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GEOMETRIC MEASURE OF ENTANGLEMENT AND U-EIGENVALUES OF TENSORS*

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Abstract. We study tensor analysis problems motivated by the geometric measure of quantum entanglement. We define the concept of the unitary eigenvalue (U-eigenvalue) of a complex tensor, the unitary symmetric eigenvalue (US-eigenvalue) of a symmetric complex tensor, and the best complex rank-one approximation. We obtain an upper bound on the number of distinct US-eigenvalues of symmetric tensors and count all US-eigenpairs with nonzero eigenvalues of symmetric tensors. We convert the geometric measure of the entanglement problem to an algebraic equation system problem. A numerical example shows that a symmetric real tensor may have a best complex rank-one approximation that is better than its best real rank-one approximation, which implies that the absolute-value largest Z-eigenvalue is not always the geometric measure of entanglement.

Key words. unitary eigenvalue (U-eigenvalue), Z-eigenvalue, symmetric real tensor, geometric measure of entanglement, the best rank-one approximation

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1. Introduction.

Entanglement of compound systems is a key resource in quantum information processing. In many practical applications it is of fundamental importance to know whether a state is entangled or not. However, this information is often not sufficient, and it is also required to know how much a state is entangled. A useful tool for quantifying the amount of entanglement of a state is given by the so-called entanglement measures [1, 2].

A widely used entanglement measure is provided by the geometric measure of entanglement [3, 4, 5] that is defined for a pure state. The geometric measure of entanglement was first proposed by Shimony [6] and extended to multipartite systems by Wei and Goldbart [7]. It has applications in various different topics, including many body physics [4], local discrimination, quantum computation, condensed matter systems, entanglement witnesses, and the study of quantum channel capacities. The geometric measure of entanglement is nothing but the injective tensor norm itself [8], which appears in the theory of operator algebra [9] and has now become increasingly important in theoretical physics—particularly in quantum channel capacities [10, 11, 12, 13, 14, 15, 16, 17, 18].

A tensor is a multidimensional array [19]. Tensor decompositions originated with Hitchcock in 1927 [20], and the idea of a multiway model is attributed to Cattell in 1944 [21]. Recently, interest in tensor decompositions has expanded to other fields. Examples include signal processing [22, 23], numerical linear algebra [24, 25], computer

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vision [26, 27], numerical analysis, data mining, graph analysis, neuroscience, and more. A book has appeared very recently on multiway data analysis [28]. Many tensor decompositions have been intensively studied under the name of rank-one approximation to high-order tensors [25, 29, 30, 31].

Eigenvalues of higher-order tensors were introduced in 2005 by Qi [32] and Lim [33] and have attracted much attention in the literature [34]. E-eigenvalues and E-eigenvectors as well as Z-eigenvalues and Z-eigenvectors were introduced and discussed. Examples include properties of algorithms for finding Z-eigenvalues and Z-eigenvectors [34, 35, 36, 37, 38], roots of E-characteristic polynomials [39, 40, 41], and the best rank-one approximation [42, 43].

In fact, the geometric measure of entanglement problem is a multiway optimization problem, as well as a tensor decomposition problem or a rank-one approximation to high-order tensors problem [8, 43, 44, 45]. Recently, it was shown that the geometric measure of a symmetric pure state with nonnegative amplitudes is equal to the largest Z-eigenvalue of the underlying nonnegative tensor [44]. A natural conjecture is that when the underlying symmetrical tensor is real, the geometric measure of the symmetric pure state is also equal to the largest Z-eigenvalue of the tensor.

However, the geometric measure of entanglement is very different from the corresponding largest Z-eigenvalue. In this paper, we study tensor analysis problems motivated by the geometric measure of entanglement. The paper is organized as follows. In section 2, by reviewing the geometric measure of entanglement problem, we introduce the concept of the unitary eigenvalue (U-eigenvalue) of a tensor and the unitary symmetric eigenvalue (US-eigenvalue) of a symmetric tensor. In section 3, we obtain an upper bound on the number of distinct US-eigenvalues of symmetric tensors and count all eigenpairs with nonzero eigenvalues of symmetric tensors. In section 4, we convert the geometric measure of entanglement problem to an algebraic equation system problem. A numerical example shows that some symmetric real tensors have better complex rank-one approximations than real rank-one approximations. In other words, not all the absolute-value largest Z-eigenvalues of symmetric real tensors are the geometric measures of symmetric pure states. Hence, the above conjecture is false.

2. Preliminaries.

2.1. Geometric measure of entanglement. A 1-partite pure state $|x\rangle$ is a complex unit vector in $X = \mathbb{C}^n$, $\langle x|$ its conjugate transpose, and $\langle x|y\rangle$ the Hermitian inner product. Assume that $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ is an orthonormal basis of X. That is, $\langle e_i|e_j\rangle = \delta_{ij}$. Denote $|x\rangle$ and $|y\rangle$ as

$$|x\rangle = \sum_{i=1}^{n} x_i|e_i\rangle, \quad |y\rangle = \sum_{i=1}^{n} y_i|e_i\rangle,$$

where $x_i,\ y_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$. Define

$$\langle x|y\rangle := \sum_{i=1}^{n} x_i^* y_i,\ \||x\rangle\| := \sqrt{\langle x|x\rangle} = \sqrt{\sum_{i=1}^{n} x_i^* x_i},$$

where x^* denotes the complex conjugate of x.

A d-partite pure state $|\psi\rangle$ of a composite quantum system can be regarded as a normalized element in a Hilbert tensor product space $H = \otimes_{k=1}^{d} H_k$, where $H_k = \mathbb{C}^{n_k}$ for $k = 1, 2, \ldots, d$. Assume that $\{|e_{i_k}^{(k)}\rangle : i_k = 1, 2, \ldots, n_k\}$ is an orthonormal basis of H_k. Then $\{|e^{(1)}_{i_1} e^{(2)}_{i_2} \cdots e^{(d)}_{i_d}\rangle : i_k = 1, 2, \ldots, n_k;\ k = 1, 2, \ldots, d\}$ is an orthonormal basis of H. $|\psi\rangle$ is denoted as

$$|\psi\rangle := \sum_{i_1,\ldots,i_d=1}^{n_1,\ldots,n_d} \chi_{i_1\cdots i_d} |e^{(1)}_{i_1} \cdots e^{(d)}_{i_d}\rangle,$$

where $\chi_{i_1\cdots i_d} \in \mathbb{C}$. Assume that

$$|\varphi\rangle = \sum_{i_1,\ldots,i_d=1}^{n_1,\ldots,n_d} y_{i_1\cdots i_d} |e^{(1)}_{i_1} \cdots e^{(d)}_{i_d}\rangle,$$

$$\langle\psi|\varphi\rangle := \sum_{i_1,\ldots,i_d=1}^{n_1,\ldots,n_d} \chi^*_{i_1\cdots i_d} y_{i_1\cdots i_d}, \quad || |\varphi\rangle || := \sqrt{\langle\varphi|\varphi\rangle}.$$

A separable (i.e., Hartree) d-partite pure state is denoted as

$$|\phi\rangle := \otimes_{k=1}^{d} |\phi^{(k)}\rangle,$$

the index $k = 1, \ldots, d$ labels the parts, and

$$|\phi^{(k)}\rangle := \sum_{i_k=1}^{n_k} x^{(k)}_{i_k} |e^{(k)}_{i_k}\rangle.$$

Denote by $Separ(H)$ the set of all separable pure states $|\phi\rangle$ in H, subject to the constraint $\langle\phi|\phi\rangle = 1$. The geometric measure of a given d-partite pure state $|\psi\rangle$ is defined as [7]

$$\min_{|\phi\rangle \in Separ(H)} || |\psi\rangle - |\phi\rangle ||.$$

Sometimes, the geometric measure for pure states is also taken as

$$\min_{|\phi\rangle \in Separ(H)} \frac{1}{2} || |\psi\rangle - |\phi\rangle ||^2 = 1 - G(\psi),$$

where $G(\psi)$ is the maximal overlap:

$$G(\psi) := \max_{|\phi\rangle \in Separ(H)} |\langle\psi|\phi\rangle|.$$

To actually find the nearest separable state, one arrives at the nonlinear eigenproblem for the stationary $|\phi\rangle$:

$$\sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_d=1}^{n_1,\ldots,n_{k-1},n_{k+1},\ldots,n_d} \chi^*_{i_1\cdots i_d}\ x^{(1)}_{i_1} \cdots x^{(k-1)}_{i_{k-1}}\ x^{(k+1)}_{i_{k+1}} \cdots x^{(d)}_{i_d} = \lambda x^{(k)*}_{i_k},$$

$$\sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_d=1}^{n_1,\ldots,n_{k-1},n_{k+1},\ldots,n_d} \chi_{i_1\cdots i_d}\ x^{(1)*}_{i_1} \cdots x^{(k-1)*}_{i_{k-1}}\ x^{(k+1)*}_{i_{k+1}} \cdots x^{(d)*}_{i_d} = \lambda x^{(k)}_{i_k},$$

where the eigenvalue λ is associated with the Lagrange multiplier enforcing the constraint $\langle\phi|\phi\rangle = 1$.

In the basis-independent form,

$$\langle\psi|(\otimes_{j=1,j\neq k}^{d}|\phi^{(j)}\rangle) = \lambda\langle\phi^{(k)}|,$$

$$(\otimes_{j=1,j\neq k}^{d}\langle\phi^{(j)}|)|\psi\rangle = \lambda|\phi^{(k)}\rangle.$$

The largest, $|\lambda|_{\max}$, which we call the *entanglement eigenvalue*, corresponds to the closest separable state and is equal to the maximal overlap $G(\psi)$.

2.2. Unitary eigenvalues of complex tensors. A d-array of complex numbers representing a d-order tensor will be denoted by $\mathcal{T} = [\chi_{i_1\cdots i_d}] \in H = \mathbb{C}^{n_1\times\cdots\times n_d}$. For $x, y \in \mathbb{C}^{n_i},\ i = 1, 2, \ldots, d$, define the inner product and norm as

$$\langle x, y\rangle = x^{*T}y = \sum_{i=1}^{n_i} x_i^* y_i,$$

$$||x|| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n_i} x_i^* x_i} = \sqrt{\sum_{i=1}^{n_i} |x_i|^2},$$

where x^* denotes the complex conjugate of x.

For $A, B \in H$, define the inner product and norm as

$$\langle A, B\rangle \equiv A^*B := \sum_{i_1,\ldots,i_d=1}^{n_1,\ldots,n_d} A_{i_1\cdots i_d}^* B_{i_1\cdots i_d},$$

$$||A|| := \sqrt{\langle A, A\rangle}.$$

A rank-one tensor is defined as $\otimes_{i=1}^{d}x^{(i)} \in H$, where $x^{(i)} \in \mathbb{C}^{n_i}$. Define

$$\langle A, \otimes_{i=1}^{d}x^{(i)}\rangle \equiv A^*x^{(1)}\cdots x^{(d)} := \sum_{i_1,\ldots,i_d=1}^{n_1,\ldots,n_d} A_{i_1\cdots i_d}^* x_{i_1}^{(1)}\cdots x_{i_d}^{(d)}.$$

By the tensor product, $\langle\mathcal{T}, \otimes_{i=1,i\neq k}^{d}x^{(i)}\rangle$ and $\langle\otimes_{i=1,i\neq k}^{d}x^{(i)}, \mathcal{T}\rangle$ for the vector $x^{(i)} \in \mathbb{C}^{n_i}$ denote vectors in $\mathbb{C}^{n_k}$, whose i_kth components are

$$\langle\mathcal{T}, \otimes_{i=1,i\neq k}^{d}x^{(i)}\rangle_{i_k} := \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_d=1}^{n_1,\ldots,n_{k-1},n_{k+1},\ldots,n_d} \chi_{i_1\cdots i_k\cdots i_d}^*\ x_{i_1}^{(1)}\cdots x_{i_{k-1}}^{(k-1)}x_{i_{k+1}}^{(k+1)}\cdots x_{i_d}^{(d)},$$

$$\langle\otimes_{i=1,i\neq k}^{d}x^{(i)}, \mathcal{T}\rangle_{i_k} := \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_d=1}^{n_1,\ldots,n_{k-1},n_{k+1},\ldots,n_d} \chi_{i_1\cdots i_k\cdots i_d}\ x_{i_1}^{(1)*}\cdots x_{i_{k-1}}^{(k-1)*}x_{i_{k+1}}^{(k+1)*}\cdots x_{i_d}^{(d)*}.$$

We call a number $\lambda \in \mathbb{C}$ a *unitary eigenvalue (U-eigenvalue)* of $\mathcal{T}$ if λ and a rank-one tensor $\otimes_{i=1}^{d}x^{(i)} \in H$ are solutions of the following equation system:

$$\left\{\begin{array}{l} \langle\mathcal{T}, \otimes_{i=1,i\neq k}^{d}x^{(i)}\rangle = \lambda {x^{(k)}}^*, \\ \langle\otimes_{i=1,i\neq k}^{d}x^{(i)}, \mathcal{T}\rangle = \lambda x^{(k)}, \qquad k = 1, 2, \ldots, d, \\ ||x^{(i)}|| = 1,\ i = 1, 2, \ldots, d. \end{array}\right. \tag{2.1}$$

In this case, the largest $|\lambda|$ is the entanglement eigenvalue, and the corresponding rank-one tensor $\otimes_{i=1}^d x^{(i)}$ is the closest separable state.

A tensor $\mathcal{S} = [s_{i_1 \cdots i_d}] \in \mathbb{C}^{n \times \cdots \times n}$ is called *symmetric* if its entries $s_{i_1 \cdots i_d}$ are invariant under any permutation of their indices. Denote by $Sym(d, n)$ all symmetric d-order n-dimensional tensors. If all entries of $\mathcal{S}$ are real, we call it a *symmetric real tensor.* Let $x \in \mathbb{C}^n$. Simply denote the rank-one tensor $\otimes_{i=1}^d x$ as x^d and

$$\mathcal{S}^* x^d := \sum_{i_1,\ldots,i_d=1}^{n_1,\ldots,n_d} \mathcal{S}^*_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_d}.$$

By the tensor product, $\mathcal{S}^* x^{d-1}$ for a vector $x \in \mathbb{C}^n$ denotes a vector in $\mathbb{C}^n$, whose kth component is

$$(\mathcal{S}^* x^{d-1})_k := \sum_{i_2,\ldots,i_d=1}^{n} \mathcal{S}^*_{k i_2 \cdots i_d} x_{i_2} \cdots x_{i_d}.$$

We call a number $\lambda \in \mathbb{C}$ a *unitary symmetric eigenvalue (US-eigenvalue)* of $\mathcal{S}$ if λ and a nonzero vector $x \in \mathbb{C}^n$ are solutions of the following equation system:

$$\left\{ \begin{array}{l} \mathcal{S}^* x^{d-1} = \lambda x^*, \\ \mathcal{S} x^{*d-1} = \lambda x, \\ ||x|| = 1. \end{array} \right. \tag{2.2}$$

In this case, we say that x is a *unitary symmetric eigenvector (US-eigenvector)* of the tensor $\mathcal{S}$ associated with the US-eigenvalue λ. We call (λ, x) a *US-eigenpair* of $\mathcal{S}$. The largest $|\lambda|$ is the entanglement eigenvalue. The corresponding rank-one tensor $\otimes_{i=1}^d x$ is the closest symmetric separable state.

THEOREM 1. *Assume that complex d-order tensors $A, B, \mathcal{T} \in H = \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$. Then the following hold:*

(a) $\langle A, B \rangle = \langle B, A \rangle^*$*;*

(b) *all U-eigenvalues are real numbers;*

(c) *the US-eigenpair (λ, x) to a complex symmetric d-order tensor $\mathcal{S}$ can also be defined by the following equation system:*

$$\left\{ \begin{array}{l} \mathcal{S}^* x^{d-1} = \lambda x^*, \\ ||x|| = 1, \ \lambda \in \mathbb{R}, \end{array} \right. \tag{2.3}$$

or

$$\left\{ \begin{array}{l} \mathcal{S} x^{*\ d-1} = \lambda x, \\ ||x|| = 1, \ \lambda \in \mathbb{R}. \end{array} \right. \tag{2.4}$$

Proof. (a) The result can be directly deduced by the definition of the inner product of tensors.

(b) Assume that $\mathcal{T}$ has a U-eigenvalue λ and a corresponding rank-one tensor $\otimes_{i=1}^d x^{(i)}$. Then by (2.1)

$$\langle \mathcal{T}, \otimes_{i=1}^d x^{(i)} \rangle = \lambda, \quad \langle \otimes_{i=1}^d x^{(i)}, \mathcal{T} \rangle = \lambda.$$

By Theorem 1(a), it follows that $\lambda = \lambda^*$. Hence, λ is a real number.

(c) It follows by (b) and (2.2). This completes the proof. ∎

Remark. US-eigenpairs are closely related to Z-eigenpairs defined by Qi [32], eigenpairs defined by Kolda and Mayo [34], and Q-eigenpairs defined by Zhang and Qi [45]. They are defined as follows.

A *Z-eigenpair* (λ, u) to a symmetric real tensor $\mathcal{S}$ is defined by

$$\begin{cases} \mathcal{S}u^{d-1} = \lambda u, \\ ||u|| = 1,\ \lambda \in \mathbb{R},\ u \in \mathbb{R}^n. \end{cases} \tag{2.5}$$

An *eigenpair* (λ, u) to a symmetric real tensor $\mathcal{S}$ is defined by

$$\begin{cases} \mathcal{S}u^{d-1} = \lambda u, \\ ||u|| = 1,\ \lambda \in \mathbb{C},\ u \in \mathbb{C}^n. \end{cases} \tag{2.6}$$

A *Q-eigenpair* (λ, u) to a symmetric complex tensor $\mathcal{S}$ is defined by

$$\begin{cases} \mathcal{S}u^{d-1} = \lambda u^*, \\ ||u|| = 1,\ \lambda \in \mathbb{R},\ u \in \mathbb{C}^n. \end{cases} \tag{2.7}$$

It is clear that a Z-eigenpair is a special case of a US-eigenpair, eigenpair, and Q-eigenpair. If an eigenpair (λ, u) is defined by Kolda and Mayo, then the eigenvalue $\lambda = \mathcal{S}u^*u^{d-1} \neq \mathcal{S}u^d$ unless u is a real vector. The definition of Q-eigenpairs is very close to the definition of US-eigenpairs. It is easy to deduce that if (λ, u) is a Q-eigenpair, then (λ, u^*) is a US-eigenpair, which means that the closest symmetric separable state is $\otimes_{i=1}^d u^*$, not $\otimes_{i=1}^d u$. Hence, we define the U-eigenpairs and US-eigenpairs by the inner product in this paper.

THEOREM 2. *Assume that a d-order n-dimensional symmetric complex tensor $\mathcal{S} \in Sym(d, n)$. Then the following hold:*

(a) *if $d \geq 3$, d is an odd integer, and $\lambda \neq 0$, then the system* (2.2) *is equivalent to*

$$\mathcal{S}^* x^{d-1} = x^*,\ x \neq \mathbf{0}, \tag{2.8}$$

and the number of US-eigenpairs of (2.2) *is the double of the number of solutions of* (2.8)*;*

(b) *if $d \geq 3$, d is an even integer, and $\lambda \neq 0$, then the system* (2.2) *is equivalent to*

$$\mathcal{S}^* x^{d-1} = \pm x^*,\ x \neq \mathbf{0}, \tag{2.9}$$

and the number of US-eigenpairs of (2.2) *is equal to the number of solutions of* (2.9).

Proof. (a) Let (λ, x) be a US-eigenpair of $\mathcal{S}$. Then λ is a real number. If $\lambda \neq 0$, since $d > 2$ and d is odd, then $\hat{\lambda} \equiv \frac{1}{\lambda^{1/(d-2)}}$ is a real number. Let $y = \hat{\lambda} x$. Substituting y into (2.3), it follows that

$$\mathcal{S}^* y^{d-1} = y^*.$$

On the other hand, for each solution $y (\neq \mathbf{0})$ of (2.8), let

$$(\lambda, x) = \pm(1/||y||^{d-2}, y/||y||).$$

Then (λ, x) is a US-eigenpair of $\mathcal{S}$. Hence (2.3) and (2.8) are equivalent. And the number of eigenpairs of (2.3) is double the number of solutions of (2.8).

(b) Let (λ, x) be a US-eigenpair of $\mathcal{S}$. Then λ is a real number. If $\lambda \neq 0$, since $d > 2$ and d is even, then $\hat{\lambda} \equiv \frac{1}{|\lambda|^{1/(d-2)}}$ is a real number. Let $y = \hat{\lambda} x$. Substituting y into (2.3), it follows that if $\lambda > 0$, then $\mathcal{S}^* y^{d-1} = y^*$; if $\lambda < 0$, then $\mathcal{S}^* y^{d-1} = -y^*$.

On the other hand, for each solution $y (\neq \mathbf{0})$ of (2.9), if $\mathcal{S}^* y^{d-1} = y^*$, let $(\lambda, x) = (1/||y||^{d-2}, y/||y||)$; if $\mathcal{S}^* y^{d-1} = -y^*$, let $(\lambda, x) = (-1/||y||^{d-2}, y/||y||)$. It is easy to deduce that (λ, x) is a US-eigenpair of $\mathcal{S}$. Hence (2.3) and (2.9) are equivalent. And the number of eigenpairs of (2.3) is equal to the number of solutions of (2.9).

This completes the proof. □

2.3. Best real rank-one approximation and Z-eigenvalues. Mathematicians have always studied the best rank-one tensor approximation problems over the real field [19, 47, 46, 42, 48]. Assume that a d-order real tensor $\mathcal{T} = [\chi_{i_1 \cdots i_d}] \in H = \mathbb{R}^{n_1 \times \cdots \times n_d}$. If there exist a scalar λ and d unit-norm vectors $u^{(i)} \in R^{n_i}$, $i = 1, 2, \ldots, d$, such that the rank-one tensor $\bar{\mathcal{T}} \triangleq \lambda \prod_{i=1}^{d} u^{(i)}$ minimizes the least-squares cost function

$$f(\bar{\mathcal{T}}) = \|\mathcal{T} - \bar{\mathcal{T}}\|^2$$

over the manifold of rank-one tensors, then $\lambda \prod_{j=1}^{d} u^{(j)}$ is said to be the best real rank-one approximation to tensor $\mathcal{T}$. Similarly, given a symmetric real tensor $\mathcal{T} \in \mathbb{R}^{n \times \cdots \times n}$, if there exist a scalar λ and unit-norm vector $u \in R^n$ such that the symmetric rank-one tensor $\bar{\mathcal{T}} \triangleq \lambda u^d$ minimizes the least-squares cost function

$$f(\bar{\mathcal{T}}) = \|\mathcal{T} - \bar{\mathcal{T}}\|^2$$

over the manifold of symmetric rank-one tensors, then λu^d is said to be the best symmetric real rank-one approximation to tensor $\mathcal{T}$. It is well known that

$$|\lambda| = \max_{||u||=1} |\langle \mathcal{T}, u^d \rangle| = \max_{||u||=1} \left| \sum_{i_1, \ldots, i_d = 1}^{n_1, \ldots, n_d} T_{i_1 \cdots i_d} u_{i_1} \cdots u_{i_d} \right|.$$

Friedland [46] and Zhang, Ling, and Qi [48] showed that the best real rank-one approximation to a symmetric real tensor, which in principle can be nonsymmetric, can be chosen symmetric. Furthermore, the best symmetric real rank-one approximation to a symmetric tensor is unique if the tensor does not lie on a certain real algebraic variety. According to [42, 46, 48], λu^d is the best real rank-one approximation of $\mathcal{T}$ if and only if λ is a Z-eigenvalue of $\mathcal{T}$ with the largest absolute value, while u is a real Z-eigenvector of $\mathcal{T}$ associated with the Z-eigenvalue λ.

2.4. Best complex rank-one approximation and US-eigenvalues. However, physicists always study quantum entanglement problems over the complex field. Similarly, for a symmetric multipartite pure states, Hayashi et al. [8] indicated that the closest product state in terms of the fidelity (which is a distance measure describing how close two given quantum states are; for details, see [49]) can be chosen as a symmetric product state for symmetric pure states whose amplitudes are all nonnegative in a computational basis. Moreover, Hübener et al. [3] claimed that the closest product state to a symmetric entangled multiparticle state is also symmetric up to a phase. Hence, in this paper, we focus on the best symmetric complex rank-one approximation of symmetric tensors, i.e., the closest symmetric product state problems.

Given a d-order n-dimensional symmetric complex tensor $\mathcal{S}$, if there exists a rank-one tensor $\otimes_{i=1}^{d} u^{(i)}$, where $u^{(i)} \in \mathbb{C}^n$ and $||u^{(i)}|| = 1$, that maximizes the function $|\langle \mathcal{S}, \otimes_{i=1}^{d} u^{(i)} \rangle| = |\mathcal{S}^* u^{(1)} \cdots u^{(d)}|$ for unit-norm vectors $u^{(i)} \in \mathbb{C}^n$, then we call the rank-one tensor $\lambda \otimes_{i=1}^{d} u^{(i)}$ the best complex rank-one approximation to the tensor $\mathcal{S}$, where $\lambda = |\mathcal{S}^* u^{(1)} \cdots u^{(d)}|$. Similarly, if there exists a unit-norm vector $u \in \mathbb{C}^n$ that maximizes the function $|\mathcal{S}^* u^d|$ for each unit-norm vector $x \in \mathbb{C}^n$, then we call the rank-one tensor $\lambda \otimes_{i=1}^{d} u$ the best complex symmetric rank-one approximation to the tensor $\mathcal{S}$, where $\lambda = |\mathcal{S}^* u^d|$. We denote the maximum u as

$$G(\mathcal{S}) := \max_{u \in \mathbb{C}^n, ||u||=1} |\mathcal{S}^* u^d|. \tag{2.10}$$

For these two kinds of best complex rank-one approximations, by [3], we know that for a symmetric complex tensor, its best symmetric complex rank-one approximation is also its best complex rank-one approximation. Hence $G(\mathcal{S})$ is the largest absolute value of US-eigenvalues of $\mathcal{S}$. It is shown that the geometric measure of a symmetric pure state with nonnegative amplitudes is equal to the largest Z-eigenvalue of the underlying nonnegative tensor [44]. In other words, the best real rank-one approximation to a real nonnegative symmetric tensor is its best complex rank-one approximation. One may ask whether the best real rank-one approximation to a symmetric real tensor is always its best complex rank-one approximation.

3. US-eigenpairs of symmetric tensors.

Here, we study US-eigenpairs.

THEOREM 3 (Takagi's factorization). *Let $A \in \mathbb{C}^{n\times n}$ be a symmetric complex matrix. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = U^T \mathrm{diag}(\lambda_1, \ldots, \lambda_n) U, \ \text{where } \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

By Takagi's factorization, it is easy to obtain the following result.

THEOREM 4. *Let $A \in \mathbb{C}^{n\times n}$ be a symmetric complex matrix. Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix such that*

$$A^* = U^T \mathrm{diag}(\lambda_1, \ldots, \lambda_n) U, \ \text{where } \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

Let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$, $i = 1, \ldots, n$. $(\lambda_i,\ U^{*T}\mathbf{e}_i)$ and $(-\lambda_i,\ \sqrt{-1}U^{*T}\mathbf{e}_i)$, $i = 1, \ldots, n$, are then both US-eigenpairs of A. The number of distinct US-eigenvalues is at most $2n$.

Proof. Let $x_i = U^{*T}\mathbf{e}_i$, $i = 1, \ldots, n$. Then $\mathbf{e}_i = Ux_i$. Since

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\mathbf{e}_i = \lambda_i \mathbf{e}_i = \lambda_i \mathbf{e}_i^*,$$

it follows that

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)(Ux_i) = \lambda_i (Ux_i)^* = \lambda_i (U^* x_i^*),$$

$$U^T \mathrm{diag}(\lambda_1, \ldots, \lambda_n) U x_i = \lambda_i x_i^*,$$

$$A^* x_i = \lambda_i x_i^*,$$

which means that $(\lambda_i,\ U^{*T}\mathbf{e}_i)$ is a US-eigenpair of A.

Furthermore, since $A^*(\sqrt{-1}x_i) = \sqrt{-1}A^*x_i = \sqrt{-1}\lambda_i x_i^* = -\lambda_i(\sqrt{-1}x_i)^*$, then $(-\lambda_i,\ \sqrt{-1}U^{*T}\mathbf{e}_i)$ is also a US-eigenpair of A. Hence, $\pm\lambda_i$, $i = 1, \ldots, n$, are US-eigenvalues of A.

On the other hand, assume that (λ, x) is a US-eigenpair of A. Then

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)(Ux) = \lambda(Ux)^*.$$

Let $(y_1, y_2, \ldots, y_n)^T = Ux \neq \mathbf{0}$. From the above formulation, we have that

$$\lambda_i y_i = \lambda y_i^*, \quad i = 1, 2, \ldots, n.$$

It follows that $|\lambda| = |\lambda_i|$, i.e., $\lambda = \pm\lambda_i$ if $y_i \neq 0$ for some $i = 1, 2, \ldots, n$. Hence, the number of distinct US-eigenvalues is at most $2n$. ∎

If $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1},\ 1 \leq k \leq n$, then, corresponding to the linearity of the space of solutions, all US-eigenvectors with respect to λ_1 are unit vectors in $\mathrm{span}\{U^{*T}\mathbf{e}_1, \ldots, U^{*T}\mathbf{e}_k\}$. Hence, we have the following result.

THEOREM 5. *If $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1},\ 1 \leq k \leq n$, then the set of all US-eigenvectors with respect to λ_1 is*

$$USeig(A, \lambda_1) \equiv \left\{ \frac{\sum_{i=1}^k \alpha_i U^{*T}\mathbf{e}_i}{||\sum_{i=1}^k \alpha_i U^{*T}\mathbf{e}_i||} : \alpha_i \in \mathbb{R},\ i = 1, \ldots, k,\ \sum_{i=1}^k \alpha_i^2 \neq 0 \right\};$$

the set of all US-eigenvectors with respect to $-\lambda_1$ is

$$USeig(A, -\lambda_1) \equiv \left\{ \frac{\sum_{i=1}^k \alpha_i \sqrt{-1} U^{*T}\mathbf{e}_i}{||\sum_{i=1}^k \alpha_i \sqrt{-1} U^{*T}\mathbf{e}_i||} : \alpha_i \in \mathbb{R},\ i = 1, \ldots, k,\ \sum_{i=1}^k \alpha_i^2 \neq 0 \right\}.$$

Proof. Let $x_i = U^{*T}e_i,\ i = 1, \ldots, k$. Then $A^* x_i = \lambda_1 x_i^*$. For each $\alpha_i \in \mathbb{R},\ i = 1, \ldots, k$ with $\sum_{i=1}^k \alpha_i^2 \neq 0$, it follows that

$$A^* \left(\sum_{i=1}^k \alpha_i x_i \right) = \sum_{i=1}^k A^*(\alpha_i x_i) = \lambda_1 \sum_{i=1}^k \alpha_i x_i^* = \lambda_1 \left(\sum_{i=1}^k \alpha_i x_i \right)^*.$$

Hence $\frac{\sum_{i=1}^k \alpha_i x_i}{||\sum_{i=1}^k \alpha_i x_i||}$ is also a US-eigenvector of A with respect to λ_1. The first result follows. For the $-\lambda_1$ case, similarly, the second result follows. This completes the proof. ∎

THEOREM 6. *Let $d \geq 3,\ n \geq 2$ be integers, and let $\mathcal{S} \in Sym(d, n)$. If (2.8) has finitely many solutions, then the following hold:*

(a) *if d is odd, the number of nonzero solutions of (2.8) is at most $\frac{(d-1)^{2n}-1}{d-2}$;*

(b) *if d is even, the number of nonzero solutions of (2.9) is at most $\frac{2((d-1)^{2n}-1)}{d-2}$;*

(c) *$\mathcal{S}$ has at most $\frac{2((d-1)^{2n}-1)}{d(d-2)}$ distinct nonzero US-eigenvalues;*

(d) *for nonzero US-eigenvalues, all the US-eigenpairs of $\mathcal{S}$ are as follows:*

$$\begin{cases} \pm\left(\frac{1}{||x||^{d-2}}, \frac{x}{||x||}\right) & \text{if } d \text{ is odd,} \\ \left(\frac{1}{||x||^{d-2}}, \frac{x}{||x||}\right) \text{ and } \left(\frac{-1}{||x||^{d-2}}, \frac{e^{\pi\sqrt{-1}/d}x}{||x||}\right) & \text{if } d \text{ is even,} \end{cases}$$

where x is a solution of (2.8).

Proof. (a) Assume that d is odd. Let $x = y + z\sqrt{-1},\ y, z \in \mathbb{R}^n$. Then (2.8) can be rewritten as follows:

$$\begin{cases} \mathrm{Re}(\mathcal{S}^*(y + z\sqrt{-1})^{d-1}) = y, \\ \mathrm{Im}(\mathcal{S}^*(y + z\sqrt{-1})^{d-1}) = -z. \end{cases} \tag{3.1}$$

We first convert (3.1) into the following homogeneous system:

$$\begin{cases} \mathrm{Re}(\mathcal{S}^*(y+z\sqrt{-1})^{d-1}) = t^{d-2}y, \\ \mathrm{Im}(\mathcal{S}^*(y+z\sqrt{-1})^{d-1}) = -t^{d-2}z, \end{cases} \tag{3.2}$$

where each homogeneous polynomial is of degree $d-1$. Bezout's theorem says that there are generically $(d-1)^{2n}$ solutions in the projective space $\mathbb{P}^{2n}$ (see [40, Remark 2.1]). If we remove the trivial solution $(0,0,t)$ with $t \neq 0$, this leaves $(d-1)^{2n}-1$ solutions.

Assume that (t,y,z) is a nonzero solution of (3.2) with $t \neq 0$. Then $(\frac{y}{t},\frac{z}{t})$ is a nonzero solution of (3.1). Let $\xi = e^{\frac{2\pi\sqrt{-1}}{d-2}}$, where $e^{\sqrt{-1}\theta} := \cos\theta + \sqrt{-1}\sin\theta$; this is an Euler formulation. Then $(\xi^k t, y, z)$ for $k = 0,1,d-3$ are distinct solutions of (3.2), and $(\frac{y}{\xi^k t},\frac{z}{\xi^k t})$ for $k=0,1,d-3$ are distinct solutions of (3.1). However, there exists at most one k for $k=0,1,d-3$ such that $(\frac{y}{\xi^k t},\frac{z}{\xi^k t})$ is a real solution of (3.1). Hence, (3.1) has at most $\frac{(d-1)^{2n}-1}{d-2}$ distinct nonzero real solutions. It follows that (2.8) has at most $\frac{(d-1)^{2n}-1}{d-2}$ distinct nonzero solutions.

(b) This result follows similarly to the proof of (a).

(c) Assume that d is odd. If $x \neq \mathbf{0}$ is a solution of (2.8), then ηx is also a solution of (2.8) if

$$\eta^{d-1} = \eta^*, \quad \eta(\neq 0) \in \mathbb{C}. \tag{3.3}$$

It follows that $\eta^d = |\eta|^2$. Since $d > 2$, hence $|\eta| = 1$. It follows that $\eta^d = 1$. Hence (3.3) has d distinct solutions. By Theorem 2(a), we know that all these pairs $\pm(1/||x||^{d-2}, \eta x/||x||)$ are US-eigenpairs of $\mathcal{S}$. Hence, $\mathcal{S}$ has at most $\frac{2((d-1)^{2n}-1)}{d(d-2)}$ distinct nonzero US-eigenvalues.

Assume that d is even. Let $\eta = e^{\pi\sqrt{-1}/d}$. It is clear that if $x \in \mathbb{C}^n$ is a nonzero solution of (2.8), i.e., $\mathcal{S}^* x^{d-1} = x^*$, then ηx is a solution of $\mathcal{S}^* x^{d-1} = -x^*$. Hence, $(\frac{1}{||x||^{d-2}},\frac{x}{||x||})$ and $(\frac{-1}{||x||^{d-2}},\frac{\eta x}{||x||})$ are US-eigenpairs of $\mathcal{S}$. It also follows that $\mathcal{S}$ has at most $\frac{2((d-1)^{2n}-1)}{d(d-2)}$ distinct nonzero US-eigenvalues.

(d) Hence, all the US-eigenpairs of $\mathcal{S}$ are as follows:

$$\begin{cases} \pm(\frac{1}{||x||^{d-2}},\frac{x}{||x||}) & \text{if } d \text{ is odd,} \\ (\frac{1}{||x||^{d-2}},\frac{x}{||x||}) \text{ and } (\frac{-1}{||x||^{d-2}},\frac{e^{\pi\sqrt{-1}/d}x}{||x||}) & \text{if } d \text{ is even,} \end{cases}$$

where x is a solution of (2.8). This completes the proof. ∎

NOTE 1. *Let $\mathcal{S}$ be the symmetric $2\times2\times2\times2$ tensor whose nonzero entries are*

$$\mathcal{S}_{1111} = 2,\ \mathcal{S}_{1112} = -1,\ \mathcal{S}_{1122} = -1,\ \mathcal{S}_{1222} = -2,\ \mathcal{S}_{2222} = 1.$$

The number of nonzero solutions of the equation system (2.8) *is* 40, *which shows that the bound is tight.*

NOTE 2. *Cartwright and Sturmfels in* [40] *showed that every symmetric tensor has finite E-eigenvalues (which is introduced by Qi in* [32]*). At the same time, they indicated that the magnitudes of the eigenvalues with $x \cdot \bar{x} = 1$ may still be an infinite set (see Example* 5.8 *of* [40]*), which implies that the system $\mathcal{S}x^{d-1} = x$ has infinite nonzero solutions, where $\mathcal{S}$ is a symmetric $3\times3\times3$ tensor whose nonzero entries are* $\mathcal{S}_{111} = 2$, $\mathcal{S}_{122} = \mathcal{S}_{212} = \mathcal{S}_{221} = \mathcal{S}_{133} = \mathcal{S}_{313} = \mathcal{S}_{331} = 1$.

NOTE 3. *Let $\mathcal{S}$ be the symmetric $3\times3\times3$ tensor as in Note* 2. *Then $x = (-0.5, a\sqrt{-1}, \sqrt{1-a^2}\sqrt{-1})$ for all $0 < a < 1$ are nonzero solutions of $\mathcal{S}x^{d-1} = x^*$. It implies that* (2.8) *may have infinite nonzero solutions.*

4. Best symmetric rank-one approximation of symmetric tensors.

THEOREM 7. *Let $\mathcal{S}$ be a symmetric complex tensor. Let λ be a US-eigenvalue of $\mathcal{S}$. Then the following hold:*

(a) *$-\lambda$ is also a US-eigenvalue of $\mathcal{S}$;*

(b) *$G(\mathcal{S}) = \lambda_{\max}$.*

Proof. (a) By Theorems 4 and 6, we know that if λ is a US-eigenvalue of $\mathcal{S}$, then $-\lambda$ is also a US-eigenvalue of $\mathcal{S}$.

(b) Since $G(\mathcal{S}) = |\lambda|_{\max}$, by Theorem 7(a), it follows that $G(\mathcal{S}) = \lambda_{\max}$. ∎

4.1. Best symmetric rank-one approximation of symmetric matrices.

THEOREM 8. *Let $A \in \mathbb{C}^{n\times n}$ be a symmetric complex matrix. Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix such that*

$$A^* = U^T \mathrm{diag}(\lambda_1, \dots, \lambda_n)U, \ \ \textit{where}\ \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

Then for all $x \in USeig(A, \lambda_1) \cup USeig(A, -\lambda_1)$ and $\eta \in \mathbb{C}$ with $|\eta| = 1$, $(\eta x) \otimes (\eta x)$ is the best symmetric rank-one approximation of A.

Proof. If $x \in USeig(A, \lambda_1) \cup USeig(A, -\lambda_1)$, $\eta \in \mathbb{C}$, and $|\eta| = 1$, then $|A^*\ (\eta x)^2| = |\eta^2 A^* x^2| = |A^* x^2| = \lambda_1 = G(A)$. Hence, $(\eta x)\otimes(\eta x)$ is the symmetric best rank-one approximation of A. ∎

4.2. Best symmetric rank-one approximation of order $d \geq 3$.

The best symmetric rank-one approximation problem is to find a unit-norm vector $\hat{x} \in \mathbb{C}^n$ such that

$$\mathrm{Q1}:\ \mathcal{S}^* \hat{x}^d = \max\{|\mathcal{S}^* x^d| : x \in \mathbb{C}^n,\ ||x|| = 1\}.$$

By Theorem 7, introducing the US-eigenvalue method, Q1 is equivalent to the following problem:

$$\mathrm{Q2}:\ \max\{|\lambda| : \mathcal{S}^* x^{d-1} = \lambda x^*,\ x \in \mathbb{C}^n,\ ||x|| = 1,\ \lambda \in \mathbb{R}\}.$$

THEOREM 9. *Let $\mathcal{S} \in Sym(d, n)$. Then the following hold:*

(a) *the best symmetric rank-one approximation problem is equivalent to the following optimization problem:*

$$\mathrm{Q3}:\ \min\{||x|| : \mathcal{S}^* x^{d-1} = x^*,\ x \neq \mathbf{0} \in \mathbb{C}^n\};$$

(b) *if $\tilde{x} \in \mathbb{C}^n$ is a solution of* Q3*, then $G(\mathcal{S}) = 1/||\tilde{x}||^{d-2}$, and $\frac{\eta}{||\tilde{x}||^{2d-2}}\tilde{x}^d$ is the best symmetric rank-one approximation of $\mathcal{S}$ for each $\eta \in \mathbb{C}$ satisfying $|\eta| = 1$.*

Proof. (a) First, the best symmetric rank-one approximation problem Q1 is equivalent to Q2. Second, by Theorem 6(d), we know that if x is a nonzero solution of $\mathcal{S}^* x^{d-1} = x^*$, then eigenpairs of $\mathcal{S}$ are $\pm(\frac{1}{||x||^{d-2}}, \frac{x}{||x||})$ if d is odd or $(\frac{1}{||x||^{d-2}}, \frac{x}{||x||})$ and $(\frac{-1}{||x||^{d-2}}, \frac{e^{\pi\sqrt{-1}/d}x}{||x||})$ if d is even. Hence, Q3 is equivalent to Q2. It follows that Q3 is equivalent to Q1.

(b) Assume that $\tilde{x} \in \mathbb{C}^n$ is a solution of Q3. Then

$$G(\mathcal{S}) = 1/||\tilde{x}||^{d-2} = \left|\mathcal{S}^*\ \left(\frac{\tilde{x}}{||\tilde{x}||}\right)^d\right|.$$

If $\eta \in \mathbb{C}$ satisfying $|\eta| = 1$, then

$$\left|\mathcal{S}^* \left(\frac{\eta\tilde{x}}{||\tilde{x}||}\right)^d\right| = |\eta^d| \left|\mathcal{S}^* \left(\frac{\tilde{x}}{||\tilde{x}||}\right)^d\right| = \left|\mathcal{S}^* \left(\frac{\tilde{x}}{||\tilde{x}||}\right)^d\right| = G(\mathcal{S}),$$

$$G(\mathcal{S}) \left(\frac{\eta\tilde{x}}{||\tilde{x}||}\right)^d = \frac{\eta}{||\tilde{x}||^{2d-2}}\tilde{x}^d.$$

Hence, $\frac{\eta}{||\tilde{x}||^{2d-2}}\tilde{x}^d$ is the best symmetric rank-one approximation of $\mathcal{S}$. This completes the proof. □

The problem of finding eigenpairs is equivalent to solving a polynomial system $\mathcal{S}^* x^{d-1} = x^*$; i.e., the eigenpairs are points on an algebraic variety. If there are finitely many eigenpairs, one algorithm for the best rank-one approximation is to compute the variety (solve the system) and then sort the finite solutions by their eigenvalue. Theorem 9 converts the best symmetric rank-one approximation problem to a problem for solving an algebraic equation system.

Let $x = y + z\sqrt{-1}$, $y, z \in \mathbb{R}^n$. Then Q3 is equivalent to the following problem:

$$\text{Q4} : \min \left\{ y^T y + z^T z : \left\{ \begin{array}{l} \text{Re } \mathcal{S}^*(y + z\sqrt{-1})^{d-1} = y, \\ \text{Im } \mathcal{S}^*(y + z\sqrt{-1})^{d-1} = -z, \end{array} \right. \quad y \neq \mathbf{0} \text{ or } z \neq \mathbf{0} \in \mathbb{R}^n \right\}.$$

Example 1. Assume that $\mathcal{S}$ is a symmetric real tensor with $d = 3$ and $n = 2$. Then Q4 is equivalent to the following optimization problem:

$$\min \quad y^T y + z^T z$$

$$\text{subject to} \sum_{j=1,k=1}^{2} \mathcal{S}_{ijk}(y_j y_k - z_j z_k) = y_i, \quad i = 1, 2, \tag{4.1}$$

$$\sum_{j=1,k=1}^{2} \mathcal{S}_{ijk}(2y_j z_k) = -z_i, \quad i = 1, 2, \tag{4.2}$$

$$y \neq \mathbf{0} \quad \text{or} \quad z \neq \mathbf{0} \in \mathbb{R}^2.$$

By solving equation systems (4.1) and (4.2), one obtains the positive US-eigenvalues and their US-eigenvectors of $\mathcal{S}$ and then gets the largest eigenvalue and the best rank-one approximation.

For example, we especially compute US-eigenpairs of two 3rd-order 2-dimensional symmetric real tensors; see Tables 1 and 2. Observing numerical examples, we also find the following interesting results:

(1) Let $\eta = (-1 + \sqrt{-3})/2$. If u is a US-eigenvector of $\mathcal{S}$ corresponding to a US-eigenvalue λ, then u, ηu, $\eta^2 u$ and u^*, $(\eta u)^*$, $(\eta^2 u)^*$ are US-eigenvectors of $\mathcal{S}$ corresponding to λ. Indeed, by the proof of Theorem 6, we know that if η is a dth root of unity, i.e., $\eta^d = 1$, then $\eta^k u$ $(k = 1, 2, \ldots, d)$ are US-eigenvectors of $\mathcal{S}$ corresponding to λ. Furthermore, if $\mathcal{S}$ is real, then u^* is also a US-eigenvector corresponding to λ. Hence, there exist $2d$ different US-eigenvectors with the same US-eigenvalue. But if $\mathcal{S}$ is complex, then u^* is not a US-eigenvector and the result cannot hold.

(2) If u is a real US-eigenvector, then $u = u^*$, $\eta u = (\eta^2 u)^*$, and $\eta^2 u = (\eta u)^*$. Hence, there are three different US-eigenvectors if one of u, ηu, and $\eta^2 u$ is a real vector.

TABLE 1
US-eigenpairs of S with $S_{111} = 2,\ S_{112} = 1,\ S_{122} = -1,\ S_{222} = 1.$

	Eigenvalues($\lambda > 0$)	Eigenvectors(u)
1	0.326409	$(-0.188256 - 0.32607\sqrt{-1}, -0.463206 - 0.802296\sqrt{-1})^T$
2	0.326409	$(-0.188256 + 0.32607\sqrt{-1}, -0.463206 + 0.802296\sqrt{-1})^T$
3	2.12132	$(0.541675 - 0.454519\sqrt{-1}, 0.664463 - 0.241845\sqrt{-1})^T$
4	2.12132	$(-0.541675 + 0.454519\sqrt{-1}, 0.664463 + 0.241845\sqrt{-1})^T$
5	2.12132	$(-0.122788 - 0.696364\sqrt{-1}, -0.541675 + 0.454519\sqrt{-1})^T$
6	2.12132	$(-0.122788 + 0.696364\sqrt{-1}, -0.541675 - 0.454519\sqrt{-1})^T$
7	2.12132	$(0.664463 - 0.241845\sqrt{-1}, -0.122788 + 0.696364\sqrt{-1})^T$
8	2.12132	$(0.664463 + 0.241845\sqrt{-1}, -0.122788 - 0.696364\sqrt{-1})^T$
9	2.17445	$(0.253551 - 0.439164\sqrt{-1}, -0.430943 + 0.746415\sqrt{-1})^T$
10	2.17445	$(0.253551 + 0.439164\sqrt{-1}, -0.430943 - 0.746415\sqrt{-1})^T$
11	2.35468	$(-0.48629 - 0.842279\sqrt{-1}, -0.116283 - 0.201409\sqrt{-1})^T$
12	2.35468	$(-0.48629 + 0.842279\sqrt{-1}, -0.116283 + 0.201409\sqrt{-1})^T$
13	0.326409	$(0.376513, 0.926411)^T$
14	2.17445	$(-0.507103, 0.861886)^T$
15	2.35468	$(0.97258, 0.232567)^T$

TABLE 2
US-eigenpairs of S with $S_{111} = 2,\ S_{112} = -1,\ S_{122} = -2,\ S_{222} = 1.$

	Eigenvalues($\lambda > 0$)	Eigenvectors(u)
1	2.23607	$(-0.494041 - 0.855703\sqrt{-1}, 0.0769673 + 0.133311\sqrt{-1})^T$
2	2.23607	$(-0.494041 + 0.855703\sqrt{-1}, 0.0769673 - 0.133311\sqrt{-1})^T$
3	2.23607	$(0.180365 - 0.312401\sqrt{-1}, -0.466335 + 0.807716\sqrt{-1})^T$
4	2.23607	$(0.180365 + 0.312401\sqrt{-1}, -0.466335 - 0.807716\sqrt{-1})^T$
5	2.23607	$(0.313676 - 0.543303\sqrt{-1}, 0.389368 - 0.674405\sqrt{-1})^T$
6	2.23607	$(0.313676 + 0.543303\sqrt{-1}, 0.389368 + 0.674405\sqrt{-1})^T$
7	3.16228	$(-0.443605 - 0.550649\sqrt{-1}, -0.550649 + 0.443605\sqrt{-1})^T$
8	3.16228	$(-0.443605 + 0.550649\sqrt{-1}, -0.550649 - 0.443605\sqrt{-1})^T$
9	3.16228	$(-0.255074 - 0.659498\sqrt{-1}, 0.659498 - 0.255074\sqrt{-1})^T$
10	3.16228	$(-0.255074 + 0.659498\sqrt{-1}, 0.659498 + 0.255074\sqrt{-1})^T$
11	3.16228	$(0.698679 - 0.108848\sqrt{-1}, -0.108848 - 0.698679\sqrt{-1})^T$
12	3.16228	$(0.698679 + 0.108848\sqrt{-1}, -0.108848 + 0.698679\sqrt{-1})^T$
13	2.23607	$(-0.627352, -0.778736)^T$
14	2.23607	$(-0.360729, 0.932671)^T$
15	2.23607	$(0.988081, -0.153935)^T$

(3) There is at least one real US-eigenvector of $\mathcal{S}$, since a Z-eigenpair is a special US-eigenpair and Z-eigenpairs always exist for symmetric real tensors.

(4) The best real rank-one approximation is sometimes also the best complex rank-one approximation even if the tensor is not a symmetric nonnegative real tensor; see Table 1. On the other hand, a symmetric real tensor may have the best complex rank-one approximation that is better than its best real rank-one approximation; see Table 2. In other words, the absolute-value largest Z-eigenvalue is sometimes not its largest US-eigenvalue.

5. Conclusion. We define the concepts of U-eigenvalues, US-eigenvalues, and best complex rank-one approximations motivated by the geometric measure of entanglement problem with the inner product method, compare US-eigenpairs to Z-eigenpairs, eigenpairs (defined by Kolda and Mayo), and Q-eigenpairs, and claim that the largest US-eigenvalue of a symmetric complex tensor is its maximal overlap $G(\mathcal{S})$. We obtain some properties of US-eigenvalues and US-eigenpairs to symmetric tensors. We convert the geometric measure of the entanglement problem to an algebraic equation system problem. However, much is still unknown about US-eigenpairs and the best complex rank-one approximations. For example, what is the necessary and sufficient condition for the equality of the largest absolute Z-eigenvalue and the largest US-eigenvalue to a symmetric real tensor? How can one generalize the computation method of the best real rank-one approximation to the complex case, such as SS-HOPM? Can the concept of U-eigenpair be used in the entanglement in many-body systems research? These are all potential topics of future research.

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