lower order, is also positive semi-definite (positive definite, negative semi-definite, negative definite, or SOS, respectively). The inheritance property established in [18] can be regarded as a special case of this inheritance property. Furthermore, in this case, we show that the extremal H-eigenvalues of the higher order tensor are bounded by the extremal H-eigenvalues of the lower order tensor, multiplied with some constants.

In [12], it was proved that strong Hankel tensors are SOS tensors, but no concrete SOS decomposition was given. In this paper, by using the inheritance property described above, we give a concrete sum-of-squares decomposition for a strong Hankel tensor.

The second inheritance property of Hankel tensors we will establish in this paper is an extension of the inheritance property established in [18] to the odd-order case. Normally, positive semi-definiteness and the SOS property are only well-defined for even order tensors. By [17], an even order symmetric tensor is positive semi-definite if and only if it has no negative H-eigenvalues. In this paper, we will show that if the associated Hankel matrix of a Hankel tensor has no negative (or non-positive, or positive, or nonnegative) eigenvalues, then the Hankel tensor has also no negative (non-positive, positive, or nonnegative, respectively) H-eigenvalues. In this case, we show that the extremal H-eigenvalues of the Hankel tensor are also bounded by the extremal eigenvalues of the associated Hankel matrix, multiplied with some constants.

Finally, we raise the third inheritance property of Hankel tensors as a conjecture.

This paper is organized as follows. In Sect. 2, we first introduce some basic concepts and properties of Hankel tensors. Then by using a convolution formula, we show that if a lower-order Hankel tensor is positive semi-definite (or positive definite, negative semi-definite, negative definite, or SOS), then its associated higher-order Hankel tensor with the same generating vector and a multiple order, is also positive semi-definite (positive definite, negative semi-definite, negative definite, or SOS, respectively). In this case, some inequalities to bound the extremal H-eigenvalues of the higher order tensor by the extremal H-eigenvalues of the lower order tensor, multiplied with some constants, are given. Based on this inheritance property, we give a concrete sum-of-squares decomposition for each strong Hankel tensor. In Sect. 3, we investigate some structure-preserving Vandermonde decompositions of some particular Hankel tensors, and we prove that each strong Hankel tensor admits an augmented Vandermonde decomposition with all positive coefficients. With this tool, we show that if the associated Hankel matrix of a Hankel tensor has no negative (non-positive, positive, or nonnegative) eigenvalues, then the Hankel tensor has also no negative (non-positive, positive, or nonnegative, respectively) H-eigenvalues, i.e., the second inheritance property of Hankel tensors holds. In this case, we show that the extremal H-eigenvalues of the Hankel tensor are also bounded by the extremal eigenvalues of the associated Hankel matrix, multiplied with some constants. Numerical examples are given in Sect. 4. The third inheritance property of Hankel tensors is raised in Sect. 5 as a conjecture.

2 The first inheritance property of Hankel tensors

This section is devoted to the first inheritance property of Hankel tensors. We will prove that if a lower-order Hankel tensor is positive semi-definite or SOS, then a Hankel tensor with the same generating vector and a high multiple order is also positive semi-definite or SOS, respectively.

2.1 Hankel tensor-vector products

We shall have a close look at the nature of the Hankel structure first. In matrix theory, the multiplications of most structured matrices, such as Toeplitz, Hankel, Vandermonde, and Cauchy matrices, with vectors have their own analytic interpretations. Olshevsky and Shokrollahi [14] listed several important connections between fundamental analytic algorithms and structured matrix-vector multiplications. They claimed that there is a close relationship between Hankel matrices and discrete convolutions. And we will see shortly that this is also true for Hankel tensors.

We first introduce some basic facts about discrete convolutions. Let two vectors $\mathbf{u} \in \mathbb{C}^{n_1}$ and $\mathbf{v} \in \mathbb{C}^{n_2}$. Then their *convolution* $\mathbf{w} = \mathbf{u} * \mathbf{v} \in \mathbb{C}^{n_1+n_2-1}$ is a longer vector defined by

$$w_k = \sum_{j=0}^{k} u_j v_{k-j}, \quad k = 0, 1, \ldots, n_1 + n_2 - 2,$$

where $u_j = 0$ when $j \geq n_1$ and $v_j = 0$ when $j \geq n_2$. Denote $p_{\mathbf{u}}(\xi)$ and $p_{\mathbf{v}}(\xi)$ as the polynomials whose coefficients are $\mathbf{u}$ and $\mathbf{v}$, respectively, i.e.,

$$p_{\mathbf{u}}(\xi) = u_0 + u_1\xi + \cdots + u_{n_1-1}\xi^{n_1-1}, \quad p_{\mathbf{v}}(\xi) = v_0 + v_1\xi + \cdots + v_{n_2-1}\xi^{n_2-1}.$$

Then we can verify easily that $\mathbf{u} * \mathbf{v}$ consists of the coefficients of the product $p_{\mathbf{u}}(\xi) \cdot p_{\mathbf{v}}(\xi)$. Another important property of discrete convolutions is that

$$\mathbf{u} * \mathbf{v} = V^{-1}\left(V \begin{bmatrix} \mathbf{u} \\ \mathbf{0} \end{bmatrix} .* V \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}\right)$$

for an arbitrary $(n_1 + n_2 - 1)$-by-$(n_1 + n_2 - 1)$ nonsingular Vandermonde matrix V. In applications, the Vandermonde matrix is often taken as the Fourier matrices, since we have fast algorithms for discrete Fourier transforms. Similarly, if there are more vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, then their convolution is equal to

$$\mathbf{u}_1 * \mathbf{u}_2 * \cdots * \mathbf{u}_m = V^{-1}\left(V \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{0} \end{bmatrix} .* V \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{0} \end{bmatrix} .* \cdots .* V \begin{bmatrix} \mathbf{u}_m \\ \mathbf{0} \end{bmatrix}\right), \tag{2.1}$$

where V is a nonsingular Vandermonde matrix.

Ding, Qi, and Wei [7] proposed a fast scheme for multiplying a Hankel tensor by vectors. The main approach is embedding a Hankel tensor into a larger anti-circulant tensor, which can be diagonalized by the Fourier matrices. A special mth-order N-dimensional Hankel tensor $\mathscr{C}$ is called an *anti-circulant tensor*, if its generating vector has a period N. Let the first N components of its generating vector be $\mathbf{c} = [c_0, c_1, \ldots, c_{N-1}]^\top$. Then the generating vector of $\mathscr{C}$ has the form

$$[c_0, c_1, \ldots, c_{N-1}, \ldots, c_0, c_1, \ldots, c_{N-1}, c_0, c_1, \ldots, c_{N-m}]^\top \in \mathbb{C}^{m(N-1)+1}.$$

Thus we often call the vector $\mathbf{c}$ the *compressed generating vector* of the anti-circulant tensor $\mathscr{C}$. Ding, Qi, and Wei proved in [7, Theorem 3.1] that an mth-order N-dimensional anti-circulant tensor can be diagonalized by the N-by-N Fourier matrix, i.e.,

$$\mathscr{C} = \mathscr{D} \times_1 F_N \times_2 F_N \cdots \times_m F_N,$$

where $F_N = \left(\exp(\frac{2\pi\imath}{N} jk)\right)_{j,k=0}^{N-1}$ $(\imath = \sqrt{-1})$ is the N-by-N Fourier matrix, and $\mathscr{D}$ is a diagonal tensor with diagonal entries $\text{ifft}(\mathbf{c}) = F_N^{-1}\mathbf{c}$ (here, "ifft" is an abbreviation of "inverse fast Fourier transform"). Then given m vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m \in \mathbb{C}^N$, we can calculate the anti-circulant tensor-vector product by

$$\mathscr{C} \times_1 \mathbf{y}_1 \times_2 \mathbf{y}_2 \cdots \times_m \mathbf{y}_m = (F_N^{-1}\mathbf{c})^\top \big(F_N\mathbf{y}_1. * F_N\mathbf{y}_2. * \cdots . * F_N\mathbf{y}_m\big),$$

where ".∗" is a Matlab-type notation for multiplying two vectors component-by-component, and $F_N\mathbf{y}_k$ and $F_N^{-1}\mathbf{c}$ can be realized via fft and ifft, respectively.

Let $\mathscr{H}$ be an mth-order Hankel tensor of size $n_1 \times n_2 \times \cdots \times n_m$ and $\mathbf{h}$ be its generating vector. Taking the vector $\mathbf{h}$ as the compressed generating vector, we can form an anti-circulant tensor $\mathscr{C}_\mathscr{H}$ of order m and dimension $N = n_1+\cdots+n_m-m+1$. Interestingly, we find that the Hankel tensor $\mathscr{H}$ is exactly the first leading principal subtensor of $\mathscr{C}_\mathscr{H}$, that is, $\mathscr{H} = \mathscr{C}_\mathscr{H}(1 : n_1, 1 : n_2, \ldots, 1 : n_m)$. Hence, the Hankel tensor-vector product $\mathscr{H} \times_1 \mathbf{x}_1 \times_2 \mathbf{x}_2 \cdots \times_m \mathbf{x}_m$ is equal to the anti-circulant tensor-vector product

$$\mathscr{C}_\mathscr{H} \times_1 \begin{bmatrix}\mathbf{x}_1\\ \mathbf{0}\end{bmatrix} \times_2 \begin{bmatrix}\mathbf{x}_2\\ \mathbf{0}\end{bmatrix} \cdots \times_m \begin{bmatrix}\mathbf{x}_m\\ \mathbf{0}\end{bmatrix},$$

where $\mathbf{0}$ denotes an all-zero vector of appropriate size. Thus it can be computed via

$$\mathscr{H} \times_1 \mathbf{x}_1 \times_2 \mathbf{x}_2 \cdots \times_m \mathbf{x}_m = (F_N^{-1}\mathbf{h})^\top \left(F_N \begin{bmatrix}\mathbf{x}_1\\ \mathbf{0}\end{bmatrix} . * F_N \begin{bmatrix}\mathbf{x}_2\\ \mathbf{0}\end{bmatrix} . * \cdots . * F_N \begin{bmatrix}\mathbf{x}_m\\ \mathbf{0}\end{bmatrix}\right).$$

Particularly, when $\mathscr{H}$ is square and all the vectors are the same, i.e., $n := n_1 = \cdots = n_m$ and $\mathbf{x} := \mathbf{x}_1 = \cdots = \mathbf{x}_m$, the homogeneous polynomial can be evaluated via

$$\mathscr{H}\mathbf{x}^m = (F_N^{-1}\mathbf{h})^\top \left(F_N \begin{bmatrix}\mathbf{x}\\ \mathbf{0}\end{bmatrix}\right)^{[m]}, \tag{2.2}$$

where $N = mn-m+1$, and $\mathbf{v}^{[m]} = [v_1^m, v_2^m, \ldots, v_N^m]^\top$ stands for the componentwise mth power of the vector $\mathbf{v}$. Moreover, this scheme has an analytic interpretation.

Comparing (2.2) and (2.1), we can write immediately that

$$\mathscr{H}\mathbf{x}^m = \mathbf{h}^\top (\underbrace{\mathbf{x} * \mathbf{x} * \cdots * \mathbf{x}}_{m}) =: \mathbf{h}^\top \mathbf{x}^{*m}, \tag{2.3}$$

since $F_N = F_N^\top$. Employing this convolution formula for Hankel tensor-vector products, we can derive the inheritability of positive semi-definiteness and the SOS property of Hankel tensors from the lower order to the higher order.

2.2 Lower-order implies higher-order

Use $\mathscr{H}_m$ to denote an mth-order n-dimensional Hankel tensor with the generating vector $\mathbf{h} \in \mathbb{R}^{mn-m+1}$, where m is even and $n = qk - q + 1$ for some integers q and k. Then by the convolution formula (2.3), we have $\mathscr{H}_m\mathbf{x}^m = \mathbf{h}^\top\mathbf{x}^{*m}$ for an arbitrary vector $\mathbf{x} \in \mathbb{C}^n$. Assume that $\mathscr{H}_{qm}$ is a (qm)th-order k-dimensional Hankel tensor that shares the same generating vector $\mathbf{h}$ with $\mathscr{H}_m$. Similarly, it holds that $\mathscr{H}_{qm}\mathbf{y}^{qm} = \mathbf{h}^\top\mathbf{y}^{*qm}$ for an arbitrary vector $\mathbf{y} \in \mathbb{C}^k$.

If $\mathscr{H}_m$ is positive semi-definite, then it is equivalent to $\mathscr{H}_m\mathbf{x}^m = \mathbf{h}^\top\mathbf{x}^{*m} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Note that $\mathbf{y}^{*qm} = (\mathbf{y}^{*q})^{*m}$. Thus for an arbitrary vector $\mathbf{y} \in \mathbb{R}^k$, we have

$$\mathscr{H}_{qm}\mathbf{y}^{qm} = \mathbf{h}^\top \mathbf{y}^{*qm} = \mathbf{h}^\top (\mathbf{y}^{*q})^{*m} = \mathscr{H}_m (\mathbf{y}^{*q})^m \geq 0.$$

Therefore, the higher-order but lower-dimensional Hankel tensor $\mathscr{H}_{qm}$ is also positive semi-definite. Furthermore, if $\mathscr{H}_m$ is positive definite, i.e., $\mathscr{H}_m\mathbf{x}^m > 0$ for all nonzero vector $\mathbf{x} \in \mathbb{R}^n$, then $\mathscr{H}_{qm}$ is also positive definite. We may also derive the negative definite and negative semi-definite cases similarly.

If $\mathscr{H}_m$ is SOS, then there are some multivariate polynomials $p_1, p_2, \ldots, p_r$ such that for any $\mathbf{x} \in \mathbb{R}^n$

$$\mathscr{H}_m\mathbf{x}^m = \mathbf{h}^\top \mathbf{x}^{*m} = p_1(\mathbf{x})^2 + p_2(\mathbf{x})^2 + \cdots + p_r(\mathbf{x})^2.$$

Thus we have for any $\mathbf{y} \in \mathbb{R}^k$

$$\mathscr{H}_{qm}\mathbf{y}^{qm} = \mathscr{H}_m(\mathbf{y}^{*q})^m = p_1(\mathbf{y}^{*q})^2 + p_2(\mathbf{y}^{*q})^2 + \cdots + p_r(\mathbf{y}^{*q})^2. \tag{2.4}$$

From the definition of discrete convolutions, we know that $\mathbf{y}^{*q}$ is also a multivariate polynomial about $\mathbf{y}$. Therefore, the higher-order Hankel tensor $\mathscr{H}_{qm}$ is also SOS. Moreover, the SOS rank, i.e., the minimum number of squares in the sum-of-squares representations [5], of $\mathscr{H}_{qm}$ is no larger than the SOS rank of $\mathscr{H}_m$. Hence, we summarize the inheritance properties of positive semi-definiteness and the SOS property in the following theorem.

Theorem 2.1 *If an* mth-*order Hankel tensor is positive/negative (semi-)definite, then the* (qm)th-*order Hankel tensor with positive integer* q *and the same generating vector is also positive/negative (semi-)definite. If an* mth-*order Hankel tensor is SOS, then the* (qm)th-*order Hankel tensor with the same generating vector is also SOS with no larger SOS rank.*

Let $\mathscr{T}$ be an mth-order n-dimensional tensor and $\mathbf{x} \in \mathbb{C}^n$. Recall that $\mathscr{T}\mathbf{x}^{m-1}$ is a vector with $(\mathscr{T}\mathbf{x}^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} \mathscr{T}_{ii_2\ldots i_m} x_{i_2}\ldots x_{i_m}$. If there is a real scalar λ and a nonzero $\mathbf{x} \in \mathbb{R}^n$ such that $\mathscr{T}\mathbf{x}^{m-1} = \lambda\mathbf{x}^{[m-1]}$, where $\mathbf{x}^{[m-1]} := [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^\top$, then we call λ an *H-eigenvalue* of the tensor $\mathscr{T}$ and $\mathbf{x}$ a corresponding *H-eigenvector*. This concept was first introduced by Qi [17], and H-eigenvalues are shown to be essential for investigating tensors. By [17, Theorem 5], we know that an even order symmetric tensor is positive (semi-)definite if and only if all its H-eigenvalues are positive (nonnegative). Applying the convolution formula, we can further obtain a quantitative result about the extremal H-eigenvalues of Hankel tensors. Let $\|\cdot\|_p$ be the p-norm of vectors, i.e., $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$.

Theorem 2.2 *Let* $\mathscr{H}_m$ *and* $\mathscr{H}_{qm}$ *be two Hankel tensors with the same generating vector of order* m *and* qm*, respectively, where* m *is even. Denote the minimal and the maximal H-eigenvalue of a tensor as* $\lambda_{\min}(\cdot)$ *and* $\lambda_{\max}(\cdot)$*, respectively. Then*

$$\lambda_{\min}(\mathscr{H}_{qm}) \geq \begin{cases} c_1 \cdot \lambda_{\min}(\mathscr{H}_m), & \text{if } \mathscr{H}_{qm} \text{ is positive semi-definite}, \\ c_2 \cdot \lambda_{\min}(\mathscr{H}_m), & \text{otherwise}; \end{cases}$$

and

$$\lambda_{\max}(\mathscr{H}_{qm}) \leq \begin{cases} c_1 \cdot \lambda_{\max}(\mathscr{H}_m), & \text{if } \mathscr{H}_{qm} \text{ is negative semi-definite}, \\ c_2 \cdot \lambda_{\max}(\mathscr{H}_m), & \text{otherwise}, \end{cases}$$

where $c_1 = \min_{\mathbf{y}\in\mathbb{R}^k} \|\mathbf{y}^{*q}\|_m^m / \|\mathbf{y}\|_{qm}^{qm}$ *and* $c_2 = \max_{\mathbf{y}\in\mathbb{R}^k} \|\mathbf{y}^{*q}\|_m^m / \|\mathbf{y}\|_{qm}^{qm}$ *are positive constants depending on* m, n, *and* q.

Proof Since $\mathscr{H}_m$ and $\mathscr{H}_{qm}$ are even order symmetric tensors, from [17, Theorem 5] we have

$$\lambda_{\min}(\mathscr{H}_m) = \min_{\mathbf{x}\in\mathbb{R}^n} \frac{\mathscr{H}_m\mathbf{x}^m}{\|\mathbf{x}\|_m^m}, \quad \lambda_{\min}(\mathscr{H}_{qm}) = \min_{\mathbf{y}\in\mathbb{R}^k} \frac{\mathscr{H}_{qm}\mathbf{y}^{qm}}{\|\mathbf{y}\|_{qm}^{qm}},$$

where $n = qk - q + 1$.

If $\mathscr{H}_{qm}$ is positive semi-definite, i.e., $\mathscr{H}_{qm}\mathbf{y}^{qm} \geq 0$ for all $\mathbf{y} \in \mathbb{R}^k$, then we denote $c_1 = \min_{\mathbf{y}\in\mathbb{R}^k} \|\mathbf{y}^{*q}\|_m^m / \|\mathbf{y}\|_{qm}^{qm}$, which is a constant depending only on m, n, and q. Then by the convolution formula proposed above, we have

$$\begin{aligned}\lambda_{\min}(\mathscr{H}_{qm}) &\geq c_1 \cdot \min_{\mathbf{y}\in\mathbb{R}^k} \frac{\mathscr{H}_{qm}\mathbf{y}^{qm}}{\|\mathbf{y}^{*q}\|_m^m} = c_1 \cdot \min_{\mathbf{y}\in\mathbb{R}^k} \frac{\mathscr{H}_m(\mathbf{y}^{*q})^m}{\|\mathbf{y}^{*q}\|_m^m} \geq c_1 \cdot \min_{\mathbf{x}\in\mathbb{R}^n} \frac{\mathscr{H}_m\mathbf{x}^m}{\|\mathbf{x}\|_m^m} \\ &= c_1 \cdot \lambda_{\min}(\mathscr{H}_m).\end{aligned}$$

If $\mathscr{H}_{qm}$ is not positive semi-definite, then we denote $c_2 = \max_{\mathbf{y}\in\mathbb{R}^k} \|\mathbf{y}^{*q}\|_m^m / \|\mathbf{y}\|_{qm}^{qm}$. Let $\widehat{\mathbf{y}}$ be a vector in $\mathbb{R}^k$ such that $\lambda_{\min}(\mathscr{H}_{qm}) = \mathscr{H}_{qm}\widehat{\mathbf{y}}^{qm} / \|\widehat{\mathbf{y}}\|_{qm}^{qm} < 0$. Then

$$\lambda_{\min}(\mathscr{H}_{qm}) \geq c_2 \cdot \frac{\mathscr{H}_{qm}\widehat{\mathbf{y}}^{qm}}{\|\widehat{\mathbf{y}}^{*q}\|_m^m} = c_2 \cdot \frac{\mathscr{H}_m(\widehat{\mathbf{y}}^{*q})^m}{\|\widehat{\mathbf{y}}^{*q}\|_m^m} \geq c_2 \cdot \min_{\mathbf{x}\in\mathbb{R}^n} \frac{\mathscr{H}_m\mathbf{x}^m}{\|\mathbf{x}\|_m^m} = c_2 \cdot \lambda_{\min}(\mathscr{H}_m).$$

Thus we obtain a lower bound of the minimal H-eigenvalue of $\mathscr{H}_{qm}$, no matter whether this tensor is positive semi-definite or not. The proof of the upper bound of the maximal H-eigenvalue of $\mathscr{H}_{qm}$ is similar. □

2.3 SOS decomposition of strong Hankel tensors

When the lower order m in Theorem 2.1 equals 2, i.e., matrix case, the $(2q)$th-order Hankel tensor sharing the same generating vector with this positive semi-definite Hankel matrix is called a *strong Hankel tensor*. We shall discuss strong Hankel tensors in detail in later sections. Now we focus on how to write out an SOS decomposition of a strong Hankel tensor following the formula (2.4). Li, Qi, and Xu [12] showed that even order strong Hankel tensors are SOS. However, their proof is not constructive, and no concrete SOS decomposition is given.

For an arbitrary Hankel matrix H generated by $\mathbf{h}$, we can compute its *Takagi factorization* efficiently by the algorithm proposed by Browne, Qiao, and Wei [4], where only the generating vector rather than the whole Hankel matrix is required to store. The Takagi factorization can be written as $H = UDU^\top$, where $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r]$ is a column unitary matrix ($U^*U = I$) and $D = \text{diag}(d_1, d_2, \ldots, d_r)$ is a diagonal matrix. When the matrix is real, the Takagi factorization is exactly the singular value decomposition of the Hankel matrix H. Furthermore, when H is positive semi-definite, the diagonal matrix D has nonnegative diagonal entries. Thus the polynomial $\mathbf{x}^\top H\mathbf{x}$ can be expressed as a sum of squares $p_1(\mathbf{x})^2 + p_2(\mathbf{x})^2 + \cdots + p_r(\mathbf{x})^2$, where

$$p_k(\mathbf{x}) = d_k^{1/2}\mathbf{u}_k^\top \mathbf{x}, \quad k = 1, 2, \ldots, r.$$

Following the formula (2.4), the $2q$-degree polynomial $\mathscr{H}_{2q}\mathbf{y}^{2q}$ can also be written as a sum of squares $q_1(\mathbf{y})^2 + q_2(\mathbf{y})^2 + \cdots + q_r(\mathbf{y})^2$, where

$$q_k(\mathbf{y}) = d_k^{1/2}\mathbf{u}_k^\top \mathbf{y}^{*q}, \quad k = 1, 2, \ldots, r.$$

Recall that any homogenous polynomial is associated with a symmetric tensor. And an interesting observation is that the homogenous polynomial $q_k(\mathbf{y})$ is associated with a qth-order Hankel tensor generated by $d_k^{1/2}\mathbf{u}_k$. Thus we determine an SOS decomposition of a strong Hankel tensor $\mathscr{H}_{2q}$ by r vectors $d_k^{1/2}\mathbf{u}_k$ ($k = 1, 2, \ldots, r$). And we summarize the above procedure in Algorithm 1.

Algorithm 1 An SOS decomposition of a strong Hankel tensor.

Input: The generating vector $\mathbf{h}$ of a strong Hankel tensor;
Output: An SOS decomposition $q_1(\mathbf{y})^2 + q_2(\mathbf{y})^2 + \cdots + q_r(\mathbf{y})^2$ of this Hankel tensor;
1: Compute the Takagi factorization of the Hankel matrix generated by $\mathbf{h}$: $H = UDU^\top$;
2: $\mathbf{q}_k = d_k^{1/2}\mathbf{u}_k$ for $k = 1, 2, \ldots, r$;
3: Then $\mathbf{q}_k$ generates a qth-order Hankel tensor $\mathscr{Q}_k$ as the coefficient tensor of each term $q_k(\cdot)$ in the SOS decomposition for $k = 1, 2, \ldots, r$;

Example 2.1 The first example is a 4th-order 3-dimensional Hankel tensor $\mathscr{H}$ generated by $[1, 0, 1, 0, 1, 0, 1, 0, 1]^\top \in \mathbb{R}^9$. The Takagi factorization of the Hankel matrix generated by the same vector is

$$\begin{bmatrix} 1&0&1&0&1\\ 0&1&0&1&0\\ 1&0&1&0&1\\ 0&1&0&1&0\\ 1&0&1&0&1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \cdot \begin{bmatrix} 3&0\\ 0&2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}}\\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Thus by Algorithm 1, an SOS decomposition of $\mathscr{H}\mathbf{y}^4$ is obtained:

$$\left(\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \cdot \begin{bmatrix} 1&0&1\\ 0&1&0\\ 1&0&1 \end{bmatrix} \cdot \begin{bmatrix} y_1\\ y_2\\ y_3 \end{bmatrix}\right)^2 + \left(\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \cdot \begin{bmatrix} 0&1&0\\ 1&0&1\\ 0&1&0 \end{bmatrix} \cdot \begin{bmatrix} y_1\\ y_2\\ y_3 \end{bmatrix}\right)^2$$
$$= (y_1^2 + y_2^2 + y_3^2 + 2y_1y_3)^2 + (2y_1y_2 + 2y_2y_3)^2.$$

However, the SOS decomposition is not unique, since $\mathscr{H}\mathbf{y}^4$ can also be written as $\frac{1}{2}(y_1 + y_2 + y_3)^4 + \frac{1}{2}(y_1 - y_2 + y_3)^4$.

3 The second inheritance property of Hankel tensors

In this section, we prove the second inheritance property of Hankel tensors, i.e., if the associated Hankel matrix of a Hankel tensor has no negative (non-positive, positive, or nonnegative) eigenvalues, then that Hankel tensor has no negative (non-positive, positive, or nonnegative, respectively) H-eigenvalues. A basic tool to prove this is the augmented Vandermonde decomposition with positive coefficients.

3.1 Strong Hankel tensors

Recall that the associated Hankel matrix of a strong Hankel tensor is positive semi-definite. When m is even, we immediately know that an even order strong Hankel tensor must be positive semi-definite by Theorem 2.1, which has been proved in [18, Theorem 3.1].

Qi [18] also introduced the *Vandermonde decomposition* of a Hankel tensor

$$\mathscr{H} = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k^{\circ m}, \tag{3.1}$$

where $\mathbf{v}_k$ is in the Vandermonde form $\left[1, \xi_k, \xi_k^2, \ldots, \xi_k^{n-1}\right]^\top$, $\mathbf{v}^{\circ m} := \underbrace{\mathbf{v} \circ \mathbf{v} \circ \cdots \circ \mathbf{v}}_{m}$ is a rank-one tensor, and the outer product is defined by

$$(\mathbf{v}_1 \circ \mathbf{v}_2 \circ \cdots \circ \mathbf{v}_m)_{i_1 i_2 \ldots i_m} = (\mathbf{v}_1)_{i_1} (\mathbf{v}_2)_{i_2} \cdots (\mathbf{v}_m)_{i_m}.$$

The Vandermonde decomposition is equivalent to the factorization of the generating vector of $\mathscr{H}$, i.e.,

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_r \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_r \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{mn-m} & \xi_2^{mn-m} & \cdots & \xi_r^{mn-m} \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix}.$$

Since the above Vandermonde matrix is nonsingular if and only if $r = mn - m + 1$ and $\xi_1, \xi_2, \ldots, \xi_r$ are mutually distinct, every Hankel tensor must have such a Vandermonde decomposition with the number of terms r in (3.1) no larger than $mn - m + 1$. When the Hankel tensor is positive semi-definite, we desire that all the coefficients α_k are positive, so that each term in (3.1) is a rank-one positive semi-definite Hankel tensor when m is even. Moreover, a real square Hankel tensor $\mathscr{H}$ is called a *complete Hankel tensor*, if all the coefficients α_k in one of its Vandermonde decompositions are positive [18].

Nevertheless, the set of all complete Hankel tensors is not "complete". Li, Qi, and Xu showed in [12, Corollary1] that the mth-order n-dimensional complete Hankel tensor cone is not closed and its closure is the mth-order n-dimensional strong Hankel tensor cone. An obvious counterexample is $\mathbf{e}_n^{\circ m}$, where $\mathbf{e}_n = [0, \dots, 0, 1]^\top$. Since all the Vandermonde vectors begin with unity and α_k ($k = 1, 2, \dots, r$) are positive, the positive semi-definite Hankel tensor $\mathbf{e}_n^{\circ m}$ is not a complete Hankel tensor.

Fortunately, $\alpha \mathbf{e}_n^{\circ m}$ is the only kind of rank-one non-complete Hankel tensors.

Proposition 3.1 *If* $\mathbf{v}^{\circ m}$ *is a rank-one Hankel tensor, then*

$$\mathbf{v} = \alpha[1, \xi, \xi^2, \dots, \xi^{n-1}]^\top \quad \textit{or} \quad \alpha[0, 0, \dots, 0, 1]^\top.$$

Proof If $v_0 \neq 0$, then we can assume that $v_0 = 1$ without loss of generality. Denote $v_1 = \xi$. Then from the definition of Hankel tensors, we have $v_0^{m-2} v_1 v_{k-1} = v_0^{m-1} v_k$. Hence, we can easily write $v_k = \xi^k$ for $k = 0, 1, \dots, n-1$, i.e., $\mathbf{v} = [1, \xi, \xi^2, \dots, \xi^{n-1}]^\top$.

If $v_0 = 0$ and $k < n-1$, then we have (i) $v_{k-1}^{(m-1)/2} v_k v_{k+1}^{(m-1)/2} = v_k^m$ for m is odd, and (ii) $v_{k-1}^{m/2} v_{k+1}^{m/2} = v_k^m$ for m is even and $k < n-1$. It implies that $u_k = 0$ for $k = 0, 1, \dots, n-2$, so $\mathbf{v} = \alpha[0, 0, \dots, 0, 1]^\top$. □

We will shortly show that if we add the term $\mathbf{e}_n^{\circ m}$ into the basis, then all the strong Hankel tensors can be decomposed into an *augmented Vandermonde decomposition*

$$\mathscr{H} = \sum_{k=1}^{r-1} \alpha_k \mathbf{v}_k^{\circ m} + \alpha_r \mathbf{e}_n^{\circ m}.$$

Note that $\frac{1}{\xi^{n-1}}[1, \xi, \xi^2, \dots, \xi^{n-1}]^\top \to \mathbf{e}_n$ when $\xi \to \infty$. The cone of Hankel tensors with an augmented Vandermonde decomposition is actually the closure of the cone of complete Hankel tensors. When a Hankel tensor $\mathscr{H}$ has such an augmented Vandermonde decomposition, its associated Hankel matrix H also has a corresponding decomposition

$$H = \sum_{k=1}^{r-1} \alpha_k \widetilde{\mathbf{v}}_k^{\circ 2} + \alpha_r \mathbf{e}_{(n-1)m/2+1}^{\circ 2},$$

where $\widetilde{\mathbf{v}}_k = \left[1, \xi_k, \xi_k^2, \dots, \xi_k^{(n-1)m/2}\right]^\top$ and $\mathbf{v}^{\circ 2}$ is exactly $\mathbf{v}\mathbf{v}^\top$, and vice versa. Therefore, if a positive semi-definite Hankel tensor has an augmented Vandermonde decomposition with all positive coefficients, then it is also a strong Hankel tensor, that is, its associated Hankel matrix must be positive semi-definite. Furthermore, when we obtain an augmented Vandermonde decomposition of its associated Hankel matrix, we can induce an augmented Vandermonde decomposition of the original Hankel tensor straightforwardly. Hence, we begin with the positive semi-definite Hankel matrices.

3.2 A general vandermonde decompostion of Hankel matrices

We shall introduce the algorithm for a general Vandermonde decomposition of an arbitrary Hankel matrix proposed by Boley, Luk, and Vandevoorde [2] in this subsection. Let's begin with a nonsingular Hankel matrix $H \in \mathbb{C}^{r \times r}$. After we solve the

Yule-Walker equation [9, Chapter 4.7]:

$$\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{r-1} \\ h_1 & h_2 & h_3 & \cdots & h_r \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ h_{r-2} & h_{r-1} & h_r & \cdots & h_{2r-3} \\ h_{r-1} & h_r & h_{r+1} & \cdots & h_{2r-2} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{r-1} \end{bmatrix} = \begin{bmatrix} h_r \\ h_{r+1} \\ \vdots \\ h_{2r-2} \\ \gamma \end{bmatrix},$$

we obtain an r term recurrence for $k = r, r+1, \ldots, 2r-2$, i.e.,

$$h_k = a_{r-1}h_{k-1} + a_{r-2}h_{k-2} + \cdots + a_0 h_{k-r}.$$

Denote C the companion matrix [9, Chapter 7.4.6] corresponding to the polynomial $p(\lambda) = \lambda^r - a_{r-1}\lambda^{r-1} - \cdots - a_0\lambda^0$, i.e.,

$$C = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_0 & a_1 & \cdots & a_{r-2} & a_{r-1} \end{bmatrix}.$$

Let the Jordan canonical form of C be $C = V^\top J V^{-\top}$, where $J = \operatorname{diag}\{J_1, J_2, \ldots, J_s\}$ and J_l is the $k_l \times k_l$ Jordan block corresponding to eigenvalue λ_l. Moreover, the nonsingular matrix V has the form

$$V = \left[\mathbf{v}, J^\top \mathbf{v}, (J^\top)^2 \mathbf{v}, \ldots, (J^\top)^{r-1}\mathbf{v}\right],$$

where $\mathbf{v} = [\mathbf{e}_{k_1,1}^\top, \mathbf{e}_{k_2,1}^\top, \ldots, \mathbf{e}_{k_s,1}^\top]^\top$ is a vector partitioned conformably with J and $\mathbf{e}_{k_l,1}$ is the first k_l-dimensional unit coordinate vector. This kind of V is often called a *confluent Vandermonde matrix*. When the multiplicities of all the eigenvalues of C equal one, the matrix V is exactly a Vandermonde matrix.

Denote $\mathbf{h}_0$ as the first column of H and $\mathbf{w} = V^{-\top}\mathbf{h}_0$. There exists a unique block diagonal matrix $D = \operatorname{diag}\{D_1, D_2, \ldots, D_s\}$, which is also partitioned conformably with J, satisfying

$$D\mathbf{v} = \mathbf{w} \quad \text{and} \quad DJ^\top = JD.$$

Moreover, each block D_l is a k_l-by-k_l upper anti-triangular Hankel matrix. If we partition $\mathbf{w} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s]^\top$ conformably with J, then the lth block is determined by

$$D_l = \begin{bmatrix} (\mathbf{w}_l)_1 & (\mathbf{w}_l)_2 & \cdots & (\mathbf{w}_l)_{k_l} \\ (\mathbf{w}_l)_2 & \cdots & (\mathbf{w}_l)_{k_l} & 0 \\ \vdots & \iddots & \iddots & \vdots \\ (\mathbf{w}_l)_{k_l} & 0 & \cdots & 0 \end{bmatrix}.$$

Finally, we obtain a general Vandermonde decomposition of the full-rank Hankel matrix

$$H = V^\top D V.$$

If the leading $r \times r$ principal submatrix, i.e., $H(1:r, 1:r)$, of an $n \times n$ rank-r Hankel matrix H is nonsingular, then H admits the Vandermonde decomposition $H = (V_{r\times n})^\top D_{r\times r} V_{r\times n}$, which is induced by the decomposition of the leading $r \times r$ principal submatrix.

Nevertheless, this generalized Vandermonde decomposition is insufficient for discussing the positive definiteness of a real Hankel matrix, since the factors V and D could be complex even though H is a real matrix. We shall modify this decomposition into a general real Vandermonde decomposition. Assume that two eigenvalues λ_1 and λ_2 of C form a pair of conjugate complex numbers. Then the corresponding parts in D and V are also conjugate, respectively. That is,

$$\begin{bmatrix} V_1^\top & V_2^\top \end{bmatrix} \cdot \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1^\top & \bar{V}_1^\top \end{bmatrix} \cdot \begin{bmatrix} D_1 & \\ & \bar{D}_1 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ \bar{V}_1 \end{bmatrix}.$$

Note that

$$\begin{bmatrix} \mathbf{u} + \mathbf{v}\imath & \mathbf{u} - \mathbf{v}\imath \end{bmatrix} \cdot \begin{bmatrix} a + b\imath & \\ & a - b\imath \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}^\top + \mathbf{v}^\top\imath \\ \mathbf{u}^\top - \mathbf{v}^\top\imath \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \cdot 2 \begin{bmatrix} a & -b \\ -b & -a \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}^\top \\ \mathbf{v}^\top \end{bmatrix}.$$

Denote the jth column of $V_1^\top$ as $\mathbf{u}_j + \mathbf{v}_j\imath$ and the jth entry of the first column of D_1 is $a_j + b_j\imath$, where $\imath = \sqrt{-1}$ and $\mathbf{u}_j, \mathbf{v}_j, a_j, b_j$ are all real. Then

$$\begin{aligned} &\begin{bmatrix} V_1^\top & V_2^\top \end{bmatrix} \cdot \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{v}_1 & \dots & \mathbf{u}_{k_1} & \mathbf{v}_{k_1} \end{bmatrix} \cdot \begin{bmatrix} \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{k_1} \\ \Lambda_2 & \cdots & \Lambda_{k_1} & \mathbf{O} \\ \vdots & \iddots & \iddots & \vdots \\ \Lambda_{k_l} & \mathbf{O} & \cdots & \mathbf{O} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{v}_1^\top \\ \vdots \\ \mathbf{u}_{k_1}^\top \\ \mathbf{v}_{k_1}^\top \end{bmatrix}, \end{aligned}$$

where the 2-by-2 block Λ_j is

$$\Lambda_j = 2 \begin{bmatrix} a_j & -b_j \\ -b_j & -a_j \end{bmatrix}.$$

If we perform the same transformations to all the conjugate eigenvalue pairs, then we obtain a real decomposition of the real Hankel matrix $H = \widehat{V}^\top \widehat{D} \widehat{V}$. Here, each diagonal block of $\widehat{D}$ corresponding to a real eigenvalue of C is an upper anti-triangular Hankel matrix, and each corresponding to a pair of conjugate eigenvalues is an upper anti-triangular block Hankel matrix with 2-by-2 blocks.

We claim that if the Hankel matrix H is positive semi-definite, then all the eigenvalues of C are real and of multiplicity one. This can be seen by recognizing that the following three cases of the diagonal blocks of $\widehat{D}$ cannot be positive semi-definite: (1) an anti-upper triangular Hankel block whose size is larger than 1, (2) a 2-by-2 block $\Lambda_j = 2\begin{bmatrix} a_j & -b_j \\ -b_j & -a_j \end{bmatrix}$, and (3) a block anti-upper triangular Hankel block with the blocks in case (2). Therefore, when a real rank-r Hankel matrix H is positive semi-definite and its leading $r \times r$ principal submatrix is positive definite, the block diagonal matrix $\widehat{D}$ in the generalized real Vandermonde decomposition must be diagonal. Hence this Hankel matrix admits a Vandermonde decomposition with r terms and positive coefficients:

$$H = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k \mathbf{v}_k^\top, \quad \alpha_k > 0, \quad \mathbf{v}_k = \left[1, \xi_k, \ldots, \xi_k^{n-1}\right]^\top.$$

This result for positive definite Hankel matrices is known [23, Lemma 0.2.1].

3.3 An augmented vandermonde decomposition of Hankel tensors

However, the associated Hankel matrix of a Hankel tensor does not necessarily have a full-rank leading principal submatrix. Thus we shall study whether a positive semi-definite Hankel matrix can always decomposed into the form

$$H = \sum_{k=1}^{r-1} \alpha_k \mathbf{v}_k \mathbf{v}_k^\top + \alpha_r \mathbf{e}_n \mathbf{e}_n^\top, \quad \alpha_k \geq 0.$$

We first need a lemma about the rank-one modifications on a positive semi-definite matrix. Denote the range and the kernel of a matrix A as $\mathrm{Ran}(A)$ and $\mathrm{Ker}(A)$, respectively.

Lemma 3.1 *Let A be a positive semi-definite matrix with* $\mathrm{rank}(A) = r$. *Then there exists a unique $\alpha > 0$ such that $A - \alpha \mathbf{u}\mathbf{u}^\top$ is positive semi-definite with* $\mathrm{rank}(A - \alpha \mathbf{u}\mathbf{u}^\top) = r - 1$, *if and only if $\mathbf{u}$ is in the range of A.*

Proof The condition $\mathrm{rank}(A - \alpha \mathbf{u}\mathbf{u}^\top) = \mathrm{rank}(A) - 1$ obviously indicates that $\mathbf{u} \in \mathrm{Ran}(A)$. Thus we only need to prove the "if" part of the statement.

Let the nonzero eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_r$ and the corresponding eigenvectors be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$, respectively. Since $\mathbf{u} \in \mathrm{Ran}(A)$, we can write $\mathbf{u} = \mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_r \mathbf{x}_r$. Note that $\mathrm{rank}(A - \alpha \mathbf{u}\mathbf{u}^\top) = \mathrm{rank}(A) - 1$ also implies $\dim \mathrm{Ker}(A - \alpha \mathbf{u}\mathbf{u}^\top) = \dim \mathrm{Ker}(A) + 1$, and equivalently there exists a unique subspace $\mathrm{span}\{\mathbf{p}\}$ such that $A\mathbf{p} = \alpha \mathbf{u}(\mathbf{u}^\top \mathbf{p}) \neq \mathbf{0}$. Write $\mathbf{p} = \eta_1 \mathbf{x}_1 + \eta_2 \mathbf{x}_2 + \cdots + \eta_r \mathbf{x}_r$. Then there exists a unique linear combination and a unique scalar α satisfying

$$\eta_i = \mu_i / \lambda_i \ (i = 1, 2, \ldots, r), \quad \alpha = \left(\mu_1^2/\lambda_1 + \cdots + \mu_r^2/\lambda_r\right)^{-1}.$$

Then we verify the positive semi-definiteness of $A - \alpha \mathbf{u}\mathbf{u}^\top$. For any $\mathbf{x} = \xi_1 \mathbf{x}_1 + \xi_2 \mathbf{x}_2 + \cdots + \xi_r \mathbf{x}_r$ in the range of A, we have

$$\mathbf{x}^\top A \mathbf{x} = \xi_1^2 \lambda_1 + \cdots + \xi_r^2 \lambda_r, \quad \mathbf{u}^\top \mathbf{x} = \mu_1 \xi_1 + \cdots + \mu_r \xi_r.$$

Along with the expression of α, the Hölder inequality indicates that $\mathbf{x}^\top A \mathbf{x} \geq \alpha(\mathbf{u}^\top \mathbf{x})^2$, i.e., the rank-$(r-1)$ matrix $A - \alpha \mathbf{u}\mathbf{u}^\top$ is also positive semi-definite. □

The following theorem tells that the leading $(r-1) \times (r-1)$ principal submatrix of a rank-r positive semi-definite Hankel matrix is always full-rank, even when the leading $r \times r$ principal submatrix is rank deficient.

Theorem 3.1 *Let $H \in \mathbb{R}^{n \times n}$ be a positive semi-definite Hankel matrix with* $\operatorname{rank}(H) = r$. *If the last column $H(:, n)$ is linearly dependent of the first $n-1$ columns $H(:, 1:n-1)$, then the leading $r \times r$ principal submatrix $H(1:r, 1:r)$ is positive definite. If $H(:, n)$ is linearly independent of $H(:, 1:n-1)$, then the leading $(r-1) \times (r-1)$ principal submatrix $H(1:r-1, 1:r-1)$ is positive definite.*

Proof We apply the mathematical induction on the dimension n. First, the statement is apparently true for 2×2 positive semi-definite Hankel matrices. Assume that the statement holds for $(n-1) \times (n-1)$ Hankel matrices, then we consider the $n \times n$ case.

Case 1: When the last column $H(:, n)$ is linearly dependent of the first $n-1$ columns $H(:, 1:n-1)$, the submatrix $H(1:n-1, 1:n-1)$ is also a rank-r positive semi-definite Hankel matrix. Then from the induction hypothesis, $H(1:r, 1:r)$ is full rank if $H(1:n-1, n-1)$ is linearly dependent of $H(1:n-1, 1:n-2)$, and $H(1:r-1, 1:r-1)$ is full rank otherwise. We shall show that the column $H(1:n-1, n-1)$ is always linearly dependent of $H(1:n-1, 1:n-2)$.

Assuming on the contrary, the leading $(r-1) \times (r-1)$ principal submatrix $H(1:r-1, 1:r-1)$ is positive definite, and the rank of $H(1:n-2, 1:n-1)$ is $r-1$. Since the column $H(:, n)$ is linear dependent of the previous $(n-1)$ columns, the rank of $H(1:n-2, :)$ is also $r-1$. Thus the rectangular Hankel matrix $H(1:n-2, :)$ has a Vandermonde decomposition

$$H(1:n-2, :) = \sum_{k=1}^{r-1} \alpha_k \begin{bmatrix} 1 \\ \xi_k \\ \vdots \\ \xi_k^{n-3} \end{bmatrix} \begin{bmatrix} 1 & \xi_k & \cdots & \xi_k^{n-2} & \xi_k^{n-1} \end{bmatrix}.$$

Since $H(n-1, n-1) = H(n-2, n)$, the square Hankel matrix $H(1:n-1, 1:n-1)$ has a corresponding decomposition

$$H(1:n-1, 1:n-1) = \sum_{k=1}^{r-1} \alpha_k \begin{bmatrix} 1 \\ \xi_k \\ \vdots \\ \xi_k^{n-2} \end{bmatrix} \begin{bmatrix} 1 & \xi_k & \cdots & \xi_k^{n-2} \end{bmatrix}.$$

This contradicts with $\operatorname{rank}\big(H(1:n-1, 1:n-1)\big) = r$. Therefore, $H(1:n-1, n-1)$ must be linearly dependent of $H(1:n-1, 1:n-2)$. Hence, the leading principal submatrix $H(1:r, 1:r)$ is positive definite.

Case 2: When the last column $H(:, n)$ is linearly independent of the first $n-1$ columns $H(:, 1:n-1)$, it is equivalent to that $\mathbf{e}_n$ is in the range of H. Thus, from Lemma 3.1, there exists a scalar α_r such that $H - \alpha_r \mathbf{e}_n \mathbf{e}_n^\top$ is rank-$(r-1)$ and also positive semi-definite. Referring back to Case 1, we know that the leading principal submatrix $H(1:r-1, 1:r-1)$ is positive definite. □

Following the above theorem, when $H(:, n)$ is linearly dependent of $H(:, 1:n-1)$, the leading $r \times r$ principal submatrix $H(1:r, 1:r)$ is positive definite. Thus H has a Vandermonde decomposition with all positive coefficients

$$H = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k \mathbf{v}_k^\top, \quad \alpha_k > 0, \quad \mathbf{v}_k = \big[1, \xi_k, \ldots, \xi_k^{n-1}\big]^\top.$$

When $H(:, n)$ is linearly independent of $H(:, 1:n-1)$, the leading $(r-1)\times(r-1)$ principal submatrix $H(1:r-1, 1:r-1)$ is positive definite. Thus H has an augmented Vandermonde decomposition with positive coefficients:

$$H = \sum_{k=1}^{r-1} \alpha_k \mathbf{v}_k \mathbf{v}_k^\top + \alpha_r \mathbf{e}_n \mathbf{e}_n^\top, \quad \alpha_k > 0, \quad \mathbf{v}_k = \big[1, \xi_k, \ldots, \xi_k^{n-1}\big]^\top.$$

Combining the definition of strong Hankel tensors and the analysis at the end of Sect. 3.1, we arrive at the following theorem.

Theorem 3.2 *Let $\mathscr{H}$ be an mth-order n-dimensional Hankel tensor and the rank of its associated Hankel matrix be r. Then it is a strong Hankel tensor if and only if it admits a Vandermonde decomposition with positive coefficients:*

$$\mathscr{H} = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k^{\circ m}, \quad \alpha_k > 0, \quad \mathbf{v}_k = \big[1, \xi_k, \ldots, \xi_k^{n-1}\big]^\top, \tag{3.2}$$

or an augmented Vandermonde decomposition with positive coefficients:

$$\mathscr{H} = \sum_{k=1}^{r-1} \alpha_k \mathbf{v}_k^{\circ m} + \alpha_r \mathbf{e}_n^{\circ m}, \quad \alpha_k > 0, \quad \mathbf{v}_k = \big[1, \xi_k, \ldots, \xi_k^{n-1}\big]^\top. \tag{3.3}$$

After Theorem 3.2, the strong Hankel tensor cone is understood thoroughly. The polynomials induced by strong Hankel tensors are not only positive semi-definite and sum-of-squares, as proved in [18, Theorem 3.1] and [12, Corollary 2], but also sum-of-mth-powers. The detailed algorithm for computing an augmented Vandermonde decomposition of a strong Hankel tensor is displayed as follows.

Algorithm 2 Augmented Vandermonde decomposition of a strong Hankel tensor.

Input: The generating vector $\mathbf{h}$ of a strong Hankel tensor;
Output: Coefficients α_k; Poles ξ_k;
1: Compute the Takagi factorization of the Hankel matrix H generated by $\mathbf{h}$: $H = UDU^\top$;
2: Recognize the rank r of H and whether $\mathbf{e}_n$ is in the range of U;
3: **If** $r < n$, **then**
4: **If** $\mathbf{e}_n \in \mathrm{Ran}(U)$, **then**
5: $\xi_r = \mathrm{Inf}$;
6: $\alpha_r = \left(\sum_{j=1}^r U(n,j)^2 / D(j,j)\right)^{-1}$;
7: Apply Algorithm 2 for the strong Hankel tensor generated by $[h_0, \dots, h_{mn-m-1}, h_{mn-m} - \alpha_r]$ to compute α_k and ξ_k for $k = 1, 2, \dots, r-1$;
8: **ElseIf** $\mathbf{e}_n \notin \mathrm{Ran}(U)$, **then**
9: $\mathbf{a} = U(1:r, 1:r)^{-\top} D(1:r, 1:r)^{-1} U(1:r, 1:r)^{-1} \mathbf{h}(r : 2r-1)$;
10: **EndIf**
11: **Else**
12: $\mathbf{a} = U(1:r, 1:r)^{-\top} D(1:r, 1:r)^{-1} U(1:r, 1:r)^{-1} [\mathbf{h}(r : 2r-2)^\top, \gamma]^\top$, where γ is arbitrary;
13: **EndIf**
14: Compute the roots $\xi_1, \xi_2, \dots, \xi_r$ of the polynomial

$$p(\xi) = \xi^r - a_{r-1}\xi^{r-1} - \cdots - a_0\xi^0;$$

15: Solve the Vandermonde system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_r \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^{r-1} & \xi_2^{r-1} & \cdots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{r-1} \end{bmatrix};$$

16: **return** α_k, ξ_k for $k = 1, 2, \dots, r$;

Example 3.1 An mth-order n-dimensional Hilbert tensor (see [21]) is defined by

$$\mathscr{H}(i_1, i_2, \dots, i_m) = \frac{1}{i_1 + i_2 + \cdots + i_m + 1}, \quad i_1, \dots, i_m = 0, 1, \dots, n-1.$$

Apparently, a Hilbert tensor is a special Hankel tensor with the generating vector $\left[1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{mn-m+1}\right]^\top$. Moreover, its associated Hankel matrix is a Hilbert matrix, which is well-known to be positive definite [21]. Thus a Hilbert tensor must be a strong Hankel tensor.

We take the 4th-order 5-dimensional Hilbert tensor, which is generated by $\left[1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{17}\right]^\top$, as the second example. Applying Algorithm 2 and taking $\gamma = \frac{1}{18}$ in the algorithm, we obtain a standard Vandermonde decomposition of

$$\mathscr{H} = \sum_{k=1}^{9} \alpha_k \mathbf{v}_k^{\circ 4}, \quad \mathbf{v}_k = \left[1, \xi_k, \dots, \xi_k^{n-1}\right]^\top,$$

where α_k and ξ_k are displayed in the following table.

k	1	2	3	4	5	6	7	8	9
ξ_k	0.9841	0.9180	0.8067	0.6621	0.5000	0.3379	0.1933	0.0820	0.0159
α_k	0.0406	0.0903	0.1303	0.1562	0.1651	0.1562	0.1303	0.0903	0.0406

From the computational result, we can see that a Hilbert tensor is actually a nonnegative strong Hankel tensor with a nonnegative Vandermonde decomposition.

Then we test some strong Hankel tensors without standard Vandermonde decompositions.

Example 3.2 Randomly generate two real scalars ξ_1, ξ_2. Then we construct a 4th-order n-dimensional strong Hankel tensor by

$$\mathscr{H} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^{\circ 4} + \begin{bmatrix} 1 \\ \xi_1 \\ \vdots \\ \xi_1^{n-2} \\ \xi_1^{n-1} \end{bmatrix}^{\circ 4} + \begin{bmatrix} 1 \\ \xi_2 \\ \vdots \\ \xi_2^{n-2} \\ \xi_2^{n-1} \end{bmatrix}^{\circ 4} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^{\circ 4}.$$

For instance, we set the size $n = 10$ and apply Algorithm 2 to obtain an augmented Vandermonde decomposition of $\mathscr{H}$. We repeat this experiment for $10,000$ times, and the average relative error between the computational solutions $\widehat{\xi}_1, \widehat{\xi}_2$ and the exact solution ξ_1, ξ_2 is 4.7895×10^{-12}. That is, our algorithm recover the augmented Vandermonde decomposition of $\mathscr{H}$ accurately.

3.4 The second inheritance property of Hankel tensors

Employing the augmented Vandermonde decomposition, we can prove the following theorem.

Theorem 3.3 *If a Hankel matrix has no negative (non-positive, positive, or nonnegative) eigenvalues, then all of its associated higher-order Hankel tensors have no negative (non-positive, positive, or nonnegative, respectively) H-eigenvalues.*

Proof This statement for even order case is a direct corollary of Theorem 2.1.

Suppose that the Hankel matrix has no negative eigenvalues. When the order m is odd, decompose an mth-order strong Hankel tensor $\mathscr{H}$ into $\mathscr{H} = \sum_{k=1}^{r-1} \alpha_k \mathbf{v}_k^{\circ m} + \alpha_r \mathbf{e}_n^{\circ m}$ with $\alpha_k \geq 0$ $(k = 1, 2, \ldots, r)$. Then for an arbitrary vector $\mathbf{x}$, the first entry of $\mathscr{H}\mathbf{x}^{m-1}$ is

$$(\mathscr{H}\mathbf{x}^{m-1})_1 = \sum_{k=1}^{r-1} (\mathbf{v}_k)_1 \cdot \alpha_k (\mathbf{v}_k^\top \mathbf{x})^{m-1} = \sum_{k=1}^{r-1} \alpha_k (\mathbf{v}_k^\top \mathbf{x})^{m-1} \geq 0.$$

If $\mathscr{H}$ has no H-eigenvalues, then the theorem is proven. Assume it has at least one H-eigenvalue λ and let $\mathbf{x}$ be a corresponding H-eigenvector. Then when $x_1 \neq 0$, from the definition we have

$$\lambda = (\mathscr{H}\mathbf{x}^{m-1})_1 / x_1^{m-1} \geq 0,$$

since m is odd. When $x_1 = 0$, we know that $(\mathscr{H}\mathbf{x}^{m-1})_1$ must also be zero, thus all the item $\alpha_k(\mathbf{v}_k^\top \mathbf{x})^{m-1} = 0$ for $k = 1, 2, \ldots, r-1$. So the tensor-vector product

$$\mathscr{H}\mathbf{x}^{m-1} = \sum_{k=1}^{r-1} \mathbf{v}_k \cdot \alpha_k(\mathbf{v}_k^\top \mathbf{x})^{m-1} + \mathbf{e}_n \cdot \alpha_r x_n^{m-1} = \mathbf{e}_n \cdot \alpha_r x_n^{m-1},$$

and it is apparent that $\mathbf{x} = \mathbf{e}_n$ and $\lambda = \alpha_r x_n^{m-1}$, where the H-eigenvalue λ is nonnegative. The other cases can be proved similarly. □

When the Hankel matrix has no negative eigenvalue, it is positive semi-definite, i.e., the associated Hankel tensors are strong Hankel tensors, which may be of either even or odd order. Thus, we have the following corollary.

Corollary 3.1 [12] *Strong Hankel tensors have no negative H-eigenvalues.*

We also have a quantitative version of the second inheritance property.

Theorem 3.4 *Let $\mathscr{H}$ be an mth-order n-dimensional Hankel tensor, and H be its associated Hankel matrix. If H is positive semi-definite, then*

$$\lambda_{\min}(\mathscr{H}) \geq c \cdot \lambda_{\min}(H),$$

where $c = \min_{\mathbf{y}\in\mathbb{R}^n} \|\mathbf{y}^{*\frac{m}{2}}\|_2^2 / \|\mathbf{y}\|_m^m$ if m is even, and $c = \min_{\mathbf{y}\in\mathbb{R}^n} \|\mathbf{y}^{*\frac{m-1}{2}}\|_2^2 / \|\mathbf{y}\|_{m-1}^{m-1}$ if m is odd, both dependent on m and n. If H is negative semi-definite, then

$$\lambda_{\max}(\mathscr{H}) \leq c \cdot \lambda_{\max}(H).$$

Proof When the minimal eigenvalue of H equals 0, the above equality holds for any nonnegative c. Moreover, when the order m is even, Theorem 2.2 gives the constant c. Thus, we need only to discuss the situation that H is positive definite and m is odd.

Since H is positive definite, the Hankel tensor $\mathscr{H}$ has a standard Vandermonde decomposition with positive coefficients:

$$\mathscr{H} = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k^{\circ m}, \quad \alpha_k > 0, \quad \mathbf{v}_k = \left[1, \xi_k, \ldots, \xi_k^{n-1}\right]^\top,$$

where $\xi_1, \xi_2, \ldots, \xi_r$ are mutually distinct. Then by the proof of Theorem 3.3, for any nonzero $\mathbf{x} \in \mathbb{R}^n$,

$$(\mathscr{H}\mathbf{x}^{m-1})_1 = \sum_{k=1}^{r} \alpha_k(\mathbf{v}_k^\top \mathbf{x})^{m-1} > 0,$$

since $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r]$ spans the whole space. So if λ and $\mathbf{x}$ are an H-eigenvalue and a corresponding H-eigenvector of $\mathscr{H}$, then λ must be positive and the first entry x_1 of $\mathbf{x}$ must be nonzero.

Let $\mathbf{h} \in \mathbb{R}^{m(n-1)+1}$ be the generating vector of both the Hankel matrix H and the Hankel tensor $\mathscr{H}$. Denote $\mathbf{h}_1 = \mathbf{h}\big(1 : (m-1)(n-1)+1\big)$, which generates a Hankel matrix H_1 and an $(m-1)$st-order Hankel tensor $\mathscr{H}_1$. Note that H_1 is a leading principal submatrix of H and $\mathscr{H}_1$ is exactly the first row tensor of $\mathscr{H}$, i.e., $\mathscr{H}(1, :, :, \ldots, :)$. Then we have

$$\lambda = \frac{(\mathscr{H}\mathbf{x}^{m-1})_1}{x_1^{m-1}} = \frac{\mathscr{H}_1\mathbf{x}^{m-1}}{x_1^{m-1}} \geq \frac{\mathscr{H}_1\mathbf{x}^{m-1}}{\|\mathbf{x}\|_{m-1}^{m-1}}.$$

Now $m-1$ is an even number, then we know from Theorem 2.2 that there is a constant c such that $\lambda_{\min}(\mathscr{H}_1) \geq c \cdot \lambda_{\min}(H_1)$. Therefore, for each existing H-eigenvalue λ of $\mathscr{H}$, we obtain

$$\lambda \geq \lambda_{\min}(\mathscr{H}_1) \geq c \cdot \lambda_{\min}(H_1) \geq c \cdot \lambda_{\min}(H).$$

The last inequality holds because H_1 is a principal submatrix of H [9, Theorem 8.1.7]. □

When m is even, the parameter c in the theorem can be rewritten into

$$c = \min_{\mathbf{y}\in\mathbb{R}^n} \frac{\|\mathbf{y}^{*\frac{m}{2}}\|_2^2}{\|\mathbf{y}\|_m^m} = \frac{1}{N}\left(\min_{\mathbf{y}\in\mathbb{R}^n} \frac{\|\widehat{F}_N\mathbf{y}\|_m}{\|\mathbf{y}\|_m}\right)^m = \frac{N^{m-1}}{\|\widehat{F}_N^*\|_m^m},$$

where $N = (n-1)m/2 + 1$ and $\widehat{F}_N = F_N(:, 1 : n)$. Moreover, the matrix m-norm in the above equation can be computed by algorithms such as in [10].

It is unclear whether we have a similar quantitative form of the extremal H-eigenvalues of a Hankel tensor when its associated Hankel matrix has both positive and negative eigenvalues.

4 The third inheritance property of Hankel tensors

We have proved two inheritance properties of Hankel tensors in this paper. We now raise the third inheritance property of Hankel tensors as a conjecture.

Conjecture If a lower-order Hankel tensor has no negative H-eigenvalues, then its associated higher-order Hankel tensor with the same generating vector, where the higher order is a multiple of the lower order, also has no negative H-eigenvalues.

We see that the first inheritance property of Hankel tensors established in Sect. 2 is only a special case of this inheritance property, i.e., the lower-order Hankel tensor is of even-order. At this moment, we are unable to prove or to disprove this conjecture if the lower-order Hankel tensor is of odd-order. However, if this conjecture is true, then it is of significance. If the lower-order Hankel tensor is of odd-order while the higher-order Hankel tensor is of even-order, then the third inheritance property would provide a new way to identify some positive semi-definite Hankel tensors and a link between odd-order symmetric tensors of no negative H-eigenvalues and positive semi-definite symmetric tensors.

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