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$\{0, 1\}$ completely positive tensors and multi-hypergraphs

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A B S T R A C T

Completely positive graphs have been employed to associate with completely positive matrices for characterizing the intrinsic zero patterns. As tensors have been widely recognized as a higher-order extension of matrices, the multi-hypergraph, regarded as a generalization of graphs, is then introduced to associate with tensors for the study of complete positivity. To describe the dependence of the corresponding zero pattern for a special type of completely positive tensors—the $\{0, 1\}$ completely positive tensors, the completely positive multi-hypergraph is defined. By characterizing properties of the associated multi-hypergraph, we provide necessary and sufficient conditions for any $(0, 1)$ associated tensor to be $\{0, 1\}$ completely positive. Furthermore, a necessary and sufficient condition for a uniform multi-hypergraph to be a completely positive multi-hypergraph is proposed as well.

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1. Introduction

Completely positive matrices (cp matrices) [6,28], as a special type of nonnegative matrices, have wide applications in combinatorial theory including the study of block designs [16], and in optimization especially in creating convex formulations of NP-hard problems, such as the quadratic assignment problem in combinatorial optimization and the polynomial optimization problems [1–3,17,31]. The verification of cp matrices is generally NP-hard unless for small scale matrices. For example, all $n \times n$ nonnegative symmetric positive semidefinite matrices (usually called the doubly nonnegative (dnn) matrices) are cp-matrices whenever $n \leq 4$ [7]. For general case, it is obvious that cp matrices are dnn, but not always true conversely [4,13,27]. It depends on some inherited zero pattern which cp matrices possess. To describe this dependence, the tool of graphs was employed and the completely positive graph (cp graph) was introduced which has all its nonnegative associated matrices being cp. Among all those properties on cp-graphs, one of the most important and well-known is a graph to be completely positive if and only if it does not have an odd cycle of length greater than 4 [19]. This gives us a very efficient way to verify cp-matrices in terms of cp graphs.

Recently, the concept of cp matrix has been extended to the higher order cp tensor, which admits its definition in a pretty natural way as initiated by Qi et al. in [25]. Analog to the matrix case, the cp tensors were employed to reformulate polynomial optimization problems [24]. Numerical optimization for the best fit of completely positive tensors with given length of decomposition was formulated as a nonnegative constrained least-squares problem in Kolda's paper [18]. For the verification of cp tensors, an efficient approach in terms of truncated moment sequences for checking completely positive tensors was proposed and an optimization algorithm based on semidefinite relaxation for completely positive tensor decomposition was established by Fan and Zhou in [14]. This approach was later accelerated with some preprocessing steps by Luo and Qi in [21]. Some structured and geometrical properties on general cp tensors were also discussed in [21,26].

Inspired by the technique of using cp graphs for the characterization of cp matrices, we employ the multi-hypergraph as a tool to describe the inherited zero pattern for cp tensors, which can further assist with the verification of cp-tensors. Multi-hypergraphs appeared in the literature at least in 1988 or even earlier. Here we use the definitions in [23]. Due to complexity of cp-tensors for general higher order cases, we will focus on a special type of cp-tensors called the $\{0,1\}$ cp tensor, which is exactly a higher order extension of the $\{0,1\}$ cp matrix that has been well studied in [8,9] motivated by the applications in many fields such as the pattern recognition [20]. In order to verify $\{0,1\}$ cp tensors, we first build up the correspondence between multi-hypergraphs and symmetric tensors which are called the associated tensors. The $(0,1)$ associated tensor is also defined which is uniquely determined by the corresponding multi-hypergraph. Based on the aforementioned one-to-one relationship, we establish the necessary and sufficient conditions for a $(0,1)$ associated tensor to be $\{0,1\}$ cp in terms of some structure property possessed by the corresponding uniform multi-hypergraph. For general $\{0,1\}$ cp tensors which are not necessarily to be $(0,1)$ tensors, the cp multi-hypergraph is introduced and the necessary and sufficient condition of this type of multi-hypergraph is proposed. All of these can not only be served for verification for cp tensors, but also build up a bridge between tensor analysis and multi-hypergraph theory.

The rest of the paper is organized as follows. In Section 2, we introduce the associated tensors for multi-hypergraphs. Some related concepts and properties are also presented. In Section 3, the $\{0,1\}$ cp tensors is introduced and the equivalence conditions for $(0,1)$ associated tensors of multi-hypergraphs to be $\{0,1\}$ cp are proposed. In Section 4, the cp multi-hypergraph is defined in terms of $\{0,1\}$ cp tensors, and the necessary and sufficient condition of cp multi-hypergraphs is established.

Throughout the paper we denote by $[n]$ the set $\{1,2,\ldots,n\}$ for a positive integer n, $|S|$ for the cardinality of set (or multiset) S, and Z_+^n for the set of nonnegative integral vectors of dimension n. Denote by $\mathcal{T}_{m,n}$ the set of all mth order n-dimensional real tensors, and $\mathcal{S}_{m,n}$ the set of all mth order n-dimensional symmetric tensors. Denote R^n the real n-dimensional Euclidean space and R_+^n the set of all nonnegative vectors in R^n. Let $\mathcal{F} := \{0,1\}$ and denote by $\mathcal{F}_{m,n}$ the set of all mth order n-dimensional real tensors whose elements are either 1 or 0, and by $\mathbb{SF}_{m,n}$ the set of all symmetric tensors in $\mathcal{F}_{m,n}$. As convention we denote

$$S(m,n) := \{\tau = (i_1, i_2, \ldots, i_m) : i_1, i_2, \ldots, i_m \in [n]\}$$

for the index set of an element of an mth order tensor. For a vector $\mathbf{x} \in R^n$, we use $\mathtt{supp}(\mathbf{x})$ to denote the *support* of $\mathbf{x}$, i.e., the index set of the nonzero coordinates of $\mathbf{x}$.

2. The multi-hypergraph and its associated tensor

In this section, the multi-hypergraph and its associated tensors are recalled and introduced, and some related concepts and properties are presented.

Definition 2.1 *(Definition 7, [23]).* Let $V = \{v_1, v_2, \ldots, v_n\}$. A *multi-hypergraph* $\mathcal{P}$ is a pair $(V, \mathbb{E})$, where $\mathbb{E} = \{E_1, \ldots, E_N\}$ a set of multisets of V. The elements of V are called the *vertices* and the elements of $\mathbb{E}$ are called the *edges*. Moreover, a multi-hypergraph $\mathcal{P}$ is called an $n \times N$ *multi-hypergraph* if $|V| = n$, $|\mathbb{E}| = N$.

Definition 2.2 *(Definition 8, [23]).* A multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$ is called *m-uniform* $(m \geq 2)$ if for all $E \in \mathbb{E}$, the cardinal number of the multiset of E is m (including repeated memberships).

In this paper, we are interested in m-uniform multi-hypergraph. For simplicity, let $V = [n]$. The associated tensor of an m-uniform multi-hypergraph is defined as follows. Unless otherwise stated, we will use $\{i_1, \cdots, i_m\}$ to denote the multiset including repeated memberships throughout the paper.

Definition 2.3. Let $V = [n]$. A tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathcal{S}_{m,n}$ is said to be an associated tensor with the m-uniform multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$ if for all $(i_1, \cdots, i_m) \in S(m,n)$, $a_{i_1 \cdots i_m} \neq 0$ when the multiset $\{i_1, \cdots, i_m\}$ forms an edge in $\mathbb{E}$, and $a_{i_1 \cdots i_m} = 0$ otherwise.

Let $\alpha \in \mathbb{E}$. We use $B(\alpha)$ to denote the set consisting of all distinct elements of α and call it the *base* of α. Apparently, any hypergraph (see [5] for details) is a multi-hypergraph with $\alpha = B(\alpha)$ for each edge $\alpha \in \mathbb{E}$. Since the repetition is allowed in each edge for a general multi-hypergraph (i.e., $\mathcal{B}(\alpha) \subseteq \alpha$), some partial order can be induced for edges in terms of their bases. Let $\alpha, \beta \in \mathbb{E}$. α is said to be *majorized* (*strictly majorized*) by an edge β, denoted as $\alpha \preceq \beta$ ($\alpha \prec \beta$), if $B(\alpha) \subseteq B(\beta)$ ($B(\alpha) \subset B(\beta)$). α and β are said to be *similar*, denoted as $\alpha \sim \beta$, if both $\alpha \preceq \beta$ and $\alpha \succeq \beta$ hold. Similar edges have a common base. The majorization defines a partial order on $\mathbb{E}$ and gives a clustering of edges in $\mathbb{E}$, say $\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_r$ (possibly with some overlappings). By Zorn's Lemma, there exists at least one *maximal* (*minimal*) element in each $\mathcal{D}_i$, denoted as ℓ_i ($\mathbf{m}_i$, respectively) which satisfies

$$B(\alpha) \subseteq B(\ell_i) \ \ (B(\alpha) \supseteq B(\mathbf{m}_i), \text{ respectively}), \quad \forall \alpha \in \mathcal{D}_i$$

for each $i = 1, 2, \ldots, r$. For an m-uniform $n \times N$ multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$, it is obvious that $1 \leq |\mathbf{m}_i| \leq |\ell_i| \leq m$. Denote

$$rk(\mathcal{P}) := \max_{1 \leq i \leq r} \{|B(\ell_i)| : \ell_i \text{ is a maximal edge of } \mathcal{D}_i\},$$

and

$$ck(\mathcal{P}) := \min_{1 \leq i \leq r} \{|B(\mathbf{m}_i)| : \mathbf{m}_i \text{ is a minimal edge of } \mathcal{D}_i\}$$

$rk(\mathcal{P})$ ($ck(\mathcal{P})$, respectively) is called the *rank* (*co-rank*) of $\mathcal{P}$. For an m-uniform hypergraph $\mathcal{P}$ with no repeated vertices allowed in any of its edges, we have $rk(\mathcal{P}) = ck(\mathcal{P}) = m$.

A multi-hypergraph $\mathcal{P}$ may have several maximal (minimal) edges with different bases. But its rank (co-rank) shall be a unique number by definition. For any $\alpha \in S(m,n)$, we denote by $M(\alpha)$ the multiset generated by α and define the *complete m-multiset determined by α* as

$$\mathcal{D}_\alpha := \{\eta \in S(m,n) : M(\eta) \preceq M(\alpha)\}.$$

Lemma 2.4. *Let $|B(\alpha)| = r$ where $\alpha \in S(m,n)$. Then*

$$|\mathcal{D}_\alpha| = r^m. \tag{2.1}$$

Proof. Let $\alpha \in S(m,n)$, and $|B(\alpha)| = r$. We may assume w.l.g. that $B(\alpha) = \{s_1, s_2, \ldots, s_r\}, 1 \leq s_1 < s_2 < \ldots < s_r \leq n$. For each $\eta := (i_1, i_2, \ldots, i_m) \in \mathcal{D}_\alpha$, its coordinate i_k can be any number chosen from $B(\alpha)$ for each $k \in [m]$, and thus there are r^m choices, which leads to the desired assertion. $\square$

The following example is presented for the illustration of the above concepts.

Example 2.5. Let $\mathcal{P} = (V, \mathbb{E})$ be a 3-uniform multi-hypergraph with its associated tensor $\mathcal{A} = (A_{ijk}) \in \mathcal{S}_{3,3}$ whose nonzero elements are listed as below:

$$A_{112} = A_{122} = A_{133} = A_{113} = A_{223} = A_{111} = A_{222} = A_{333} = 1$$

Three complete 3-multisets of $\mathbb{E}$ given by the majorization are

$$\begin{aligned}
\mathcal{D}_1 &= \{\{1,1,2\}, \{1,2,1\}, \{2,1,1\}, \{1,2,2\}, \{2,1,2\}, \{2,2,1\}, \{1,1,1\}, \{2,2,2\}\}, \\
\mathcal{D}_2 &= \{\{1,1,3\}, \{1,3,1\}, \{3,1,1\}, \{1,3,3\}, \{3,1,3\}, \{3,3,1\}, \{1,1,1\}, \{3,3,3\}\}, \\
\mathcal{D}_3 &= \{\{2,2,3\}, \{2,3,2\}, \{3,2,2\}, \{2,3,3\}, \{3,2,3\}, \{3,3,2\}, \{2,2,2\}, \{3,3,3\}\}.
\end{aligned}$$

There are six maximal edges in each $\mathcal{D}_i$. In fact, all the edges but the three minimal edges $\{1,1,1\}, \{2,2,2\}, \{3,3,3\}$ are the maximal edges. So $rk(\mathcal{P}) = 2$, $ck(\mathcal{P}) = 1$. Note that $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ does not form a partition of $\mathbb{E} \subset S(3,3)$ since

$$\{1,1,1\} \in \mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset.$$

3. $\{0,1\}$ cp tensors and $(0,1)$ associated tensors

In this section, we will discuss the condition for $(0,1)$ associated tensors of uniform multi-hypergraphs to be $\{0,1\}$ cp tensors. Before stating the main theorem, the involved concepts are introduced as a start.

Definition 3.1. An mth order n-dimensional symmetric tensor $\mathcal{A}$ is called a *completely positive tensor*, or *cp tensor* for short, if $\mathcal{A}$ can be decomposed as

$$\mathcal{A} = \sum_{j=1}^{q} \mathbf{u}_j^m, \quad \mathbf{u}_j \in R_+^n, \forall j \in [q] \tag{3.1}$$

The smallest number q satisfying (3.1) is called the *cp-rank* of $\mathcal{A}$. Moreover, $\mathcal{A}$ is called to be $\{0,1\}$–*cp* if $\mathbf{u}_j \in \mathcal{F}^n$ for each $j \in [q]$ in (3.1).

Note that a $\{0,1\}$–*cp* tensor may not be a $(0,1)$ tensor, and a cp-tensor with all entries in $\mathcal{F}$ is not necessarily $\{0,1\}$–*cp*.

By Definition 2.3, it is obvious that for a given m-uniform multi-hypergraph $\mathcal{P}$, its associated tensors are infinitely many since we can put any nonzero scalars as entries in those positions corresponding to edges of the multi-hypergraph. If we further restrict the associated tensor $\mathcal{A}$ to be in $\mathbb{SF}_{m,n}$, then the correspondence turns out to be one-to-one, i.e.,

$$\{i_1, i_2, \ldots, i_m\} \in \mathbb{E} \Longleftrightarrow a_{i_1 i_2 \ldots i_m} = 1. \tag{3.2}$$

For any tensor $\mathcal{A} = (a_\sigma) \in \mathcal{T}_{m,n}$, a *tensor pattern* $\tilde{\mathcal{A}} = (\tilde{a}_\sigma) \in \mathcal{F}_{m,n}$ is defined in the way that for any $\sigma \in S(m,n)$, $\tilde{a}_\sigma = 1$ if $a_\sigma \neq 0$ and $\tilde{a}_\sigma = 0$ otherwise. Apparently, all associated tensors for an m-uniform multi-hypergraph share the same tensor pattern which is exactly the corresponding $(0,1)$ associated tensor. The pattern of a tensor $\mathcal{A}$ reflects the distribution of zero (nonzero) elements of $\mathcal{A}$ and thus can be used to characterize its spectral property e.g. [10–12,15,22,30,29] and combinatorial properties such as the irreducibility.

Definition 3.2 *(Definition 2.1, [10]).* An mth order n-dimensional real tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathcal{T}_{m,n}$ is called *reducible* if there is a nonempty proper subset $I \subset [n]$ such that

$$a_{i_1 \ldots i_m} = 0, \quad \forall i_1 \in I, \ \forall i_2, \ldots, i_m \notin I. \tag{3.3}$$

$\mathcal{A}$ is called *irreducible* if it is not reducible.

Recall that a tensor $\mathcal{A} = (A_\sigma) \in \mathcal{T}_{m,n}$ is said to be *strong symmetric* if it satisfies $A_\alpha = A_\beta$ whenever $B(\alpha) = B(\beta)$ for any $\alpha, \beta \in S(m,n)$[26]. We use $\mathbb{SSF}_{m,n}$ to denote the set of all m-order n dimensional strong symmetric real tensors. A *slice* of tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ is defined as a sub-tensor of order $m-1$ obtained from $\mathcal{A}$ with some index fixed. A *zero slice*, or a *trivial* slice, is a slice whose elements are all zeros. Given a nonempty subset $I := \{s_1, s_2, \ldots, s_r\}$ of $[n]$, a *principal subtensor* of $\mathcal{A}$ determined by I, is defined as the mth order r dimensional tensor $\mathcal{B} = (A_{i_1 i_2 \ldots i_m})$ where each i_k is constrained in I. A *zero block* is a principal subtensor whose entries are all zero. Obviously, an irreducible tensor has no zero slice nor any zero block.

Reducibility is a pattern property for tensors. By employing the permutational similarity property, we can decompose any $(0,1)$ reducible tensor into a direct sum of a finite number of low dimensional irreducible tensors and a zero tensor in the permutational similar sense. Before stating this result, some related concepts are recalled here. Let $\mathcal{A}, \mathcal{B} \in \mathcal{T}_{m,n}$. We say that $\mathcal{A}$ is *permutational similar* to $\mathcal{B}$, denoted as $\mathcal{A} \sim_p \mathcal{B}$, if there exists a permutation matrix $P \in R^{n \times n}$ such that

$$\mathcal{B} = \mathcal{A} \times_1 P \times_2 P \times_3 \cdots \times_m P,$$

where $\tilde{\mathcal{A}} := \mathcal{A} \times_k P = (\tilde{a}_{i_1 \ldots i_m}) \in \mathcal{T}_{m,n}$ is defined as

$$\tilde{a}_{i_1 \ldots i_{k-1} i_k i_{k+1} \ldots i_m} = \sum_{j=1}^{n} a_{i_1 \ldots i_{k-1} j i_{k+1} \ldots i_m} p_{i_k j}$$

Utilizing the permutational similarity of tensors, we can build up some identical relation among their corresponding multi-hypergraphs. Let $\mathcal{P}_1 = (V_1, \mathbb{E}_1)$ and $\mathcal{P}_2 = (V_2, \mathbb{E}_2)$ be two given m-uniform multi-hypergraphs with their $(0,1)$ associated tensors $\mathcal{A}$ and $\mathcal{B}$ respectively. Then $\mathcal{A} \sim_p \mathcal{B}$ if and only if there exists a bijection ϕ from V_1 to V_2 such that

$$\{i_1, i_2, \ldots, i_m\} \in \mathbb{E}_1 \mapsto \{\phi(i_1), \phi(i_2), \ldots, \phi(i_m)\} \in \mathbb{E}_2$$

that is, $\mathcal{P}(\mathcal{B})$ is the multi-hypergraph obtained from $\mathcal{P}(\mathcal{A})$ by the reordering of its vertices, and thus they are identical in this sense.

Let $\mathcal{A}_i = (a_\sigma^{(i)}) \in \mathcal{T}_{m,n_i}, i = 1, 2$ and $n_1 + n_2 = n$. The *direct sum* of $\mathcal{A}_1$ and $\mathcal{A}_2$, denoted by

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 = (a_{i_1 \ldots i_m}),$$

is defined by

$$a_{i_1 \ldots i_m} = \begin{cases} a^{(1)}_{i_1 \ldots i_m} & \text{if } i_1, \ldots, i_m \in [n_1], \\ a^{(2)}_{i_1 \ldots i_m} & \text{if } i_1, \ldots, i_m \in n_1 + [n_2], \\ 0 & \text{otherwise}. \end{cases}$$

Here $a + S$ is defined as the translation of set S, i.e., $a + S = \{a + s : s \in S\}$.

Now we are in a position to describe the decomposition for tensors in the sense of permutation similarity.

Lemma 3.3. *Let $\mathcal{A} \in \mathbb{SSF}_{m,n}$, where $m \geq 2, n \geq 1$. Then*

$$\mathcal{A} \sim_p \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \oplus \mathcal{A}_r \oplus \mathcal{O}_{r+1} \tag{3.4}$$

where $\mathcal{A}_i \in \mathbb{SSF}_{m,n_i}$ is irreducible, $\mathcal{O}_{r+1}$ is a zero tensor of order m and dimension n_{r+1}, and $n_1 + \ldots + n_{r+1} = n$.

Proof. The result is trivial if $\mathcal{A}$ is irreducible tensor. Now we assume that $\mathcal{A} \in \mathbb{SSF}_{m,n}$ is a reducible tensor. We will use induction to prove the desired statement. For $n = 1$, the reducibility implies that $\mathcal{A} = 0$. The statement holds by setting $r = 0$. Assume that for all k satisfying $1 \leq l \leq n$ with $n \geq 1$, the statement holds. They for the case of $l + 1$, there exists a nonempty subset I of $[l + 1]$ such that

$$A_{i_1 i_2 \ldots i_m} = 0, \forall i_1 \in I, i_2, \ldots, i_m \notin I \tag{3.5}$$

Let $\mathcal{P} = (V, \mathbb{E})$ be the multi-hypergraph with $\mathcal{A}$ as an associated tensor, and we assume w.l.g. that

$$I := \{k_1, k_2, \dots, k_r\}, 1 \le k_1 < k_2 < \cdots < k_r \le l+1.$$

Then we let $\phi : [l+1] \to [l+1]$ be an one-to-one correspondence such that

$$\phi(k_i) = i, \quad \forall\ i = 1, 2, \dots, r,$$

and ϕ maps $[l+1]\backslash I$ to $[l+1]\backslash[r]$. ϕ can be regarded as a permutation on $[l+1]$, and so there is a permutation matrix P corresponding to ϕ. Actually if we define $P = (p_{ij}) \in \mathcal{F}^{(l+1)\times(l+1)}$ by

$$p_{ij} = 1 \quad \texttt{iff}\ j = \phi(i)$$

for each $i \in [l+1]$. It follows readily that

$$\tilde{\mathcal{A}} := \mathcal{A} \times_1 P \times_2 P \times_3 \dots \times_m P = \mathcal{A}_{11} \oplus \mathcal{A}_{22} \tag{3.6}$$

where $\mathcal{A}_{11} \in \mathbb{SSF}_{m,r}, \mathcal{A}_{22} \in \mathbb{SF}_{m,l+1-r}$. Note that r, $l+1-r \le n$, the desired decomposition can be proved by the induction. $\square$

Lemma 3.3 shows that a strong symmetric tensor can always be decomposed into the direct sum of irreducible tensors, possibly with a zero block. But this is not true for a symmetric tensor, as is illustrated by the following counterexample.

Example 3.4. Let $\mathcal{A} = (A_{ijk})$ be an $2 \times 2 \times 2$ tensor defined by

$$A(:,:,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A(:,:,2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is not difficult to check that $\mathcal{A}$ is symmetric but none strong symmetric. Furthermore, its reducibility can be verified by taking $I = \{1\}$. However, $\mathcal{A}$ cannot be written as the direct sum $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ where $\mathcal{A}_1 = \mathcal{A}[I] = (A_{111}), \mathcal{A}_2 = (A_{222})$ are both $1 \times 1 \times 1$ tensors.

We shall note that when a tensor $\mathcal{A}$ is $\{0,1\}$–*cp*, then it must be a strong symmetric tensor[26]. Thus there is no difference between strong symmetry and symmetry when $\mathcal{A}$ is known to be $\{0,1\}$–*cp*. In fact, the tensor $\mathcal{A}$ in Example 3.4 is not a $\{0,1\}$–*cp* tensor.

The following lemma is dedicated to the necessary and sufficient conditions of $\{0,1\}$–*cp* property for irreducible $(0,1)$ tensors.

Lemma 3.5. *Let $m \ge 2$, $n \ge 1$ be two positive integers, and $\mathcal{A} \in \mathbb{SSF}_{m,n}$ be irreducible. Then the following statements are equivalent:*

(i) $\mathcal{A}$ *is* $\{0,1\}$*–cp;*
(ii) $\mathcal{A} = \mathcal{J}$ *is the all-*1 *tensor;*
(iii) the multi-hypergraph $\mathcal{P}$ *with associated tensor* $\mathcal{A}$ *is a complete block.*

Proof. If $\mathcal{A} = \mathcal{J}$, then surely $\mathcal{A}$ is $\{0,1\}$–*cp* since $\mathcal{A} = \ell^m$ with $\ell = (1,1,\dots,1)^\top$.

Conversely, we let $\mathcal{A} \in \mathbb{SF}_{m,n}$ be a $\{0,1\}$–*cp* tensor. Then $\mathcal{A}$ has a decomposition (3.1) with

$$\mathbf{u}_j = (u_{1j}, u_{2j}, \dots, u_{nj})^\top \in \mathcal{F}^n.$$

Then we have

$$a_{i_1 i_2 \dots i_m} = \sum_{j=1}^{q} u_{i_1 j} u_{i_2 j} \dots u_{i_m j}, \quad \forall (i_1, i_2, \dots, i_m) \in S(m,n).$$

We will first show that $q = 1$ in decomposition (3.1). Suppose that $q > 1$. If there exist a pair of positive integers $(s,t) : 1 \le s < t \le q$ such that

$$k \in \mathtt{supp}(\mathbf{u}_s) \cap \mathtt{supp}(\mathbf{u}_t)$$

for some $k \in [n]$, then $u_{ks} = u_{kt} = 1$. Hence we have

$$\begin{aligned} A_{kk\dots k} &= \sum_{j=1}^{q} u_{kj} u_{kj} \dots u_{kj} \\ &= \sum_{j=1}^{q} u_{kj}^m \\ &\ge u_{ks}^m + u_{kt}^m = 2 \end{aligned}$$

a contradiction to the assumption that $\mathcal{A}$ is a (0,1) tensor. Thus we have

$$\mathtt{supp}(\mathbf{u}_i) \cap \mathtt{supp}(\mathbf{u}_j) = \emptyset, \forall 1 \le i < j \le q \tag{3.7}$$

Now we define

$$\mathcal{D}_i = \{\sigma \in \mathbb{E} : B(\sigma) \subseteq \mathtt{supp}(\mathbf{u}_i)\}, \forall i = 1, 2, \dots, q$$

Then we get $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_q\}$ each a subset of $\mathbb{E}$, and

$$\mathcal{D}_i \cap \mathcal{D}_j = \emptyset, \quad \forall 1 \le i < j \le q$$

Denote $V_i = V(\mathcal{D}_i)$ and $\mathcal{P}_i := (V_i, \mathcal{D}_i)$ for $i = 1, 2, \dots, q$. Then

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_q$$

where $\mathcal{P} = (V, \mathbb{E})$ is the multi-hypergraph associated with $\mathcal{A}$. It turns that $\mathcal{A} \sim_p \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_q$ where $\mathcal{A}_i$ is the adjacency tensor of $\mathcal{P}_i$, a contradiction to the hypothesis that $\mathcal{A}$ is irreducible. Hence $q = 1$, and thus there exists a vector $\mathbf{u} = (u_1, \ldots, u_n)^\top \in \mathcal{F}^n$ such that $\mathcal{A} = \mathbf{u}^m$.

To prove that $\mathcal{A} = \mathcal{J} = \ell^m$, we need only to show that $\mathtt{supp}(\mathbf{u}) = [n]$. In fact, if $\mathtt{supp}(\mathbf{u})$ is a proper subset of $[n]$, then by setting $I = [n] \backslash \mathtt{supp}(\mathbf{u})$, we show that $\mathcal{A}$ is reducible by definition, which is a contradiction to the hypothesis. Thus $\mathtt{supp}(\mathbf{u}) = [n]$ and $\mathcal{A} = \mathcal{J}$. Thus the equivalence between (i) and (ii) is obtained.

The remaining part of the lemma is immediate by definition. □

From Lemma 3.5 and its proof, we can get the following equivalences for $\{0,1\}$–*cp* tensors.

Theorem 3.6. *Let $m \geq 2$, $n \geq 1$ be two positive integers. Suppose that $\mathcal{A} \in \mathbb{SF}_{m,n}$ have no zero blocks and is associated with multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$. Then the following are equivalent:*

(1) *$\mathcal{A}$ is $\{0,1\}$–cp tensor.*
(2) *$\mathcal{P}$ can be decomposed as the union of some complete blocks $\mathcal{P}_i$ of size n_i where $n_1 + \ldots + n_q = n$.*
(3) *$\mathcal{A}$ can be written in form (3.1) and with $\mathbf{u}_j \in \mathcal{F}^n$ satisfying $U^T U = diag(n_1, \ldots, n_q)$ where $U = [\mathbf{u}_1, \ldots, \mathbf{u}_q]$.*

Proof. To prove (1) ⇔ (2), we first let $\mathcal{A} \in \mathbb{SF}_{m,n}$ be a $\{0,1\}$–*cp* tensor. Then by Lemma 3.3 $\mathcal{A}$ can be written in form (3.4) where each $\mathcal{A}_i$ is an irreducible $\{0,1\}$–*cp* tensor of mth order n_i-dimension (no zero block there since $\mathcal{A}$ has no zero block). By Lemma 3.5, $\mathcal{A}_i$ is associated with a multi-hypergraph $\mathcal{P}_i = (V_i, \mathbb{E}_i)$ where $|V_i| = n_i$ for $i = 1, 2, \ldots, q$, $n_1 + n_2 + \ldots + n_q = n$. For each $i \in [q]$, by Lemma 3.5, $\mathcal{P}_i$ is the complete block of dimension n_i (since $\mathcal{A}_i$ is irreducible and $\{0,1\}$–*cp*). Thus (1) ⇒ (2) is proved. The proof of (2) ⇒ (1) is immediate if we note that the decomposition (3.1) holds by take $\mathtt{supp}(\mathbf{u}_i) = V_i$ for $i = 1, 2, \ldots, q$.

Now we show (1) ⇔ (3). First we assume that $\mathcal{A} \in \mathbb{SF}_{m,n}$ is $\{0,1\}$–*cp*. Then from the proof of Lemma 3.5 there exist some vectors $\mathbf{u}_j \in \mathcal{F}^n$ such that (3.1) holds, and

$$\mathtt{supp}(\mathbf{u}_i) \cap \mathtt{supp}(\mathbf{u}_j) = \emptyset, \forall 1 \leq i < j \leq q \tag{3.8}$$

It follows that $U^T U = diag(n_1, \ldots, n_q)$ for $U = [\mathbf{u}_1, \ldots, \mathbf{u}_q]$, where n_i is the positive integer described above. Thus (1) ⇒ (3) is proved. The other direction can be proved by reversing the above arguments. □

4. cp multi-hypergraphs

We define a multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$ to be a *cp* multi-hypergraph if $\mathcal{P}$ is associated with a $\{0,1\}$–*cp* tensor $\mathcal{A}$. Note that $\mathcal{A}$ is not necessarily a (0,1) tensor. For example, the following $3 \times 3 \times 3$ symmetric tensor is a $\{0,1\}$–*cp*, but not a $(0,1)$ tensor.

Example 4.1. Let $\mathcal{A} = (a_{ijk}) \in \mathcal{S}_{3,3}$ be a symmetric tensor defined as:

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathcal{A}(:,:,2) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A}(:,:,3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

We show that $\mathcal{A}$ is a $\{0,1\}$–*cp* tensor. In fact, if we let

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then we can verify by simple computation that

$$\mathcal{A} = \boldsymbol{u}_1^3 + \boldsymbol{u}_2^3 + \boldsymbol{u}_3^3$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathcal{F}^3$ are respectively the first, second and the third column of the (0,1) matrix U. Thus $\mathcal{A}$ is $\{0,1\}$–*cp* by definition. But $\mathcal{A}$ is not a (0,1) tensor since $a_{111} = a_{222} = a_{333} = 2$.

Denote $\mathcal{P}$ as the associated multi-hypergraph of $\mathcal{A}$. There are three distinct classes of edges of $\mathcal{P}$ according to majorization: $\mathcal{D}_{\alpha_1}, \mathcal{D}_{\alpha_2}, \mathcal{D}_{\alpha_3}$ where

$$\alpha_1 = \{v_1, v_1, v_2\}, \quad \alpha_2 = \{v_1, v_1, v_3\}, \quad \alpha_1 = \{v_2, v_2, v_3\}$$

each pair $(\mathcal{D}_i, \mathcal{D}_j)$ has a nonempty intersection. Note that $\mathcal{A}$ is irreducible. Thus the condition $\mathcal{A} \in \mathbb{SF}_{m,n}$ in Lemma 3.5 cannot be removed.

The following property is introduced for cp multi-hypergraphs.

Definition 4.2. A multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$ is said to possess Property $\mathbf{R}$ if $\mathcal{D}_\alpha \subseteq \mathbb{E}$ for any $\alpha \in \mathbb{E}$.

The aforementioned property is closely related to the zero-entry dominance property for tensors described formally by Luo and Qi in [21].

Definition 4.3 *(Definition 4.1, [21]).* An mth order n-dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is said to possess the *zero-entry dominance property* if for any $(i_1, \cdots, i_m) \in S(m, n)$, $a_{i_1 \cdots i_m} = 0$ implies that $a_{j_1 \cdots j_m} = 0$ for all $\{j_1, \cdots, j_m\}$ satisfying $B(\{j_1, \cdots, j_m\}) \supseteq B(\{i_1, \cdots, i_m\})$.

By direct verification, we can get the following equivalence.

Lemma 4.4. *A multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$ has the Property $\boldsymbol{R}$ if and only if its $(0,1)$ associated tensor has the zero-entry dominance property.*

It is known from [21] that any completely positive tensor possesses the zero-entry dominance property. It is worth pointing out that the zero-entry dominance property is only a necessary condition for $(0, 1)$ associated tensor to be $\{0, 1\}$ cp tensors, but far away from sufficient, even for the matrix case. For example, $A = [1\ 1\ 0; 1\ 1\ 1; 0\ 1\ 1]$ is a $(0, 1)$ matrix and satisfies the zero-entry dominance property, but it is not $\{0, 1\}$ cp since it is not positive semidefinite. Nevertheless, by invoking the above equivalence between Property $\mathbf{R}$ and the zero-entry dominance property, together with the definition of cp multi-hypergraphs, we can obtain that the Property $\mathbf{R}$ is exactly a necessary and sufficient condition for a multi-hypergraph to be cp.

Theorem 4.5. *An m-uniform multi-hypergraph is a cp multi-hypergraph if and only if it possesses Property $\boldsymbol{R}$.*

Proof. To get the necessity, by invoking Theorem 3.6, a cp multi-hypergraph can be decomposed as the union of some disjoint complete blocks $\mathcal{P}_i$'s where $\mathcal{P}_i = (V_i, \mathbb{E}_i)$ is of order n_i. Since a complete block has Property $\mathbf{R}$, $\mathcal{P}$ has Property $\mathbf{R}$. For the sufficiency, that is, given an m-uniform multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$ with Property $\mathbf{R}$, then $\mathcal{P}$ is cp, i.e., there exists a $\{0, 1\}$–*cp* tensor $\mathcal{A}$ associated with $\mathcal{P}$. Note that $\mathcal{A}$ need not be a (0,1) tensor by definition. For this purpose, we consider the set of the maximal edges of $\mathcal{P}$, and we classify them into $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ by the similarity of the edges. We denote

$$\alpha_i \in \mathcal{C}_i, \quad B_i = B(\alpha_i), \quad n_i = |B_i|, \forall i \in [r]$$

For $i \in [r]$, denote $\mathcal{D}_i = \mathcal{D}_{\alpha_i}$, which is the complete m-multisets determined by $\alpha_i \in \mathcal{C}_i$. Apparently,

$$\mathbb{E} = \bigcup_{i=1}^{r} \mathcal{D}_i \tag{4.1}$$

Denote $\mathbf{v}_i \in \mathcal{F}^n, \mathtt{supp}(\mathbf{v}_i) = B_i$ for each $i \in [r]$, and define

$$\mathcal{A} = \sum_{j=1}^{r} \mathbf{v}_j^m \tag{4.2}$$

Then $\mathcal{A}$ is $\{0,1\}$–*cp* by definition. The proof is completed if we show that $\mathcal{A}$ is associated with $\mathcal{P}$. In fact, if there is an element $a_{i_1 \ldots i_m} \neq 0$, then from (4.2) there exists $k \in [r]$ such that

$$B(\sigma) \subseteq \mathtt{supp}(\mathbf{v}_k) = B_k = B(\alpha_k)$$

It follows that $M(\sigma) \in \mathbb{E}$ since $\mathcal{P}$ has Property $\mathbf{R}$. Conversely, if $A_\sigma = 0$ for a $\sigma \in S(m,n)$, then $B(\sigma) \nsubseteq B(\alpha_i)$ for each $i \in [r]$. Thus $\sigma \notin \mathcal{D}_i$ for all $i \in [r]$. Consequently we have $M(\sigma) \notin \mathbb{E}$ by (4.1). The proof is completed. $\square$

By combining Theorem 4.5 and Lemma 2.4, we obtain

Corollary 4.6. *An m-uniform $n \times N$ cp multi-hypergraph $\mathcal{P} = (V, \mathbb{E})$ satisfies*

$$N = n_1^m + \ldots + n_r^m$$

where r is the number of connected branches of $\mathcal{P}$ and n_i is the dimension of the ith branch.

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