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Positive semi-definiteness and sum-of-squares property of fourth order four dimensional Hankel tensors

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ABSTRACT

A symmetric positive semi-definite (PSD) tensor, which is not sum-of-squares (SOS), is called a PSD non-SOS (PNS) tensor. Is there a fourth order four dimensional PNS Hankel tensor? The answer for this question has both theoretical and practical significance. Under the assumptions that the generating vector $\mathbf{v}$ of a Hankel tensor $\mathcal{A}$ is symmetric and the fifth element v_4 of $\mathbf{v}$ is fixed at 1, we show that there are two surfaces M_0 and N_0 with the elements v_2, v_6, v_1, v_3, v_5 of $\mathbf{v}$ as variables, such that $M_0 \geq N_0$, $\mathcal{A}$ is SOS if and only if $v_0 \geq M_0$, and $\mathcal{A}$ is PSD if and only if $v_0 \geq N_0$, where v_0 is the first element of $\mathbf{v}$. If $M_0 = N_0$ for a point $P = (v_2, v_6, v_1, v_3, v_5)^\top$, there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors for such v_2, v_6, v_1, v_3, v_5. Then, we call such P a PNS-free point. We prove that a 45-degree planar closed convex cone, a segment, a ray and an additional point are PNS-free. Numerical tests check various grid points and report that they are all PNS-free.

1. Introduction

In 1888, young Hilbert [1] proved that for homogeneous polynomials, only in the following three cases, a positive semi-definite (PSD) form definitely is a sum-of-squares (SOS) polynomial: (1) $m = 2$; (2) $n = 2$; (3) $m = 4$ and $n = 3$, where m is the degree of the polynomial and n is the number of variables. Hilbert proved that in all the other possible combinations of n and even m, there are PSD non-SOS (PNS) homogeneous polynomials. The most well-known PNS homogeneous polynomial is the Motzkin function [2] with $m = 6$ and $n = 3$. Other examples of PNS homogeneous polynomials were found in [3–6].

A homogeneous polynomial is uniquely corresponding to a symmetric tensor [7]. For a symmetric tensor, m is its order and n is its dimension. One important class of symmetric tensors is the Hankel tensor. Hankel tensors have important applications in signal processing [8–10], automatic control [11], and geophysics [12,13]. For example, Papy et al. [14,15] proposed a novel Hankel tensor model to analyze time-domain signals in nuclear magnetic resonance spectroscopy, which is used for brain tumor detection [16]. A fast computational framework for products of a Hankel tensor and vectors is addressed in Ding et al. [17]. In geophysics, Trickett et al. [13] established a new multidimensional seismic trace interpolator by using Hankel tensors.

In mathematical science, Luque and Thibon [18] studied the Hankel hyperdeterminants. Xu [19] studied the spectra of Hankel tensors and gave some upper bounds and lower bounds for the smallest and the largest eigenvalues. In [20], two

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classes of PSD Hankel tensors were identified. They are even order strong Hankel tensors and even order complete Hankel tensors. It was proved in [21] that complete Hankel tensors are strong Hankel tensors, and even order strong Hankel tensors are SOS tensors. It was also shown there that there are SOS Hankel tensors and PSD Hankel tensors, which are not strong Hankel tensors. Thus, a question was raised in [21]: Are all PSD Hankel tensors SOS tensors [22,23]? If there are no PSD non-SOS Hankel tensors, the problem for determining a given even order Hankel tensor is PSD or not can be answered by solving a semi-definite linear programming problem [21,24,25].

We may call the problem raised by the above question as the Hilbert–Hankel problem. In a certain sense, it is the Hilbert problem with a Hankel constraint. According to Hilbert [1,6], one case with low values of m and n, in which there are PNS homogeneous polynomials, is that $m = 6$ and $n = 3$. In [26], the Hilbert–Hankel problem with order six and dimension three was studied. Four special cases were analyzed. Thousands of random examples were checked. No PNS Hankel tensors of order six and dimension three were found in [26]. Theoretically, it is still an open problem whether there are PNS Hankel tensors of order six and dimension three or not.

According to Hilbert [1,6], another case with low values of m and n, in which there are PNS homogeneous polynomials, is that $m = n = 4$. In this paper, we consider this special case in a Hankel context. Let $\mathbf{v} = (v_0, v_1, \ldots, v_{12})^\top \in \Re^{13}$. A fourth order four dimensional *Hankel tensor* $\mathcal{A} = (a_{i_1i_2i_3i_4})$ is defined by

$$a_{i_1i_2i_3i_4} = v_{i_1+i_2+i_3+i_4-4},$$

for $i_1, i_2, i_3, i_4 = 1, 2, 3, 4$. The corresponding vector $\mathbf{v}$ that defines the Hankel tensor $\mathcal{A}$ is called the *generating vector* of $\mathcal{A}$. For $\mathbf{x} = (x_1, x_2, x_3, x_4)^\top \in \Re^4$, a Hankel tensor $\mathcal{A}$ uniquely defines a Hankel polynomial

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes 4} = \sum_{i_1,i_2,i_3,i_4=1}^{4} a_{i_1i_2i_3i_4}x_{i_1}x_{i_2}x_{i_3}x_{i_4} = \sum_{i_1,i_2,i_3,i_4=1}^{4} v_{i_1+i_2+i_3+i_4-4}x_{i_1}x_{i_2}x_{i_3}x_{i_4}. \tag{1}$$

If $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Re^4$, the Hankel tensor $\mathcal{A}$ is called *positive semi-definite* (PSD). If $f(\mathbf{x})$ can be represented as a sum of squares of quadratic homogeneous polynomials, the Hankel tensor $\mathcal{A}$ is called *sum-of-squares* (SOS). Clearly, $\mathcal{A}$ is PSD if it is SOS.

In the next section, we present some necessary conditions for the positive semi-definiteness of fourth order four dimensional Hankel tensors.

We may see that the role of v_j is symmetric in $f(\mathbf{x})$. In Section 3, we assume that

$$v_j = v_{12-j} \tag{2}$$

for $j = 0, \ldots, 5$. Under this assumption, by the results of Section 2, if $\mathcal{A}$ is PSD, we have $v_0 = v_{12} \geq 0$ and $v_4 = v_8 \geq 0$. Moreover, if $v_4 = v_8 = 0$ and $\mathcal{A}$ is PSD, $\mathcal{A}$ is SOS. Thus, we may only consider the case that $v_4 = v_8 > 0$. Since $\mathcal{A}$ is PSD or SOS or PNS if and only if $\alpha\mathcal{A}$ is PSD or SOS or PNS respectively, where α is an arbitrary positive number, we may simply assume that

$$v_4 = v_8 = 1. \tag{3}$$

Next, we show that there is a function $\eta(v_5, v_6)$ such that $\eta(v_5, v_6) \leq 1$ if $\mathcal{A}$ is PSD. We propose that there are two functions $M_0(v_2, v_6, v_1, v_3, v_5) \geq N_0(v_2, v_6, v_1, v_3, v_5)$, defined for $\eta(v_5, v_6) < 1$, such that $\mathcal{A}$ is SOS if and only if $v_0 \geq M_0$, and $\mathcal{A}$ is PSD if and only if $v_0 \geq N_0$. If $M_0 = N_0$ for some v_2, v_6, v_1, v_3, v_5, then there are no fourth order four dimensional PNS Hankel tensors for such v_2, v_6, v_1, v_3, v_5 under the symmetric assumption (2). We call such a point $P = (v_2, v_6, v_1, v_3, v_5)^\top \in \Re^5$ a *PNS-free point* of fourth order four dimensional Hankel tensors, or simply a PNS-free point. We call the set of points in $\Re^5$, satisfying $\eta(v_5, v_6) < 1$, the *effective domain* of fourth order four dimensional Hankel tensors, or simply the effective domain, and denote it by S. We show that if all the points in S are PNS-free, then there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors.

In Section 4, we show that a point P in S is PNS-free if there is a value M, such that when $v_0 = M$, $f_0(\mathbf{x}) \equiv f(\mathbf{x})$ has an SOS decomposition, and $f_0(\bar{\mathbf{x}}) = 0$ for $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)^\top \in \Re^4$ with $\bar{x}_1^2 + \bar{x}_4^2 \neq 0$. We call such a value M, such an SOS decomposition of $f_0(\mathbf{x})$, and such a vector $\bar{\mathbf{x}}$ the *critical value*, the *critical SOS decomposition* and the *critical minimizer* of $\mathcal{A}$ at P, respectively. Then, we show that the segment $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$ is PNS-free. We conjecture that this segment is the minimizer set of both M_0 and N_0. Then, we show that the 45° planar closed convex cone $C = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, b, 0, 0, 0)^\top : a \geq b \geq 1\}$, the ray $R = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, 0, 0, 0, 0)^\top : a \leq 0\}$ and the point $A = (1, 0, 0, 0, 0)^\top$ are also PNS-free. We illustrate L, C, R and A in Fig. 1.

In Section 5, numerical tests check various grid points, and find that $M_0 = N_0$ there. Thus, they are also PNS-free. Therefore, numerical tests indicate that there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors.

Some final remarks are made in Section 6.

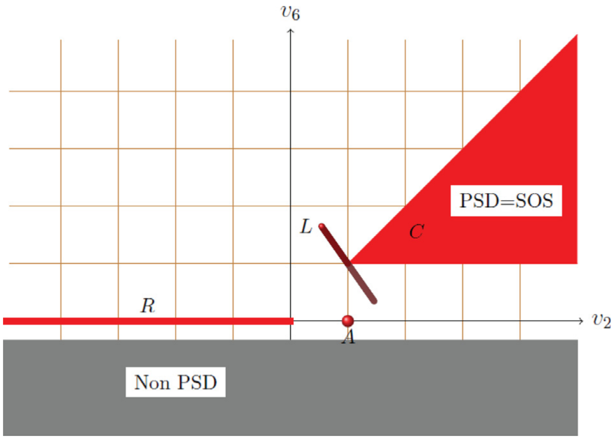

Fig. 1. The segment L, the planar closed convex cone C, the ray R and the point A.

2. Fourth order four dimensional hankel tensors

We write out (1) explicitly in terms of the coordinates of its generating vector $\mathbf{v}$:

$$\begin{aligned} f(\mathbf{x}) = {} & v_0x_1^4 + 4v_1x_1^3x_2 + v_2(4x_1^3x_3 + 6x_1^2x_2^2) + v_3(4x_1x_2^3 + 4x_1^3x_4 + 12x_1^2x_2x_3) \\ & + v_4(x_2^4 + 6x_1^2x_3^2 + 12x_1x_2^2x_3 + 12x_1^2x_2x_4) + v_5(4x_2^3x_3 + 12x_1x_2x_3^2 + 12x_1x_2^2x_4 + 12x_1^2x_3x_4) \\ & + v_6(4x_1x_3^3 + 4x_2^3x_4 + 6x_1^2x_4^2 + 6x_2^2x_3^2 + 24x_1x_2x_3x_4) \\ & + v_7(4x_2x_3^3 + 12x_2^2x_3x_4 + 12x_1x_3^2x_4 + 12x_1x_2x_4^2) + v_8(x_3^4 + 6x_2^2x_4^2 + 12x_2x_3^2x_4 + 12x_1x_3x_4^2) \\ & + v_9(4x_3^3x_4 + 4x_1x_4^3 + 12x_2x_3x_4^2) + v_{10}(4x_2x_4^3 + 6x_3^2x_4^2) + 4v_{11}x_3x_4^3 + v_{12}x_4^4. \end{aligned} \tag{4}$$

The following theorem gives some necessary conditions for fourth order four dimensional Hankel tensors being PSD. Particularly, we note that four key elements of its generating vector v_0, v_4, v_8, v_{12} must be nonnegative.

Theorem 1. *Suppose that $\mathcal{A} = (a_{i_1i_2i_3i_4})$ is a Hankel tensor generated by its generating vector $\mathbf{v} = (v_0, v_1, \ldots, v_{12})^\top \in \Re^{13}$. If $\mathcal{A}$ is a PSD (or positive definite, or SOS, or strong) Hankel tensor, then we have*

$$v_i \ge 0, \tag{5}$$

for $i = 0, 4, 8, 12,$

$$v_i + 6v_{i+2} + v_{i+4} \ge 4|v_{i+1} + v_{i+3}|, \tag{6}$$

for $i = 0, 4, 8,$

$$v_i + 6v_{i+4} + v_{i+8} \ge 4|v_{i+2} + v_{i+6}|, \tag{7}$$

for $i = 0, 4,$ and

$$v_0 + 6v_6 + v_{12} \ge 4|v_3 + v_9|. \tag{8}$$

Proof. Let $\mathbf{e}_k$ be the kth column of a 4-by-4 identity matrix, for $k = 1, 2, 3, 4$. Substituting $\mathbf{x} = \mathbf{e}_k$ to (4) for $k = 1, 2, 3, 4$, by $f(\mathbf{e}_k) \ge 0$, we have (5) for $i = 0, 4, 8, 12$.

Substituting $\mathbf{x} = \mathbf{e}_k + \mathbf{e}_{k+1}$ to (4) for $k = 1, 2, 3$, by $f(\mathbf{e}_k + \mathbf{e}_{k+1}) \ge 0$, we have

$$v_i + 4v_{i+1} + 6v_{i+2} + 4v_{i+3} + v_{i+4} \ge 0,$$

for $i = 0, 4, 8$. Substituting $\mathbf{x} = \mathbf{e}_k - \mathbf{e}_{k+1}$ to (4) for $k = 1, 2, 3$, by $f(\mathbf{e}_k - \mathbf{e}_{k+1}) \ge 0$, we have

$$v_i - 4v_{i+1} + 6v_{i+2} - 4v_{i+3} + v_{i+4} \ge 0,$$

for $i = 0, 4, 8$. Combining these two inequalities, we have (6) for $i = 0, 4, 8$.

Similarly, by $f(\mathbf{e}_k + \mathbf{e}_{k+2}) \ge 0$ and $f(\mathbf{e}_k - \mathbf{e}_{k+2}) \ge 0$ for $k = 1, 2$, we have (7) for $i = 0, 4$. By $f(\mathbf{e}_1 + \mathbf{e}_4) \ge 0$ and $f(\mathbf{e}_1 - \mathbf{e}_4) \ge 0$, we have (8). The theorem is proved. □

Whereafter, we say that a PSD Hankel tensor is SOS if a key element of its generating vector v_0, v_4, v_8, v_{12} vanishes. Before we show this, the following lemma is useful.

Lemma 1. *If a polynomial in one variable is always nonnegative:*

$$p(t) = a_0 t^{2k+1} + a_1 t^{2k} + \cdots + a_{2k+1} \geq 0, \quad \forall t \in \Re.$$

Then $a_0 = 0$.

Proof. If $a_0 > 0$, we let $t \to -\infty$ and get $p(t) \to -\infty$, which contradicts that $p(t)$ is nonnegative.

If $a_0 < 0$, we let $t \to +\infty$ and get $p(t) \to -\infty$, which also contradicts that $p(t)$ is nonnegative.

Hence, there must be $a_0 = 0$. □

Theorem 2. *Suppose the fourth order four dimensional Hankel tensor* $\mathcal{A}$ *is PSD and its generating vector is* $\mathbf{v}$. *If* $v_0 v_{12} = 0$, *then* $v_j = 0$, *for* $j = 1, \ldots, 11$, *and* $\mathcal{A}$ *is SOS.*

Proof. Without loss of generality, we assume that $v_0 = 0$.

To prove $v_1 = 0$, we take $\mathbf{x}_1 = (t, 1, 0, 0)^\top$. Then, the homogeneous polynomial (4) reduces to

$$f(\mathbf{x}_1) = 4v_1 t^3 + 6v_2 t^2 + 4v_3 t + v_4.$$

From Lemma 1, we have $v_1 = 0$ since $f(\mathbf{x}_1)$ is nonnegative. Similarly, we can prove $v_2 = v_3 = 0$ if we take $\mathbf{x}_2 = (t, 0, 1, 0)^\top$ and $\mathbf{x}_3 = (t, 0, 0, 1)^\top$ respectively.

From Theorem 1, we know $v_4 \geq 0$. When we take $\mathbf{x}_4 = (t^2, t, -\frac{1}{\sqrt{6}}, 0)^\top$, the homogeneous polynomial (4) reduces to

$$f(\mathbf{x}_4) = -\left(2\sqrt{6} - 2\right) v_4 t^4 + \mathcal{O}(t^3).$$

Let $t \to \infty$. Since $f(\mathbf{x}_4)$ is always nonnegative, we have $v_4 \leq 0$. Hence, there must be $v_4 = 0$.

If we take $\mathbf{x}_5 = (t^3, 0, t, 1)^\top$, the homogeneous polynomial (4) is

$$f(\mathbf{x}_5) = 12v_5 t^7 + \mathcal{O}(t^6).$$

From Lemma 1, we have $v_5 = 0$ since $f(\mathbf{x}_5)$ is nonnegative.

We take $\mathbf{x}_6 = (t, 0, 1, 0)^\top$. Then, the homogeneous polynomial (4) is

$$f(\mathbf{x}_6) = 4v_6 t + v_8.$$

From Lemma 1, we have $v_6 = 0$ since $f(\mathbf{x}_6)$ is nonnegative. Similarly, we can prove $v_7 = 0$ when we take $\mathbf{x}_7 = (0, t, 1, 0)^\top$.

We take $\mathbf{x}_8 = (t^4, 0, t, 1)^\top$. Then we have

$$f(\mathbf{x}_8) = 12v_8 t^5 + \mathcal{O}(t^4).$$

From Lemma 1, we have $v_8 = 0$ since the polynomial $f(\mathbf{x}_8)$ is nonnegative.

We could prove $v_9 = 0$, $v_{10} = 0$ and $v_{11} = 0$ if we take $\mathbf{x}_9 = (t, 0, 0, 1)^\top$, $\mathbf{x}_{10} = (0, t, 0, 1)^\top$ and $\mathbf{x}_{11} = (0, 0, t, 1)^\top$, respectively.

Finally, since $v_0 = v_1 = \cdots = v_{11} = 0$, we have

$$f(\mathbf{x}) = v_{12} x_4^4.$$

By Theorem 1, we get $v_{12} \geq 0$. Hence, the Hankel tensor $\mathcal{A}$ is obviously SOS. □

Theorem 3. *Suppose the fourth order four dimensional Hankel tensor* $\mathcal{A}$ *is PSD and its generating vector is* $\mathbf{v}$. *If* $v_4 v_8 = 0$, *then* $v_j = 0$ *for* $j = 1, 2, \ldots, 11$, *and* $\mathcal{A}$ *is SOS.*

Proof. By symmetry, we only need to prove this theorem under the condition $v_4 = 0$.

If we take $\mathbf{x}_1 = (1, t, 0, 0)^\top$, the homogeneous polynomial (4) reduces to

$$f(\mathbf{x}_1) = 4v_3 t^3 + 6v_2 t^2 + 4v_1 t + v_0.$$

From Lemma 1, we have $v_3 = 0$ since $f(\mathbf{x}_1)$ is nonnegative. Similarly, we can prove $v_5 = v_6 = 0$ if we take $\mathbf{x}_2 = (0, t, 1, 0)^\top$ and $\mathbf{x}_3 = (0, t, 0, 1)^\top$ respectively.

To prove $v_7 = 0$, we take $\mathbf{x}_4 = (0, t^2, t, 1)^\top$. Then, the homogeneous polynomial (4) reduces to

$$f(\mathbf{x}_4) = 16v_7 t^5 + \mathcal{O}(t^4).$$

From Lemma 1, we have $v_7 = 0$ since $f(\mathbf{x}_4)$ is nonnegative.

From Theorem 1, we know $v_8 \geq 0$. When we take $\mathbf{x}_5 = (0, -t^2, t, 1)^\top$, the homogeneous polynomial (4) reduces to

$$f(\mathbf{x}_5) = -5v_8 t^4 + \mathcal{O}(t^3).$$

Let $t \to \infty$. Since $f(\mathbf{x}_5)$ is always nonnegative, we have $v_8 \leq 0$. Hence, there must be $v_8 = 0$.

If we take $\mathbf{x}_6 = (0, 0, t, 1)^\top$, the homogeneous polynomial (4) is

$$f(\mathbf{x}_6) = 4v_9 t^3 + \mathcal{O}(t^2).$$

From Lemma 1, we have $v_9 = 0$ since $f(\mathbf{x}_6)$ is nonnegative. Similarly, we could prove $v_{10} = 0$ and $v_{11} = 0$ if we take $\mathbf{x}_7 = (0, t, 0, 1)^\top$ and $\mathbf{x}_8 = (0, 0, t, 1)^\top$, respectively.

The prove of $v_1 = 0$ and $v_2 = 0$ could be similarly obtained if we take $\mathbf{x}_9 = (1, t, 0, 0)^\top$ and $\mathbf{x}_{10} = (1, 0, t, 0)^\top$ respectively.

Finally, since $v_j = 0$ for $j = 1, \ldots, 11$, we have

$$f(\mathbf{x}) = v_0 x_1^4 + v_{12} x_4^4.$$

By Theorem 1, we get $v_0 \geq 0$ and $v_{12} \geq 0$. Hence, the Hankel tensor $\mathcal{A}$ is obviously SOS. □

3. Symmetric generating vectors

Now, we make assumptions (2) and (3). At the beginning, we consider a mini problem which is the Hankel polynomial with $x_1 = x_4 = 0$. This problem helps us to analyze the effective domain of two important surfaces M_0 and N_0.

3.1. Function η

We consider a two variable quartic polynomial

$$g(y_1, y_2) = \alpha y_1^4 + 4\beta y_1^3 y_2 + 6\gamma y_1^2 y_2^2 + 4\beta y_1 y_2^3 + \alpha y_2^4.$$

Its PSD property is completely characterized by the following theorem.

Theorem 4. *The quartic polynomial $g(y_1, y_2)$ is PSD if and only if*

$$\alpha \geq \eta(\beta, \gamma) := \begin{cases} 4|\beta| - 3\gamma & \text{if } \gamma \leq |\beta|, \\ \dfrac{3\gamma - \sqrt{9\gamma^2 - 8\beta^2}}{2} & \text{if } \gamma > |\beta|. \end{cases}$$

Proof. First, if $g(y_1, y_2)$ is PSD, from $g(1, -1) \geq 0$ and $g(1, 1) \geq 0$, we have $\alpha \geq 4|\beta| - 3\gamma$. Thus, in any case, $\eta(\beta, \gamma) \geq 4|\beta| - 3\gamma$.

Second, suppose that $\alpha \geq 4|\beta| - 3\gamma$. If $\gamma \leq 0$, we get

$$g(y_1, y_2) = (\alpha - 4|\beta| + 3\gamma)(y_1^4 + y_2^4) + 4|\beta|(y_1 + y_2)^2(y_1^2 - y_1 y_2 + y_2^2) - 3\gamma(y_1^2 - y_2^2)^2 \geq 0.$$

If $0 < \gamma \leq |\beta|$, we rewrite $g(y_1, y_2)$ as follows

$$g(y_1, y_2) = (\alpha - 4|\beta| + 3\gamma)(y_1^4 + y_2^4) + (y_1 + y_2)^2 \left[(4|\beta| - 3\gamma)(y_1^2 + y_2^2) - (4|\beta| - 6\gamma)y_1 y_2\right].$$

Since $(4|\beta| - 6\gamma)^2 - 4(4|\beta| - 3\gamma)^2 = -48|\beta|(|\beta| - \gamma) \leq 0$, it yields that $g(y_1, y_2) \geq 0$.

Finally, we consider the case $\gamma > |\beta|$. Let $\bar{\alpha} = \frac{3\gamma - \sqrt{9\gamma^2 - 8\beta^2}}{2} > 0$. Then, we have

$$g(y_1, y_2) = (\alpha - \bar{\alpha})(y_1^4 + y_2^4) + \bar{\alpha}\left(y_1^2 + \frac{2\beta}{\bar{\alpha}} y_1 y_2 + y_2^2\right)^2.$$

Obviously, if $\alpha \geq \bar{\alpha}$, $g(y_1, y_2)$ is SOS and PSD.

Next, we show that $y_1^2 + \frac{2\beta}{\bar{\alpha}} y_1 y_2 + y_2^2 = 0$ has nonzero real roots. For the convenience, we denote $t = \frac{y_1}{y_2}$ and prove that $t^2 + \frac{2\beta}{\bar{\alpha}} t + 1 = 0$ has real roots. It is easy to see that $t = 0$ is not its root. Since $\gamma > |\beta|$, we have

$$\frac{|\beta|}{\bar{\alpha}} = \frac{2|\beta|}{3\gamma - \sqrt{9\gamma^2 - 8\beta^2}} = \frac{2|\beta|\left(3\gamma + \sqrt{9\gamma^2 - 8\beta^2}\right)}{8\beta^2} \geq \frac{8|\beta|\gamma}{8\beta^2} \geq 1.$$

Hence, $|\beta| \geq \bar{\alpha}$. The discriminant of the quadratic in t is

$$\left(\frac{2\beta}{\bar{\alpha}}\right)^2 - 4 = 4\frac{\beta^2 - \bar{\alpha}^2}{\bar{\alpha}^2} \geq 0.$$

Therefore, there are nonzero (y_1, y_2) such that $g(y_1, y_2) = (\alpha - \bar{\alpha})(y_1^4 + y_2^4)$. Obviously, if $g(y_1, y_2)$ is PSD, we have $\alpha \geq \bar{\alpha}$. Thus, we say $\eta(\beta, \gamma) = \bar{\alpha}$ if $\gamma > |\beta|$. □

Then we have another necessary condition for a fourth order four dimensional Hankel tensor $\mathcal{A}$ to be PSD under assumptions (2) and (3).

Corollary 1. *Under assumptions (2) and (3), if $\mathcal{A}$ is PSD, then $\eta(v_5, v_6) \le 1$.*

Proof. Let $x_1 = x_4 = 0$, $x_2 = y_1$ and $x_3 = y_2$. By Theorem 4, we have the conclusion. □

3.2. Surfaces M_0 and N_0

We now introduce the key idea of this paper, to establish two surfaces M_0 and N_0, in the following theorem.

Theorem 5. *Suppose that assumptions (2) and (3) hold. Then, there are two functions $M_0(v_2, v_6, v_1, v_3, v_5) \ge N_0(v_2, v_6, v_1, v_3, v_5) > 0$ defined for*

$$\eta(v_5, v_6) < 1, \tag{9}$$

such that $\mathcal{A}$ is SOS if and only if $v_0 \ge M_0(v_2, v_6, v_1, v_3, v_5)$, and $\mathcal{A}$ is PSD if and only if $v_0 \ge N_0(v_2, v_6, v_1, v_3, v_5)$. If for all v_5 and v_6 satisfying (9), we have $M_0(v_2, v_6, v_1, v_3, v_5) = N_0(v_2, v_6, v_1, v_3, v_5)$, then there are no fourth order four dimensional PNS Hankel tensors under assumption (2).

Proof. Using assumptions (2) and (3), we rewrite (4) as

$$f(\mathbf{x}) = v_0(x_1^4 + x_4^4) + \bar{v}_4(x_2^4 + x_3^4) + f_1(\mathbf{x}) + f_2(\mathbf{x}),$$

where

$$f_1(\mathbf{x}) = \eta(v_5, v_6)(x_2^4 + x_3^4) + 4v_5(x_2^3x_2 + x_2x_3^3) + 6v_6x_2^2x_3^2$$

and

$$\bar{v}_4 = 1 - \eta(v_5, v_6).$$

Then $\bar{v}_4 > 0$ by (9). By Theorem 4, $f_1(\mathbf{x})$ is PSD. Since $f_1(\mathbf{x})$ has only two variables, it is also SOS by Hilbert [1,6].

We now consider terms in $f_2(\mathbf{x})$. Each monomial in $f_2(\mathbf{x})$ has at least one factor as a power of x_1 or x_4. We may order the monomials of $f_2(\mathbf{x})$. For example, consider $12v_5x_1x_2x_3^2$. Assume that it is ordered as the kth monomial of $f_2(\mathbf{x})$. Then by the arithmetic–geometric inequality, we may see that

$$-12v_5x_1x_2x_3^2 \le 3|v_5|\left(\frac{1}{\epsilon_k^3}x_1^4 + \epsilon_kx_2^4 + 2\epsilon_kx_3^4\right),$$

where ϵ_k is a small positive number. We may let ϵ_k be small enough such that the sum of the coefficients for x_2^4 on the right hand side of the above inequality for all possible k is less than $\bar{v}_4$. By symmetry, the sum of the coefficients for x_3^4 on the right hand side of the above inequality for all possible k is less than $\bar{v}_4$. We see that

$$12v_5x_1x_2x_3^2 + 3|v_5|\left(\frac{1}{\epsilon_k^3}x_1^4 + \epsilon_kx_2^4 + 2\epsilon_kx_3^4\right)$$

is a PSD diagonal minus tail form. By [27], it is SOS. Thus, as long as v_0 is big enough, when (9) is satisfied, $f(\mathbf{x})$ is SOS. From this, we see that M_0 and N_0 exist, such that they are defined as long as (9) is satisfied, $M_0 \ge N_0$, $\mathcal{A}$ is SOS if and only if $v_0 \ge M_0$, and $\mathcal{A}$ is PSD if and only if $v_0 \ge N_0$.

By Theorem 4, we now only need to consider the case that $\eta(v_5, v_6) = 1$. Suppose that for all v_5 and v_6 satisfying (9), we have $M_0(v_2, v_6, v_1, v_3, v_5) = N_0(v_2, v_6, v_1, v_3, v_5)$. Since the sets for PSD Hankel tensors and SOS Hankel tensors are closed [21], this implies that for all v_5 and v_6 satisfying $\eta(v_5, v_6) = 1$, we also have $M_0(v_2, v_6, v_1, v_3, v_5) = N_0(v_2, v_6, v_1, v_3, v_5)$, as long as N_0 is defined there. Thus, in this case, by Theorem 3, there are no fourth order four dimensional PNS Hankel tensors under assumption (2). □

For the variables of M_0 and N_0, we put v_2 and v_6 before v_1, v_3 and v_5, as v_2, v_6 play a more important role in the PSD and SOS properties of $\mathcal{A}$, comparing with v_1, v_3 and v_5.

We now regard $P = (v_2, v_6, v_1, v_3, v_5)^\top$ as a point in $\Re^5$. If $M_0(P) = N_0(P)$, P is called a *PNS-free point*. We call

$$S = \{(v_2, v_6, v_1, v_3, v_5)^\top \in \Re^5 : \eta(v_5, v_6) < 1\}$$

the *effective domain*. Theorem 5 says that if all the points in the effective domain are PNS-free, then there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors. In the next sections, we will study more on PNS-free points.

Table 1
The values of $M_0(v_2, v_6, 0, 0, 0) = N_0(v_2, v_6, 0, 0, 0)$ on some grid points.

$v_2 \setminus v_6$	−0.2	−0.1	0	0.5	1	1.5	2	4
−4.0	3.54e4	8.74e3	3.76e3	4.78e2	3.12e2	3.92e2	6.23e2	6.37e3
−2.0	2.98e4	6.77e3	2.73e3	2.75e2	1.25e2	1.70e2	3.57e2	6.11e3
−1.0	2.72e4	5.85e3	2.26e3	1.91e2	6.15e1	9.26e1	2.73e2	6.06e3
−0.5	2.59e4	5.42e3	2.04e3	1.53e2	3.78e1	6.41e1	2.48e2	6.06e3
0.0	2.46e4	4.99e3	1.82e3	1.20e2	1.96e1	4.50e1	2.39e2	6.07e3
0.5	2.34e4	4.57e3	1.62e3	8.90e1	7.058	4.18e1	2.45e2	6.09e3
1.0	2.21e4	4.17e3	1.42e3	6.21e1	1.000	4.93e1	2.56e2	6.11e3
1.5	2.09e4	3.78e3	1.23e3	3.90e1	4.191	5.69e1	2.67e2	6.14e3
2.0	1.98e4	3.41e3	1.06e3	2.02e1	8.00e0	6.46e1	2.78e2	6.16e3
3.0	1.75e4	2.70e3	7.28e2	7.16e0	1.66e1	8.01e1	3.01e2	6.21e3
4.0	1.53e4	2.04e3	4.41e2	1.23e1	2.60e1	9.60e1	3.23e2	6.25e3

4. Theoretical proofs of some PNS-free regions

4.1. Critical SOS decomposition

For the convenience, we present formally three ingredients used in theoretical proofs of this section. If a point belongs to the effective domain and enjoys these ingredients, it is PNS-free.

Definition 1. Suppose that assumptions (2) and (3) hold and $P = (v_2, v_6, v_1, v_3, v_5)^\top \in S$. Suppose that there is a number M such that $\mathcal{A}$ is SOS if $v_0 = M$, and a point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)^\top \in \Re^4$ such that $\bar{x}_1^2 + \bar{x}_4^2 > 0$ and $f_0(\bar{\mathbf{x}}) = 0$, where $f_0(\mathbf{x}) \equiv f(\mathbf{x})$ with $v_0 = M$. Then we call M the *critical value* of $\mathcal{A}$ at P, the SOS decomposition $f_0(\mathbf{x})$ the *critical SOS decomposition* of $\mathcal{A}$ at P, and $\bar{\mathbf{x}}$ the *critical minimizer* of $\mathcal{A}$ at P.

Theorem 6. *Let $P \in S$. Then P is PNS-free if $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f_0(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at P.*

Proof. Suppose that $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f_0(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at P. Then we have $M \geq M_0(P)$ by the definition of M_0. If $v_0 < M$, then

$$f(\bar{\mathbf{x}}) = (v_0 - M)(\bar{x}_1^4 + \bar{x}_4^4) + f_0(\bar{\mathbf{x}}) < 0.$$

This implies that $N_0(P) \geq M$ by the definition of N_0. But $N_0(P) \leq M_0(P)$. Thus, $M_0(P) = N_0(P) = M$, i.e., P is PNS-free. □

We believe that all the effective domain S is PNS-free. In the next four subsections, we theoretically prove that some regions of S are PNS-free.

4.2. A PNS-free segment

Professor Man Kam Kwong pointed out that $N_0(1, 1, 0, 0, 0) = 1$, $N_0(2, 1, 0, 0, 0) = 8$ and $N_0(4, 0, 0, 0, 0) = 441$, are integers. See also Table 1 in Section 6. He suggested us to considered these three points more carefully. Stimulated by Prof. Kwong's comments, we derive the results of Sections 4.2 and 4.3.

We have the following theorem.

Theorem 7. *Suppose that $P = (v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top$, where $t \in [-1, 1]$. Then, P is PNS-free, with the critical value 1 and the critical minimizer $(1, 0, -1, 0)^\top$.*

Proof. For $P = (v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top$, where $t \in [-1, 1]$, and $M = 1$, we have

$$f_0(\mathbf{x}) = \frac{1+t}{2}(x_1 + x_2 + x_3 + x_4)^4 + \frac{1-t}{2}(x_1 - x_2 + x_3 - x_4)^4$$

is SOS, and

$$f_0(1, 0, -1, 0) = 0.$$

Hence, P is PNS-free. □

By numerical experiments, we have the following conjecture.

Conjecture 1. *The segment $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$, is the minimizer set of both M_0 and N_0.*

4.3. A PNS-free planar cone

Theorem 8. *Suppose that $P = (v_2, v_6, v_1, v_3, v_5)^\top = (v_2, v_6, 0, 0, 0)^\top$ with $v_2 \geq v_6 \geq 1$. Then, P is PNS-free.*
If we parameterize $v_6 = b$ and $v_2 = (\theta + 3b - 1)(\theta^2 + (3b-2)\theta - 3b + 4)$. Then, the critical value at P is

$$M = (\theta + 3b - 1)^2(3\theta^2 + (10b - 6)\theta + 3b^2 - 10b + 9)$$

and the critical minimizer is $\bar{\mathbf{x}} = (1, 0, -(\theta + 3b - 1), 0)^\top$.

Proof. Note that for $v_2 \geq v_6 \geq 1$, we may let $v_6 = b$ and $v_2 = (\theta + 3b - 1)(\theta^2 + (3b-2)\theta - 3b + 4)$, where the parameter

$$\theta \geq \bar{\theta} = (b-1)^{\frac{1}{3}}(b+1)^{\frac{2}{3}} + (b-1)^{\frac{2}{3}}(b+1)^{\frac{1}{3}} - 2b + 1.$$

In fact, $\bar{\theta}$ is the largest real root of the cubic equation $v_2 - v_6 = 0$.

With the critical value as $M = (\theta + 3b - 1)^2(3\theta^2 + (10b - 6)\theta + 3b^2 - 10b + 9)$, the critical SOS decomposition at P is as follows

$$\begin{aligned} f_0(\mathbf{x}) = &\frac{1}{v_0}(v_0x_1^2 + 2v_2x_1x_3 + \alpha_1x_3^2)^2 + \frac{1}{v_0}(v_0x_4^2 + 2v_2x_2x_4 + \alpha_1x_2^2)^2 \\ &+ \alpha_2((\theta + 3b - 1)x_1x_3 + x_3^2)^2 + \alpha_2((\theta + 3b - 1)x_2x_4 + x_2^2)^2 \\ &+ \frac{6}{b}(x_1x_2 + x_3x_4 + bx_2x_3 + bx_1x_4)^2 + \frac{6(b^2-1)}{b}(x_1x_2 + x_3x_4)^2 + 6(v_2 - b)[x_1^2x_2^2 + x_3^2x_4^2], \end{aligned}$$

where the involved parameters are as follows:

$$\begin{aligned} \alpha_1 &= -(\theta^2 + (4b-2)\theta + 3b^2 - 4b + 1), \\ \alpha_2 &= \frac{2(\theta^2 + (4b-2)\theta + b^2 - 4b + 4)}{3\theta^2 + (10b-6)\theta + 3b^2 - 10b + 9}. \end{aligned}$$

Since $f_0(1, 0, -(\theta + 3b - 1), 0) = 0$, the corresponding critical minimizer is $\bar{\mathbf{x}} = (1, 0, -(\theta + 3b - 1), 0)^\top$. Hence, $P = (v_2, v_6, 0, 0, 0)^\top$ with $v_2 \geq v_6 \geq 1$ is PNS-free. □

The cone $C = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, b, 0, 0, 0)^\top : a \geq b \geq 1\}$ is a 45° planar closed convex cone. Its end point is just the mid point of the segment $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$, discussed in the last subsection.

4.4. A PNS-free ray

In this subsection, we show that the ray $R = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, 0, 0, 0, 0)^\top : a \leq 0\}$ is PNS-free. Let $a = -\rho$, where $\rho \geq 0$ is a constant. We report that, at a point $P = (-\rho, 0, 0, 0, 0)^\top$, $\mathcal{A}$ has the critical value

$$M = 3\sqrt[3]{\theta_1 + 32\sqrt{\theta_2}} + \frac{\theta_3}{3\sqrt[3]{\theta_1 + 32\sqrt{\theta_2}}} + 6\rho^2 + 138\rho + 609,$$

where

$$\begin{aligned} \theta_1 &:= -\rho^6 + 272\rho^5 + 12608\rho^4 + 204032\rho^3 + 1558528\rho^2 + 5750784\rho + 8290304, \\ \theta_2 &:= -(\rho + 6)^2(\rho + 4)^3(\rho^2 + 4\rho - 16)^3, \\ \theta_3 &:= 9(\rho + 8)(\rho^3 + 152\rho^2 + 1728\rho + 5120). \end{aligned}$$

The function $f_0(\mathbf{x})$ enjoys a critical SOS decomposition:

$$f_0(\mathbf{x}) = \sum_{k=1}^{5} q_k^2(\mathbf{x}),$$

where

$$\begin{aligned} q_1(\mathbf{x}) &= x_3^2 + 6x_2x_4 + \alpha_1x_1^2 + \alpha_2x_4^2, \\ q_2(\mathbf{x}) &= x_2^2 + 6x_1x_3 + \alpha_2x_1^2 + \alpha_1x_4^2, \\ q_3(\mathbf{x}) &= \alpha_3x_2x_4 + \alpha_4x_1^2 + \alpha_5x_4^2, \\ q_4(\mathbf{x}) &= \alpha_3x_1x_3 + \alpha_5x_1^2 + \alpha_4x_4^2, \\ q_5(\mathbf{x}) &= \alpha_6x_1^2 - \alpha_6x_4^2. \end{aligned}$$

The involved parameters are listed as follows:

$$\alpha_1 = -\frac{(\rho+23)M_1(-\rho) - 9\rho^3 - 21\rho^2 + 105\rho + 9}{M_1(-\rho) + 3\rho^2 + 6\rho - 33},$$
$$\alpha_2 = -3\rho,$$
$$\alpha_3 = \sqrt{-30 - 2\alpha_{15}},$$
$$\alpha_4 = \frac{6(1-\alpha_{15})}{\alpha_{33}},$$
$$\alpha_5 = \frac{16\rho}{\alpha_{33}},$$
$$\alpha_6 = \sqrt{-6\rho\alpha_{15} - \frac{192\rho(\alpha_{15}-1)}{\alpha_{33}^2}}.$$

Theorem 9. *Suppose that assumptions (2) and (3) hold. Then, for any constant $\rho \ge 0$, $P = (-\rho, 0, 0, 0, 0)^\top$ is PNS-free.*

Proof. We only need to prove that there is a critical minimizer. Let

$$\bar{\mathbf{x}} = (\alpha_{33}, \alpha_{35} + \alpha_{36}, -\alpha_{35} - \alpha_{36}, -\alpha_{33})^\top.$$

Then, we get $q_3(\bar{\mathbf{x}}) = q_4(\bar{\mathbf{x}}) = q_5(\bar{\mathbf{x}}) = 0$ immediately. Moreover, we have

$$q_1(\bar{\mathbf{x}}) = q_2(\bar{\mathbf{x}}) = (\alpha_{35} + \alpha_{36})^2 - 6(\alpha_{35} + \alpha_{36})\alpha_{33} + \alpha_{15}\alpha_{33}^2 - 3\rho\alpha_{33}^2 = 0.$$

We check the validation of the last equality by a mathematical software Maple. Hence, $f_0(\bar{\mathbf{x}}) = 0$ and $\bar{\mathbf{x}}$ is a critical minimizer at P. Hence, we get the conclusion by Theorem 6. □

4.5. A PNS-free point

We now show that the point $A = (1, 0, 0, 0, 0)^\top$ is PNS-free. In fact, the critical value at A is

$$M = 477 + 3\sqrt[3]{3906351 + 9120\sqrt{57}} + \frac{74403}{\sqrt[3]{3906351 + 9120\sqrt{57}}}.$$

The critical SOS decomposition of $f_0(\mathbf{x})$ is as follows

$$f_0(\mathbf{x}) = \sum_{k=1}^{7} q_k(\mathbf{x})^2,$$

where

$$q_1(\mathbf{x}) = x_3^2 + 6x_2x_4 - 21x_1^2 + \alpha_1 x_4^2,$$
$$q_2(\mathbf{x}) = x_2^2 + 6x_1x_3 - 21x_4^2 + \alpha_1 x_1^2,$$
$$q_3(\mathbf{x}) = 2\sqrt{3}x_2x_4 + \alpha_2 x_1^2 + \alpha_3 x_4^2,$$
$$q_4(\mathbf{x}) = 2\sqrt{3}x_1x_3 + \alpha_2 x_4^2 + \alpha_3 x_1^2,$$
$$q_5(\mathbf{x}) = \alpha_4 x_1^2 - \alpha_4 x_4^2,$$
$$q_6(\mathbf{x}) = \beta_1 x_1x_2 + \beta_2 x_1x_4,$$
$$q_7(\mathbf{x}) = \beta_1 x_3x_4 + \beta_2 x_1x_4.$$

Some involved parameters are listed as follows:

$$\beta_1 = \frac{\sqrt{-6(M_2 - 36)(3M_2 - 4336)}}{\sqrt{M_2^2 - 1302M_2 + 25056}},$$
$$\beta_2 = \frac{\beta_1(3\beta_1^2 + 116)}{\beta_1^2 + 12},$$
$$\alpha_1 = 3 - \frac{1}{2}\beta_1^2,$$
$$\alpha_2 = 22\sqrt{3} - \frac{\sqrt{3}}{6}\beta_1\beta_2,$$

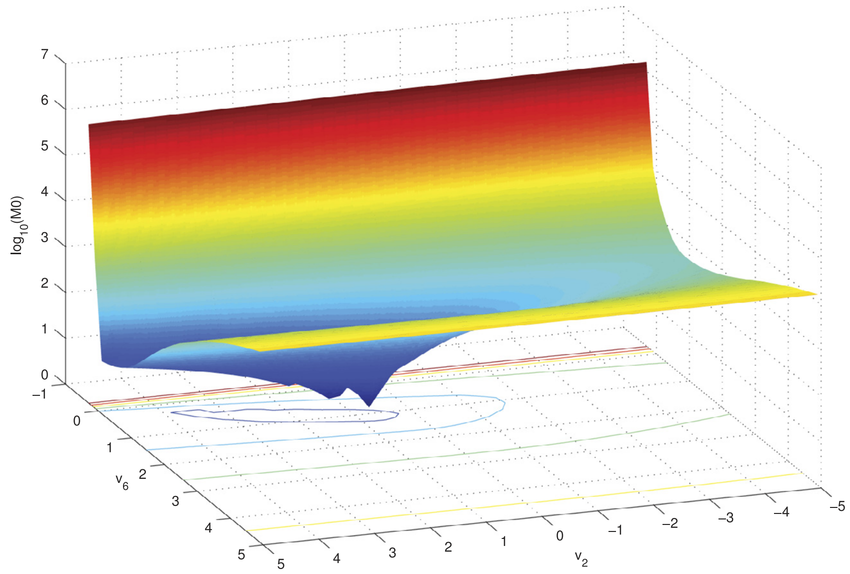

Fig. 2. The contour profile of $M_0(v_2, v_6, 0, 0, 0) = N_0(v_2, v_6, 0, 0, 0)$.

$$\alpha_3 = -\frac{8\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\beta_1^2,$$

$$\alpha_4 = \sqrt{-42\alpha_1 + 2\alpha_2\alpha_3 + \beta_2^2}.$$

Theorem 10. *Suppose that assumptions (2) and (3) hold. Then,* $A = (1, 0, 0, 0, 0)^\top$ *is PNS-free.*

Proof. Using the mathematical software Maple, we calculate

$$f(\mathbf{x}) - \sum_{k=1}^{7} q_k^2(\mathbf{x}) = \frac{-\beta_1^6 - 120\beta_1^4 + (4v_0 - 4944)\beta_1^2 + 48v_0 - 69376}{4(\beta_1^2 + 12)}(x_1^4 + x_4^4).$$

Substituting the value of $v_0 = M$ and β_1, we get $f_0(\mathbf{x}) - \sum_{k=1}^{7} q_k^2(\mathbf{x}) = 0$.

Let $\bar{\mathbf{x}} = (\beta_1, \beta_2, -\beta_2, -\beta_1)^\top$. Obviously, we obtain $q_5(\bar{\mathbf{x}}) = q_6(\bar{\mathbf{x}}) = q_7(\bar{\mathbf{x}}) = 0$. We find that $q_3(\bar{\mathbf{x}})$ and $q_4(\bar{\mathbf{x}})$ vanish if we rewrite all the parameters using β_1. Using the value of each parameter, we find that $q_1(\bar{\mathbf{x}}) = q_2(\bar{\mathbf{x}}) = 0$. Since $\bar{x}_1 = \beta_1 \approx 1.73$, $\bar{\mathbf{x}}$ is the critical minimizer. Therefore, this theorem is valid according to Theorem 6. □

5. Numerical experiments

We have proved in Section 4 that some regions are PNS-free. What about the other cases? We try to answer this problem by a numerical approach. We use the YALMIP software with an SOS module [28,29] to compute $M_0(v_2, v_6, v_1, v_3, v_5)$, which is the smallest value of v_0 such that the fourth order four dimensional Hankel tensor $\mathcal{A}$ with the generating vector $(v_0, v_1, v_2, v_3, 1, v_5, v_6, v_5, 1, v_3, v_2, v_1, v_0)^\top$ is SOS. Gloptipoly [30] and SeDuMi [31] are employed to compute $N_0(v_2, v_6, v_1, v_3, v_5)$, which is the smallest value of v_0 such that the Hankel tensor $\mathcal{A}$ is PSD.

5.1. $M_0(v_2, v_6, 0, 0, 0)$ *and* $N_0(v_2, v_6, 0, 0, 0)$

First, we focus on two elements v_2 and v_6 of generating vectors and set $v_1 = v_3 = v_5 = 0$. By Theorem 4, owing to the effective domain, we have $b > -\frac{1}{3}$. We choose $v_2 = -4, -2, -1, -0.5, 0, 0.5, 1, 1.5, 2, 3, 4$ and $v_6 = -0.2, -0.1, 0, 0.5, 1, 1.5, 2, 4$ and compute M_0 and N_0 in these grid points respectively. By our experiments, we found that these two functions are equivalent on all of the grid points. Thus, no PNS tensors are detected here. The detailed value of M_0 and N_0 are reported in Table 1.

A more intuitional profile of $M_0 = N_0$ is illustrated in Fig. 2. It is easy to see that $(v_2, v_6) = (1, 1)$ is the minimizer of both M_0 and N_0 when we set $v_1 = v_3 = v_5 = 0$.

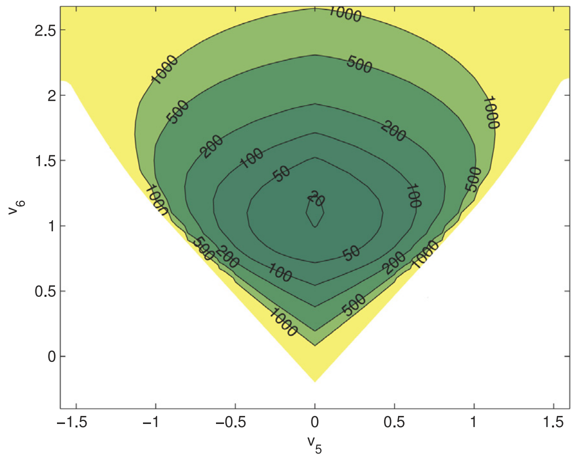

Fig. 3. The contour profile of $M_0(0, v_6, 0, 0, v_5)$.

5.2. Nonzero odd elements of the generating vectors

We consider the case that the generating vector of a fourth order four dimensional Hankel tensor has nonzero odd elements. According to Theorem 5, we say that v_5 and v_6 must satisfy $\eta(v_5, v_6) < 1$. So we study them first and set $v_1 = v_2 = v_3 = 0$. We compute a plenty of grid points with different v_5 and v_6. The function $M_0(0, v_6, 0, 0, v_5)$ is still equivalent to the function $N_0(0, v_6, 0, 0, v_5)$. That is to say, no PNS tensors are found.

The contour of $M_0(0, v_6, 0, 0, v_5) = M_0(0, v_6, 0, 0, v_5)$ is shown in Fig. 3. We could see that the nonlinear contour of $M_0 = N_0 = 500$ looks like a fire balloon.

Finally, we consider all of the elements of symmetric generating vectors of fourth order four dimensional Hankel tensors. The contours of $M_0(v_2, v_6, v_1, v_3, v_5)$ and $N_0(v_2, v_6, v_1, v_3, v_5)$ for various combinations of v_2, v_6, v_1, v_3 and v_5 are reported in Fig. 4. In all of our tests, values of the function $M_0(v_2, v_6, v_1, v_3, v_5)$ in grid points are always equivalent to the corresponding values of the function $N_0(v_2, v_6, v_1, v_3, v_5)$. So, no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors are detected.

From Figs. 3 and 4, we could say that the second element v_1 of the generating vector of a Hankel tensor affects functions $M_0(v_2, v_6, v_1, v_3, v_5)$ and $N_0(v_2, v_6, v_1, v_3, v_5)$ slightly. When we fix $v_4 = 1$, the middle element v_6 of the generating vector $\mathbf{v}$ plays a more important role since it has direct impact on the effective domain.

6. Final remarks

In this paper, we investigated the problem whether there exist fourth order four dimensional PNS Hankel tensors with symmetric generating vectors. Theoretically, we proved that such PNS Hankel tensors do not exist on the segment $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$, the cone $C = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, b, 0, 0, 0)^\top : a \geq b \geq 1\}$, the ray $R = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, 0, 0, 0, 0)^\top : a \leq 0\}$ and the point $A = (1, 0, 0, 0, 0)^\top$. The critical value on L is simply 1. It is interesting to note that the critical values on C are a polynomial in an auxiliary parameter θ with degree four. However, the critical values on R and A are irrational. This indicates that a complete proof that fourth order four dimensional PNS Hankel tensors with symmetric generating vectors do not exist may not be easy. However, numerical tests also indicate that such PNS Hankel tensors do not exist. Thus, we believe that there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors.

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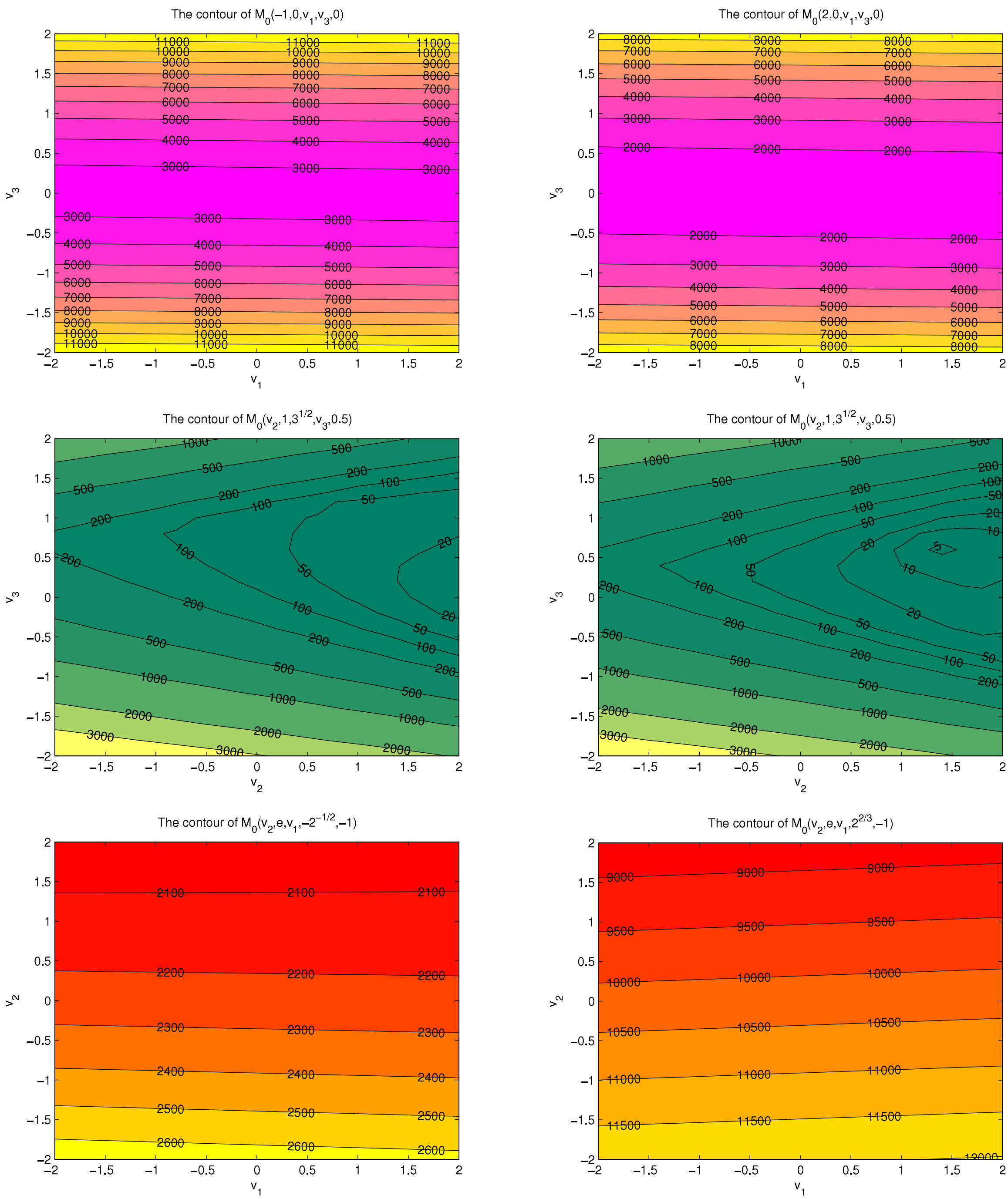

Fig. 4. The contour profiles of $M_0(v_2, v_6, v_1, v_3, v_5)$ which are equivalent to $N_0(v_2, v_6, v_1, v_3, v_5)$.

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