Z-TENSORS AND COMPLEMENTARITY PROBLEMS*

M. SEETHARAMA GOWDA†   ZIYAN LUO‡   LIQUN QI§   AND NAIHUA XIU¶

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Abstract. Tensors are multidimensional analogs of matrices. Z-tensors are tensors with non-positive off-diagonal entries. In this paper, we consider tensor complementarity problems associated with Z-tensors and describe various equivalent conditions for a Z-tensor to have the Q-property. These conditions/properties include the strong M-tensor property, the S-property, positive stable property, strict semi-monotonicity property, etc. Based on degree-theoretic ideas, we prove some refined results for even ordered tensors. We show, by an example, that a tensor complementarity problem corresponding to a strong M-tensor may not have a unique solution. A sufficient and easily checkable condition for a strong M-tensor to have unique complementarity solutions is also established.

Key Words. tensor, Z-tensor, strong M-tensor, complementarity problem, degree theory
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†Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250. Email: gowda@umbc.edu, http://www.math.umbc.edu/˜ gowda

‡The State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, P.R. China. E-mail: starkeynature@hotmail.com

§Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. E-mail: liqun.qi@polyu.edu.hk, http://www.polyu.edu.hk/ama/people/detail/1/

¶Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing, P.R.China. E-mail: nhxiu@bjtu.edu.cn

1. Introduction. A tensor is simply a multidimensional analog of a matrix. Given natural numbers $m\,(\geq 2)$ and n, an mth order, n-dimensional tensor is of the form

$$\mathcal{A} = [a_{i_1\,i_2\,i_3\,\cdots\,i_m}] \tag{1.1}$$

where $a_{i_1\,i_2\,i_3\,\cdots\,i_m} \in R,\ 1 \leq i_1, i_2, i_3, \ldots, i_m \leq n$. During the last decade, tensors have become very important in various areas. Numerous articles extending basic concepts and results of matrix theory have been written, see for example, [2, 6, 11, 12, 14, 19, 20, 21, 22]. With a view towards bringing in optimization ideas, researchers have introduced various complementarity concepts [3, 5, 13, 15, 16, 17, 18]. Given a tensor $\mathcal{A}$ in the form (1.1), we define a function $F : R^n \to R^n$ whose ith component is given by

$$F_i(x) := \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i\,i_2\,i_3 \cdots\,i_m} x_{i_2} x_{i_3} \cdots x_{i_m}.$$

This function, abbreviated by

$$F(x) = \mathcal{A}x^{m-1},$$

has a homogeneous polynomial of degree $m-1$ in each component. Corresponding to this F and any $q \in R^n$, we consider the *tensor complementarity problem* $\mathrm{TCP}(\mathcal{A}, q)$: Find $x \in R^n$ such that

$$x \geq 0,\ F(x) + q \geq 0 \text{ and } \langle x, F(x) + q \rangle = 0,$$

where $x \geq 0$ means that each component of x is nonnegative, etc. This is a generalization of the *linear complementarity problem* (corresponding to $m = 2$), a special instance of a *nonlinear complementarity problem* and a particular case of a *variational inequality problem* corresponding to the closed convex cone R^n_+. Complementarity problems and variational inequality problems have been extensively studied and there is a vast literature dealing with existence, uniqueness, computation, and applications, see for example, [7, 8]. In the last decade or so, much work has been done in extending these to symmetric cones.

Since the tensor complementarity problem is a special case of a nonlinear complementarity problem, the entire theory of nonlinear complementarity problems is applicable to tensor complementarity problems. However, because each component of $F(x)$ is a homogeneous polynomial (of the same degree), we may expect some specialized results; see [9] for an early reference where (multi)functions with certain 'homogeneity' are treated. The main questions in tensor complementarity theory are: *How do the entries of $\mathcal{A}$ influence existence, uniqueness, stability, computation, etc., and which linear complementarity concepts/results extend to tensors?*

In this article, we consider Z-tensors which are tensors with non-positive 'off-diagonal' entries. It is easy to see that such a tensor can be written as

$$\mathcal{A} = r\mathcal{I} - \mathcal{B},$$

where $r \in R$, $\mathcal{I}$ is the identity tensor and $\mathcal{B}$ is a nonnegative tensor (that is, all its entries are nonnegative). Properties of nonnegative tensors, particularly in relation to the Perron-Frobenius theorem, have been explored in several recent papers, see [2, 20, 19, 22]. If $\rho(B)$ denotes the spectral radius of $\mathcal{B}$, one says that the Z-tensor $\mathcal{A} = r\mathcal{I} - \mathcal{B}$ is an M-tensor if $r \geq \rho(B)$ and strong (or nonsingular) M-tensor if $r > \rho(B)$. Some properties of M-tensors and strong M-tensors have been discussed in [6, 21, 12]. Motivated by a paper of Luo et al. [13], here, we undertake a study of complementarity properties of Z-tensors, specifically asking when a Z-tensor $\mathcal{A}$ has the Q-property, namely, for all $q \in R^n$, TCP$(\mathcal{A}, q)$ has a solution. In addition to proving several equivalent properties, we show how degree theory offers a way of understanding the solvability of certain equations arising in Z-tensor complementarity problems.

This paper is organized as follows. In Section 2, we recall some results about nonnegative tensors. Section 3 covers a basic result about Q-tensors via degree theory. In Section 4, we consider Z-tensors and characterize the strong M-tensor property in various equivalent ways. Finally, in Section 5, we describe some refined properties of Z-tensors such as the surjectivity of the map F and the equivalence of the P-property and the strong M-tensor property.

2. Preliminaries. Throughout this paper, R^n denotes the n-dimensional Euclidean space with the usual inner product. For $x \in R^n$, we write $x \geq 0$ ($x > 0$) if all components of x are nonnegative (respectively, positive). The nonnegative orthant of R^n is denoted by R^n_+. We denote the complex n-space by C^n.
Let $\mathcal{A} = [a_{i_1\, i_2\, i_3\, \cdots\, i_m}]$ denote an mth order, n-dimensional tensor. The entries $a_{i\, i\, \cdots\, i}$, $1 \leq i \leq n$, are the 'diagonal' entries; the rest are 'off-diagonal' entries of $\mathcal{A}$. The identity tensor is one with all diagonal entries one and off-diagonal entries zero. A tensor is said to be *nonnegative* if all its entries are nonnegative.

A complex number λ is said to be an *eigenvalue* of $\mathcal{A}$ if there exists a nonzero vector $x \in C^n$ such that

$$\mathcal{A}x^{m-1} = \lambda\, x^{[m-1]},$$

where $x^{[m-1]}$ is the vector in C^n with ith component x_i^{m-1}, see [2, 14]. Define the spectrum $\sigma(\mathcal{A})$ to be the set of all eigenvalues of $\mathcal{A}$. Then, the spectral radius of $\mathcal{A}$ is defined by

$$\rho(\mathcal{A}) := \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

The following is a Perron-Frobenius type theorem for nonnegative tensors.

PROPOSITION 2.1. *([20], Theorem 2.3)*
If $\mathcal{B}$ is a nonnegative tensor, then $\rho(\mathcal{B})$ is an eigenvalue of $\mathcal{B}$ with a nonnegative eigenvector.

Next, we recall a Collatz-Wielandt type result.

PROPOSITION 2.2. *(Lemma 5.3 and Theorem 5.3, Yang-Yang [20]) Let $\mathcal{B}$ be a nonzero nonnegative mth order, n-dimensional tensor. Let $\rho(\mathcal{B})$ be its spectral radius. Then, for any $d > 0$,*

$$\min_i \frac{(\mathcal{B}d^{m-1})_i}{d_i^{m-1}} \leq \rho(\mathcal{B}) \leq \max_i \frac{(\mathcal{B}d^{m-1})_i}{d_i^{m-1}}.$$

Moreover,

$$\rho(B) = \max_{0 \neq x \geq 0} \min_{x_i > 0} \frac{(\mathcal{B}x^{m-1})_i}{x_i^{m-1}}.$$

Given an mth order, n-dimensional tensor $\mathcal{A}$, let $I \subseteq \{1, 2, \ldots, n\}$. Then, the principal subtensor of $\mathcal{A}$ corresponding to I is given by $\widetilde{\mathcal{A}} := [a_{i_1\, i_2\, i_3\, \cdots\, i_m}]$, where $i_k \in I$ for all $k = 1, 2 \ldots, m$.

The following corollary is immediate from the above proposition.

COROLLARY 2.3. *Let $\mathcal{D}$ be a principal subtensor of a nonnegative tensor $\mathcal{B}$. Then $\rho(\mathcal{D}) \leq \rho(\mathcal{B})$.*

Let $\mathcal{A}$ be a Z-tensor written in the form $\mathcal{A} = r\mathcal{I} - \mathcal{B}$, where $r \in R$ and $\mathcal{B}$ is a nonnegative tensor. We say that $\mathcal{A}$ is an M-tensor if $r \geq \rho(\mathcal{B})$ and a *strong M-tensor* if $r > \rho(\mathcal{B})$. The following result and its proof are modified versions of Theorem 3.3 in [21].

PROPOSITION 2.4. *Let $\mathcal{A} = r\mathcal{I} - \mathcal{B}$ be a Z-tensor and $\mu(\mathcal{A}) := \min_{\lambda \in \sigma(\mathcal{A})} Re(\lambda)$. Then,*

$$\mu(\mathcal{A}) = r - \rho(\mathcal{B}).$$

Moreover, $\mu(\mathcal{A})$ is a real eigenvalue of $\mathcal{A}$ corresponding to a real eigenvector.

Proof. Since $\rho(\mathcal{B})$ is a real eigenvalue of $\mathcal{B}$ corresponding to a real eigenvector, $r - \rho(\mathcal{B})$ is a real eigenvalue of $\mathcal{A}$ corresponding to (the same) real eigenvector. Hence,

$$\mu(\mathcal{A}) \leq r - \rho(\mathcal{B}).$$

On the other hand, if $\lambda \in \sigma(\mathcal{A})$, then $r - \lambda \in \sigma(\mathcal{B})$ and so,

$$r - Re(\lambda) \leq |r - \lambda| \leq \rho(\mathcal{B}).$$

This yields $r - \rho(\mathcal{B}) \leq Re(\lambda)$ and (taking the minimum over all $\lambda \in \sigma(\mathcal{A})$), $r - \rho(\mathcal{B}) \leq \mu(\mathcal{A})$. ◻

3. Q-tensors. Generalizing the concept of a Q-matrix of linear complementarity theory [4], Q-tensors were introduced in [16]. Let $\mathcal{A}$ be an mth order, n-dimensional tensor and $F(x) := \mathcal{A}x^{m-1}$. *We say that $\mathcal{A}$ is a Q-tensor if for every $q \in R^n$, TCP$(\mathcal{A}, q)$ has a solution.* Note that x is a solution of TCP$(\mathcal{A}, q)$ if and only if x is a solution of the piecewise polynomial equation

$$\min\{x, F(x) + q\} = 0.$$

Moreover, when m is even, the same x is also a solution of

$$\min\{x^{[m-1]}, F(x) + q\} = 0.$$

(Note that $\min\{x^{[m-1]}, F(x)\}$ is homogeneous of degree $m - 1$, while $\min\{x, F(x)\}$ may not be homogeneous.)

In what follows, we employ degree theoretic ideas. All necessary ideas and results concerning degree theory are given in [7], Prop. 2.1.3. The following is a basic result dealing with tensor complementarity problems. For any continuous function $f : R^n \to R^n$, suppose $f(x) = 0 \Rightarrow x = 0$. Then, the local degree of f at the origin (which equals the degree of f relative to any bounded open set containing zero) is well defined and is denoted by $\deg(f, 0)$. When this degree is nonzero, the equation $f(x) = p$ will have solutions for all p near the origin. We now apply this idea to tensor complementarity problems. Given a tensor $\mathcal{A}$, let

$$\Phi(x) := \min\{x, F(x)\}.$$

THEOREM 3.1. *Suppose that*

$$\Phi(x) = 0 \Rightarrow x = 0 \quad \textit{and} \quad \deg(\Phi, 0) \neq 0.$$

Then, $\mathcal{A}$ is a Q-tensor and TCP$(\mathcal{A}, q)$ has a nonempty compact solution for any $q \in R^n$.

Proof. For any $q \in R^n$, let $\Phi_q(x) := \min\{x, F(x)+q\}$. Then, by the nearness property of degree (see Prop. 2.1.3(c), [7]), for all q sufficiently close to zero, $\deg(\Phi_q, 0) = \deg(\Phi, 0) \neq 0$. This means that TCP$(\mathcal{A}, q)$ has a solution for all q near zero. Since $F(x)$ is positive homogeneous of degree $m-1$, by scaling, TCP$(\mathcal{A}, q)$ has a solution for all $q \in R^n$. Now we will show the compactness of the solution set of TCP$(\mathcal{A}, q)$ for any given $q \in R^n$ under the condition $\Phi(x) = 0 \Rightarrow x = 0$. To see this, first observe that the solution set of TCP$(\mathcal{A}, q)$ is closed as it is the same as that of $\min\{x,\, F(x)+q\} = 0$. The boundedness of the solution set is seen via a 'normalization argument' as follows. Suppose, if possible, for some q, the solution set of $\min\{x,\, F(x)+q\} = 0$ is unbounded. Let $x^{(k)}$ be a sequence in the solution set with $||x^{(k)}|| \to \infty$. Writing $\min\{x,\, F(x)+q\} = 0$ in terms of complementarity conditions, we see that $\min\{\lambda\, x,\, F(\lambda\, x)+\lambda^{m-1}q)\} = \min\{\lambda\, x,\, \lambda^{m-1}(F(x)+q)\} = 0$ for all $\lambda > 0$. Now replacing x by x^k, choosing $\lambda := ||x^{(k)}||^{-1}$, letting $k \to \infty$, and putting (without loss of generality) $\overline{x} := \lim \frac{x^{(k)}}{||x^{(k)}||}$, we get $\min\{\overline{x}, F(\overline{x})\} = 0$. We reach a contradiction as $||\overline{x}|| = 1$ and at the same time $\overline{x} = 0$. Thus, the nonempty solution set of TCP$(\mathcal{A}, q)$ is closed and bounded, hence compact. □

Remarks. The condition $\Phi(x) = 0 \Rightarrow x = 0$, which is equivalent to TCP$(\mathcal{A}, 0)$ having zero as the only solution, has been shown to be equivalent to the R_0-property of $\mathcal{A}$, see Proposition 3.1 (i) in [16]. The boundedness of the involved solution set of the tensor complementarity problem has been addressed under this R_0-property in Theorem 3.3, [18]. The Q-property of $\mathcal{A}$ is discussed in Theorem 3.2 of [16] in which the R-property of $\mathcal{A}$ is required. Here, differing from the R-property, the Q-property is achieved via degree theory.

COROLLARY 3.2. *Under each of the following conditions, $\mathcal{A}$ is a Q-tensor and the corresponding TCP$(\mathcal{A}, q)$ has a nonempty compact solution for any $q \in R^n$.*

- *(i) There exists a vector $d > 0$ such that for TCP$(\mathcal{A}, 0)$ and TCP$(\mathcal{A}, d)$, zero (vector) is the only solution.*
- *(ii) $\mathcal{A}$ is a strictly semi-monotone (or strictly semi-positive) tensor, that is, for each nonzero $x \geq 0$, $\max_i\, x_i(\mathcal{A}x^{m-1})_i > 0$.*
- *(iii) $\mathcal{A}$ is a strictly copositive tensor, that is, for all $0 \neq x \geq 0$, $\mathcal{A}x^m := \langle \mathcal{A}x^{m-1}, x\rangle > 0$.*
- *(iv) $\mathcal{A}$ is a positive definite tensor, that is, for all $x \neq 0$, $\mathcal{A}x^m := \langle \mathcal{A}x^{m-1}, x\rangle > 0$.*

Proof. (i) Note that this condition is precisely what is given in the well-known Karamardian's theorem. Our degree theory proof offers, in addition to existence, a stability result (in the sense that certain nonhomogeneous nonlinear complementarity problems of the form NCP(G, p) with (G, p) close to $(F, 0)$ will also have solutions). Now to show that condition (i) implies the desired results, we set up a homotopy:

$$H(x,t) := \min\{x, F(x) + td\} \quad 0 \leq t \leq 1.$$

Since TCP$(\mathcal{A}, 0)$ and TCP$(\mathcal{A}, d)$ have zero solutions, we have $H(x, 0) = 0 \Rightarrow x = 0$ and $H(x, 1) = 0 \Rightarrow x = 0$. In addition, for any t, $0 < t < 1$, $H(x, t) = 0$ implies, by scaling and using the homogeneity of F, $\min\{sx, F(sx) + d\} = 0$ for some positive s. This yields $x = 0$. Thus, the zero set of the entire homotopy reduces to just $\{0\}$. Now, by the homotopy invariance of the degree,

$$\deg(\Phi, 0) = \deg(H(x, 0), 0) = \deg(H(x, 1), 0).$$

As $H(x, 1) = \min\{x, F(x) + d\} = x$ near zero, we see that $\deg(H(x, 1), 0) = 1$. Thus, $\deg(\Phi, 0) = 1$. Now the above theorem shows that $\mathcal{A}$ is a Q-tensor and TCP$(\mathcal{A}, q)$ has a nonempty compact solution for any $q \in R^n$.

When condition (ii) holds, for any $d > 0$, TCP$(\mathcal{A}, 0)$ and TCP$(\mathcal{A}, d)$ have zero solutions. Hence $\mathcal{A}$

is a Q-tensor and the corresponding TCP$(\mathcal{A}, q)$ has a nonempty compact solution for any $q \in R^n$. It is easy to see that $(iv) \Rightarrow (iii) \Rightarrow (ii)$. Thus, the asserted conclusions hold when condition (iii) or (iv) holds. □

Remarks. It is worth pointing out that (i) of Corollary 3.2 is actually equivalent to the R-property of $\mathcal{A}$ as discussed in Proposition 3.1 (ii) and Theorem 3.2 in [16], and the Q-property under (ii) of Corollary 3.2 has been discussed in Corollary 3.3 in [16]. Besides, see [3], where conditions (iii) and (iv) are discussed in relation to the Q-property of $\mathcal{A}$.

4. Z-tensors; Some basic results. In this section, we characterize the Q-property of a Z-tensor in various equivalent ways. We start by recalling a result that says that in the case of a complementarity problem corresponding to a Z-tensor, feasibility implies solvability.

PROPOSITION 4.1. *(Corollary 1, [13]). Suppose $\mathcal{A}$ is a Z-tensor. If TCP$(\mathcal{A}, q)$ is feasible, that is, there exists $u \geq 0$ such that $\mathcal{A}u^{m-1} + q \geq 0$, then it is solvable.*

Based on this proposition, we can characterize Z-tensors having the Q-property.

THEOREM 4.2. *Suppose $\mathcal{A}$ is an mth order, n-dimensional tensor. Consider the following statements:*

(i) $\mathcal{A}$ is a Q-tensor.
(ii) For every $q \in R^n$, TCP$(\mathcal{A}, q)$ is feasible.
(iii) There exists $d > 0$ such that $\mathcal{A}d^{m-1} > 0$.

Then, $(i) \Rightarrow (ii) \Leftrightarrow (iii)$. Moreover, these statements are equivalent when $\mathcal{A}$ is a Z-tensor.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious.
The equivalence of (ii) and (iii) is established in Theorem 3.2 in [18]. When $\mathcal{A}$ is a Z-tensor, we quote the previous proposition to see that TCP$(\mathcal{A}, q)$ is solvable under (ii). □

The following result characterizes the Q-property of a Z-tensor in different ways. These conditions/properties have been discussed in various articles. We collect them together here, offer a proof for completeness and for further refined results (in the next section). Note that these results are generalizations of similar results for Z-matrices. They are also similar to the ones for Z-transformations on proper cones [10].

THEOREM 4.3. *Let $\mathcal{A}$ be a Z-tensor given by $\mathcal{A} = r\mathcal{I} - \mathcal{B}$, where $r \in R$ and B is a nonnegative tensor. Then the following statements are equivalent:*

(a) $\mathcal{A}$ is a Q-tensor.
(b) For each $q \geq 0$, there exists $x \geq 0$ such that $\mathcal{A}x^{m-1} = q$.
(c) $\mathcal{A}$ is an S-tensor, that is, there exists $d > 0$ such that $\mathcal{A}d^{m-1} > 0$.
(d) $\mathcal{A}$ is a strong M-tensor, that is, $r > \rho(\mathcal{B})$.
(e) For all $0 \neq x \geq 0$, $\max_i x_i(\mathcal{A}x^{m-1})_i > 0$.
(f) For all $q \geq 0$, zero is the only solution of TCP$(\mathcal{A}, q)$.

In addition, the above conditions are further equivalent to

(i) $\mathcal{A}$ is positive stable, that is, $\mu(\mathcal{A}) > 0$.
(ii) For all $\varepsilon \geq 0$, $(\mathcal{A} + \varepsilon\mathcal{I})x^{m-1} = 0 \Rightarrow x = 0$.
(iii) For any nonnegative diagonal tensor $\mathcal{D}$ compatible with $\mathcal{A}$, $\mathcal{A} + \mathcal{D}$ is a strong M-tensor.

Proof. $(a) \Rightarrow (b)$: Assume (a) and let $q \geq 0$. Then there exists $x \in R^n$ such that

$$x \geq 0,\, y := \mathcal{A}x^{m-1} - q \geq 0,\, \langle x, y \rangle = 0.$$

Then, by the Z-property of $\mathcal{A}$, $\langle \mathcal{A}x^{m-1}, y \rangle \leq 0$. This yields $\langle y + q, y \rangle \leq 0$ and $||y||^2 + \langle q, y \rangle \leq 0$. As

$q \geq 0$, we get $y = 0$ showing $\mathcal{A}x^{m-1} = q$.
$(b) \Rightarrow (c)$: Taking (any) $q > 0$, we get an $x \geq 0$ such that $\mathcal{A}x^{m-1} = q > 0$. By continuity, there exists $d > 0$ such that $\mathcal{A}d^{m-1} > 0$.
$(c) \Rightarrow (d)$: This comes from Proposition 2.2.
$(d) \Rightarrow (e)$: Assume (d) and suppose there exists a nonzero x with $x \geq 0$ and $x_i(\mathcal{A}x^{m-1})_i \leq 0$ for all i. without loss of generality, let $I = \{i : x_i \neq 0\}$ be $\{1, 2, \ldots, l\}$. Then, $(\mathcal{A}x^{m-1})_i \leq 0$ for all $i \in I$. Since $\mathcal{A} = r\mathcal{I} - \mathcal{B}$, considering a principal subtensor $\mathcal{D}$ of $\mathcal{B}$, we get $(r\mathcal{I} - \mathcal{D})\, y^{m-1} \leq 0$, where y is the vector formed by the x_i, $i \in I$. This leads to $r \leq \frac{(\mathcal{D}y^{m-1})_i}{y_i^{m-1}}$ for all $i \in I$ and hence to $r \leq \rho(\mathcal{D})$. As $\rho(\mathcal{D}) \leq \rho(\mathcal{B})$, this clearly is a contradiction. Hence we have $(d) \Rightarrow (e)$.
$(e) \Rightarrow (f)$: let $q \geq 0$ and let x be a solution of TCP$(\mathcal{A}, q)$. If x is nonzero, then $x_i(\mathcal{A}x^{m-1})_i > 0$ for some i and $x_i(\mathcal{A}x^{m-1} + q)_i > 0$. Thus, x cannot be complementary to $\mathcal{A}x^{m-1} + q$, yielding a contradiction.
$(f) \Rightarrow (a)$: This comes from Corollary 3.2, Item (i) by taking $q = 0$ and $q > 0$ in (f).
Now for the additional statements:
$(d) \Leftrightarrow (i)$: This comes from Proposition 2.4.
$(i) \Rightarrow (ii)$: If (ii) is false, then $\mathcal{A}$ will have a non-positive real eigenvalue, contradicting (i).
$(ii) \Rightarrow (i)$: If $\mu(\mathcal{A}) \leq 0$, then $\varepsilon := -\mu(\mathcal{A})$ will satisfy $(\mathcal{A} + \varepsilon\,\mathcal{I})x^{m-1} = 0$ for some nonzero x.
$(e) \Rightarrow (iii)$: Let $\mathcal{D}$ be any nonnegative diagonal tensor $\mathcal{D}$ (compatible with $\mathcal{A}$). Clearly, $\mathcal{A} + \mathcal{D}$ is a Z-tensor. Suppose there is a nonzero nonnegative x, with $x_i\left[(\mathcal{A} + \mathcal{D})x^{m-1}\right]_i \leq 0$ for all i. Then, $x_i(\mathcal{A}x^{m-1})_i \leq 0$ for all i, contradicting (e). Thus, $\mathcal{A} + \mathcal{D}$ satisfies a condition similar to (e) and hence a strong M-tensor.
The implication $(iii) \Rightarrow (e)$ holds by taking $\mathcal{D} = 0$. ☐

Remarks. The equivalence of (a) and (c) can also be seen by the previous theorem.
The equivalence of (e) and (f) is also given in Theorem 3.2 of [17].
When $\mathcal{A}$ is a strong M-tensor, combining Items (ii) and (iii), we get: For any nonnegative diagonal tensor $\mathcal{D}$,

$$(\mathcal{A} + \mathcal{D})x^{m-1} = 0 \Rightarrow x = 0. \tag{4.1}$$

5. Some refined results for Z-tensors. When the Z-tensor $\mathcal{A}$ is a matrix (corresponding to $m = 2$), there are more than 52 conditions equivalent to $\mathcal{A}$ being a strong M-matrix. Some generalizations of these were considered in Theorem 4.3. In what follows, we prove some refined results for even ordered tensors.

Surjectivity of the map $F(x) := \mathcal{A}x^{m-1}$.
In Theorem 4.3, Item (b), we saw that when $\mathcal{A}$ is a strong M-tensor, the equation $F(x) = q$ has a solution for every $q \geq 0$. This raises the question whether this is true for all $q \in R^n$. When $m = 2$, F is a linear map. In this case, the solvability of $F(x) = q$ for all $q \geq 0$ implies that the image of F contains an open set and hence gives the surjectivity of F. As F is linear, this gives the injectivity of F and consequently, the invertibility of F. Additionally, $F^{-1}(R^n_+) \subseteq R^n_+$. This fails when m is odd: Take $m = 3$, $n = 2$, $\mathcal{A} = \mathcal{I}$ and consider the $F(x) = (x_1^2, x_2^2)^\top$. Clearly, $F(x) = q$ is solvable for all $q \geq 0$, but not for all $q \in R^2$. However, we have the following result for even ordered tensors and a related example.

THEOREM 5.1. *Suppose $\mathcal{A}$ is a Z-tensor of even order. Then the following are equivalent:*
(a) $\mathcal{A}$ is a strong M-tensor.
(b) $F(x) = 0 \Rightarrow x = 0$ and $\deg(F, 0) = 1$.

(c) $F(x)$ is surjective.

Proof. $(a) \Rightarrow (b)$: Suppose $\mathcal{A}$ is a strong M-tensor. Then, by Item (ii) of Theorem 4.3, $F(x)=0 \Rightarrow x=0$. Thus, the local degree of F at the origin is defined. Let $\mathcal{A}=r\mathcal{I}-\mathcal{B}$, where B is nonnegative tensor with $\rho(\mathcal{B})<r$. Then, for any $t\in[0,1]$, $r\mathcal{I}-t\mathcal{B}$ is also a strong M-tensor. Thus, $(r\mathcal{I}-t\mathcal{B})x^{m-1}=0 \Rightarrow x=0$. This means that the homotopy

$$H(t,x) := (r\mathcal{I}-t\mathcal{B})x^{m-1}$$

connecting $H(0,x)=rx^{[m-1]} =: G(x)$ and $H(1,x)=F(x)$ will have its zero set $\{0\}$. This means, by the homotopy invariance of degree,

$$\deg(F,0)=\deg(G,0).$$

As m is even, the local degree of the one variable function $\phi(t)=t^{m-1}$ at zero is one; it follows from Cartesian product property of degree (see Prop. 2.1.3(h) in [7]) that $\deg(G,0)=1$. Hence, $\deg(F,0)=1$.
$(b) \Rightarrow (c)$: Given (b), by the nearness property of degree, for all q close to zero, $\deg(F-q,0)=1$. This means that the equation $F(x)-q=0$ has a solution for all such q. Since F is positive homogeneous, by scaling, we see that $F(x)-q=0$ will have a solution for all $q\in R^n$. This proves the surjectivity of F.
$(c) \Rightarrow (a)$: This follows from the equivalence of Items (b) and (d) in Theorem 4.3. ◻

The following example shows that the map F in the above theorem need not be injective and that the inclusion $F^{-1}(R^n_+)\subseteq R^n_+$ may not hold.

EXAMPLE 5.1. *Let $\mathcal{A}=[a_{i_1i_2i_3i_4}]$ be of order 4 and dimension 2 with*

$$a_{1111}=a_{2222}=1,\ a_{1112}=-2,\ a_{1122}=-\alpha,\ \text{other entries}\ \ 0,$$

where $\alpha\in\{0,4\}$. Obviously, $\mathcal{A}$ is a Z-tensor. For this tensor,

$$(\mathcal{A}x^3)_1=F_1(x)=x_1^3-2x_1^2x_2-\alpha\, x_1x_2^2 \quad \text{and} \quad (\mathcal{A}x^3)_2=F_2(x)=x_2^3,$$

where $x=(x_1,x_2)^\top\in R^2$. Now, for any nonzero $x=(x_1,x_2)^\top$, we have:
- *if $x_2\neq 0$, then $x_2\left(\mathcal{A}x^3\right)_2=x_2^4>0$;*
- *if $x_2=0$ (in which case $x_1\neq 0$), then $x_1\left(\mathcal{A}x^3\right)_1=x_1^4>0$.*

Thus, condition (e) of Theorem 4.3 holds; hence, $\mathcal{A}$ is a strong M-tensor.
When $\alpha=4$, F equals $(1,1)^\top$ at $(-1,1)^\top$ and at $(t,1)^\top$ for some $t>0$. This means that F is not surjective and the inclusion $F^{-1}(R^2_+)\subseteq R^2_+$ does not hold.

The P-property
It is well known that a Z-matrix has the P-property if and only if it is a strong M-matrix [1]. Does such a statement hold for Z-tensors? Recall that for a square real matrix A, the P-property can be described in any one of the following three equivalent ways [4]:
(i) Every principal minor of A is positive.
(ii) For each nonzero $x\in R^n$, $\max_i\, x_i(Ax)_i>0$.
(iii) For every $q\in R^n$, $\mathrm{LCP}(A,q)$ has a unique solution.

We will show below that for strong M-tensors, appropriate analogs of (i) and (ii) hold, but (iii) may fail.

Now for the positive principal minor property. While the determinant of a tensor is defined (see [11]), it is not clear how to relate the (positive) determinants with the Z-property. So, we describe the positive principal minor property in a different way. Suppose A is an invertible matrix. Then, $f(x) := Ax$ is linear and $f(x) = 0 \Rightarrow x = 0$. Thus, $\deg(f, 0)$ is defined and moreover $\deg(f, 0) = sgn\ \det(A) = 1$ if and only if the determinant of A is positive. A similar statement holds for principal submatrices of A as well. Thus, we may interpret the positive principal minor property of A by saying that $f_\alpha(y) = 0 \Rightarrow y = 0$ and $\deg(f_\alpha, 0) = 1$ for each f_α corresponding to a principal submatrix of A. We now state a generalization of this to even order Z-tensors. The one dimensional example $\mathcal{A} = [1]$ with $m = 3$, $n = 1$ and $F(x) = x^2$ shows that the result fails for odd order tensors.

THEOREM 5.2. *Suppose $\mathcal{A}$ is a Z-tensor of even order. Then $\mathcal{A}$ is a strong M-tensor if and only if for every principal subtensor $\widetilde{\mathcal{A}}$ of $\mathcal{A}$, the corresponding function $\widetilde{F}(x) := \widetilde{\mathcal{A}}x^{m-1}$ satisfies the conditions*

$$\widetilde{F}(x) = 0 \Rightarrow x = 0 \quad \textit{and} \quad \deg(\widetilde{F}, 0) = 1.$$

Proof. First assume that $\mathcal{A}$ is a strong M-tensor. Let $\mathcal{A} = r\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative tensor and $r > \rho(B)$. The case of $\mathcal{A}$ and $F(x) = \mathcal{A}x^{m-1}$ has been dealt with in the previous theorem. We assume that $\widetilde{\mathcal{A}}$ is a subtensor of $\mathcal{A}$, not equal to $\mathcal{A}$. Then there exists a proper subset I of $\{1, 2, \ldots, n\}$, which we assume without loss of generality, $I = \{1, 2, \ldots, l\}$ such that

$$\widetilde{\mathcal{A}} = [a_{j_1 j_2 \cdots j_m}],$$

where $j_k \in I$ for all $k = 1, 2, \ldots, m$. Let $\mathcal{D}$ be the subtensor of $\mathcal{B}$ corresponding to this I so that $\mathcal{C} := \widetilde{\mathcal{A}} = r\mathcal{I} - \mathcal{D}$. As $\mathcal{D}$ is a principal subtensor of $\mathcal{B}$, we must have $\rho(\mathcal{D}) \leq \rho(\mathcal{B}) < r$. Thus, $\mathcal{C}$ is a strong M-tensor. By what has been proved earlier, for $G(x) = \mathcal{C}x^{m-1}$, $x \in R^l$, $G(x) = 0 \Rightarrow x = 0$ and $\deg(G, 0) = 1$.
The converse follows from Theorem 5.1. This completes the proof. □

We now consider the P-matrix condition (ii): for each nonzero $x \in R^n$, $\max_i\ x_i(Ax)_i > 0$. Recently, Song and Qi [15] extended this to tensors: A tensor $\mathcal{A}$ is said to be a P-tensor if for any nonzero $x \in R^n$, $\max_i\ x_i(\mathcal{A}x^{m-1})_i > 0$. This was further extended in [5]: A tensor $\mathcal{A}$ is said to be an (extended) P-tensor if for any nonzero x, $\max_i\ x_i^{m-1}(\mathcal{A}x^{m-1})_i > 0$.

THEOREM 5.3. *Suppose $\mathcal{A}$ is a Z-tensor. Then the following are equivalent:*
(a) $\mathcal{A}$ is a strong M-tensor.
(b) For any nonzero x, $\max_i\ x_i^{m-1}(\mathcal{A}x^{m-1})_i > 0$.
If m is even, these are further equivalent to
(c) For any nonzero x, $\max_i\ x_i(\mathcal{A}x^{m-1})_i > 0$.

Proof. $(a) \Rightarrow (b)$: This implication comes from Proposition 4.1 in [5], whose proof is based on H-tensors and diagonal dominance ideas. Here, for completeness, we provide a (slightly different) proof. We prove the implication by induction on n. The result is clearly true for $n = 1$. Suppose (a) holds and (b) fails for some nonzero x: $x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0$ for all i. Such a condition cannot hold for any proper principal subtensor of $\mathcal{A}$ by our induction hypothesis. Thus, no component of x can be zero. Then, by putting $\alpha_i := \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}$, we see that each α_i is nonpositive. Let $\mathcal{D}$ be a nonnegative diagonal tensor with diagonal components $-\alpha_i$, so that $(\mathcal{A} + \mathcal{D})x^{m-1} = 0$. As $\mathcal{A}$ is a

strong M-tensor, (4.1) shows that this cannot happen. Thus, $(a) \Rightarrow (b)$.
$(b) \Rightarrow (a)$: If condition (b) holds for all nonzero x, it certainly holds for all nonzero nonnegative x. Consider such an x. Then there exists $x_i > 0$ with $x_i^{m-1}(\mathcal{A}x^{m-1})_i > 0$ or equivalently, $x_i(\mathcal{A}x^{m-1})_i > 0$. This implies condition (e) in Theorem 4.3. Thus, $\mathcal{A}$ is a strong M-tensor.
Now suppose that m is even. Then the signs of x_i and x_i^{m-1} are the same. Consequently, (b) and (c) are equivalent. □

Remark. When m is odd, (a) may not imply (c): Take $\mathcal{A} = [1]$ with $m = 3$, $n = 1$ and $F(x) = x^2$.

Global uniqueness
As noted previously, for a matrix A, the linear complementarity problem LCP(A, q) has a unique solution for all $q \in R^n$ if and only if A is a P-matrix. In particular, this global uniqueness property holds for a strong M-matrix. To see what happens for tensors, consider the strong M-tensor $\mathcal{A}$ of Example 5.1 with $\alpha = 0$. By Theorem 5.3, $\mathcal{A}$ is actually an (extended) P-tensor. For $q = (0, -1)^\top$, we have two solutions to TCP$(\mathcal{A}, q)$, namely, $(0, 1)^\top$ and $(2, 1)^\top$. Thus, *uniqueness of TCP solution may not prevail even for strong M-tensors (or for extended P-tensors).* This raises the question: *which strong M-tensors admit unique solutions in all related tensor complementarity problems*? In the complementarity literature, a function $f : R^n \to R^n$ is said to have the *Globally Uniquely Solvable* property (GUS-property for short) if for all $q \in R^n$, the nonlinear complementarity problem NCP(f, q) has a unique solution. Two well-known conditions implying the GUS-property are: The strong monotonicity condition (see Section 2.3 in [7]) and the 'positively bounded Jacobians' condition of Megiddo and Kojima (see Lemma 1, [3]). The GUS-property in the context of tensor complementarity problems has been addressed recently in [3]. In their conditions for the GUS-property, all involved tensors need to be symmetric and be of even order and positive semi-definite, and especially, the second-order tensor should be positive definite. Departing from these conditions, in the result below, we offer an (easily checkable) sufficient condition for a strong M-tensor to have the GUS-property.

THEOREM 5.4. *Suppose $\mathcal{A} = [a_{i_1 \cdots i_m}]$ is a strong M-tensor of order $m\,(\geq 3)$ and dimension n such that for each index i,*

$$a_{i\,i_2 \cdots i_m} = 0 \quad \textit{whenever } i_j \neq i_k \quad \textit{for some } j \neq k.$$

Then, for any $q \in R^n$, TCP$(\mathcal{A}, q)$ has a unique solution.

Proof. As $\mathcal{A}$ has the S-tensor property (see Theorem 4.3), it follows from [6, Theorem 3] and [6, Proposition 5] that there exists a positive diagonal matrix $D = Diag(d_i) \in R^{n \times n}$ such that the tensor $\bar{\mathcal{A}} = \mathcal{A}D^{m-1} := [\bar{a}_{i_1 \cdots i_m}]$, defined by

$$\bar{a}_{i_1 \cdots i_m} = a_{i_1 \cdots i_m} d_{i_1} \cdots d_{i_m}, \quad \forall\, i_1, \cdots, i_m \in \{1, \cdots, n\} \tag{5.1}$$

is strictly diagonally dominant; in fact,

$$\bar{a}_{i \cdots i} > \sum_{i_2, \cdots, i_m} |\bar{a}_{i i_2 \cdots i_m}| - \bar{a}_{i \cdots i} = -\sum_{k \neq i} \bar{a}_{ik \cdots k}, \quad \forall\, i \in \{1, \cdots, n\}, \tag{5.2}$$

Now we claim that for any given $q \in R^n$, TCP$(\bar{\mathcal{A}}, q)$ has a unique solution. As $\mathcal{A}$ is a Z-tensor with $\bar{\mathcal{A}}e^{m-1} > 0$, where e is the vector of ones in R^n, it follows that $\bar{\mathcal{A}}$ is a strong M-tensor; hence, TCP$(\bar{\mathcal{A}}, q)$ has a solution. To prove uniqueness, assume that that there exist distinct solutions $\hat{y}$ and $\tilde{y}$ of TCP$(\bar{\mathcal{A}}, q)$. That is, for any $i \in \{1, \cdots, n\}$,

$$\begin{cases} \hat{y}_i \geq 0, \ \left(\bar{\mathcal{A}}\hat{y}^{m-1} + q\right)_i \geq 0, \ \hat{y}_i \left(\bar{\mathcal{A}}\hat{y}^{m-1} + q\right)_i = 0; \\ \tilde{y}_i \geq 0, \ \left(\bar{\mathcal{A}}\tilde{y}^{m-1} + q\right)_i \geq 0, \ \tilde{y}_i \left(\bar{\mathcal{A}}\tilde{y}^{m-1} + q\right)_i = 0. \end{cases} \tag{5.3}$$

As $\hat{y} \neq \tilde{y}$, $\max_i\{|\hat{y}_i^{m-1} - \tilde{y}_i^{m-1}|\} > 0$. Let $j := \arg\max_i\{|\hat{y}_i^{m-1} - \tilde{y}_i^{m-1}|\}$, and without loss of generality, $\hat{y}_j - \tilde{y}_j > 0$. By direct calculation, we have

$$\begin{aligned}
&(\hat{y}_j - \tilde{y}_j)\left(\bar{\mathcal{A}}\hat{y}^{m-1} - \bar{\mathcal{A}}\tilde{y}^{m-1}\right)_j \\
&= (\hat{y}_j - \tilde{y}_j)\left(\bar{\mathcal{A}}\hat{y}^{m-1} + q - \bar{\mathcal{A}}\tilde{y}^{m-1} - q\right)_j \\
&= -\hat{y}_j\left(\bar{\mathcal{A}}\tilde{y}^{m-1} + q\right)_j - \tilde{y}_j\left(\bar{\mathcal{A}}\hat{y}^{m-1} + q\right)_j \\
&\leq 0.
\end{aligned}$$

On the other hand, by the imposed conditions on the entries of $\mathcal{A}$,

$$\begin{aligned}
&(\hat{y}_j - \tilde{y}_j)\left(\bar{\mathcal{A}}\hat{y}^{m-1} - \bar{\mathcal{A}}\tilde{y}^{m-1}\right)_j \\
&= (\hat{y}_j - \tilde{y}_j)\bar{a}_{j\cdots j}\left(\hat{y}_j^{m-1} - \tilde{y}_j^{m-1}\right) + \sum_{k\neq j}\bar{a}_{jk\cdots k}\left(\hat{y}_k^{m-1} - \tilde{y}_k^{m-1}\right) \\
&\geq (\hat{y}_j - \tilde{y}_j)\left(\hat{y}_j^{m-1} - \tilde{y}_j^{m-1}\right)\left(\bar{a}_{j\cdots j} + \sum_{k\neq j}\bar{a}_{jk\cdots k}\right) \\
&> 0,
\end{aligned}$$

where the first inequality follows from the definition of j and the fact that $\bar{\mathcal{A}}$ is a Z-tensor, and the last inequality follows from (5.2). This is a contradiction. Thus, TCP$(\bar{\mathcal{A}}, q)$ has a unique solution, say y^*. We can easily verify that y^* is also the unique solution to the following problem:

$$Dy \geq 0, \bar{\mathcal{A}}y^{m-1} + q \geq 0, \langle Dy, \bar{\mathcal{A}}y^{m-1} + q\rangle = 0.$$

Invoking the definition of $\bar{\mathcal{A}}$, it follows readily that Dy^* is the unique solution to TCP$(\mathcal{A}, q)$. This completes the proof. □

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