RISK MODELING IN MANAGEMENT AND ENGINEERING

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Decisions in the Face of Uncertain Outcomes

Reliable design in engineering:

a structure must be engineered to withstand various impacts
possible impacts can only be estimated
potential for failure must be kept low

Logistics in operations research:

supplies must be stockpiled to meet demands
distribution and replacement expense should be kept at bay
demands and costs are random variables, but partly guesswork

Asset management in finance:

a portfolio must be chosen with intelligent safeguarding
performance known at best from history and economic factors
the threat of losses cannot be eliminated

The inescapable modeling challenge in stochastic optimization:

appropriate formulation of **constraints** and **objectives**

Uncertain “Costs” (Viewed Abstractly)

“costs” = quantities to be minimized or kept below given levels

General “cost” expression in decision-making:

$c(x, v)$ with $x =$ **decision** vector, $v =$ **data** vector

$$x = (x_1, \ldots, x_n), \qquad v = (v_1, \ldots, v_m)$$

Stochastic uncertainty:

v is replaced by a **random variable** vector $V = (V_1, \ldots, V_m)$

then the “cost” becomes a random variable: $\underline{c}(x) = c(x, V)$

Key consequence:

the distribution of $\underline{c}(x)$ can only be **shaped** by the choice of x

but how then can constraints or minimization be understood?

A Broad Pattern for Handling Risk in Optimization

Risk measures: functionals $\mathcal{R}$ that "quantify the risk" in a random variable X by a value $\mathcal{R}(X)$ ("risk" $\neq$ "uncertainty")

Systematic prescription

Faced with an uncertain "cost" $\underline{c}(x) = c(x, V)$ articulate it numerically as $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$ for a choice of risk measure $\mathcal{R}$

Constraints: keeping $\underline{c}(x)$ "adequately" $\leq b$
modeled as: constraint $\bar{c}(x) = \mathcal{R}(\underline{c}(x)) \leq b$

Objectives: keeping $\underline{c}(x)$ "as low as reasonable"
modeled as: minimizing $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$
$=$ choosing (x, b) such that $\bar{c}(x) \leq b$ for lowest b

Supporting theory: exploration of examples and guidelines
but note: choosing $\mathcal{R}$ expresses a decision-maker's preferences

Some Familiar Approaches Subject to Pros and Cons

[**random** cost $\underline{c}(x) = c(x, V)$ reduced to a **numerical** cost $\bar{c}(x)$]

Focusing on worst cases:

$\bar{c}(x) = \sup[\underline{c}(x)]$ [taking $\mathcal{R}(X) = \sup X$ (ess. sup)]

then $\bar{c}(x) \le b \iff \underline{c}(x) \le b$ almost surely

Passing to expectations:

$\bar{c}(x) = \mu[\underline{c}(x)] = E[\underline{c}(x)]$ [taking $\mathcal{R}(X) = \mu(X) = EX$]

then $\bar{c}(x) \le b \iff \underline{c}(x) \le b$ "on average"

Adopting a safety margin:

$\bar{c}(x) = \mu[\underline{c}(x)] + \lambda\sigma[\underline{c}(x)]$ [taking $\mathcal{R}(X) = \mu X + \lambda\sigma(X)$]

then $\bar{c}(x) \le b$ unless in tail beyond λ standard deviations

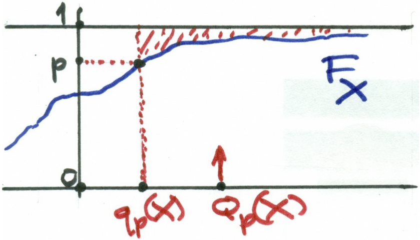
Looking at quantiles:

$\bar{c}(x) = q_p[\underline{c}(x)]$ [taking $\mathcal{R}(X) = p$-quantile of X]

then $\bar{c}(x) \le b \iff \mathrm{prob}\{\underline{c}(x) \le b\} \ge p$

Quantiles and "Superquantiles": VaR and CVaR

F_X = cumulative distribution function for random variable X



Quantile: "value-at-risk" in finance

$$q_p(X) = \mathrm{VaR}_p(X) = F_X^{-1}(p)$$

Superquantile: "conditional value-at-risk" in finance

$$Q_p(X) = \mathrm{CVaR}_p(X) = E[\,X\,|X \geq q_p(X)\,] = \frac{1}{1-p}\int_p^1 q_t(X)dt$$

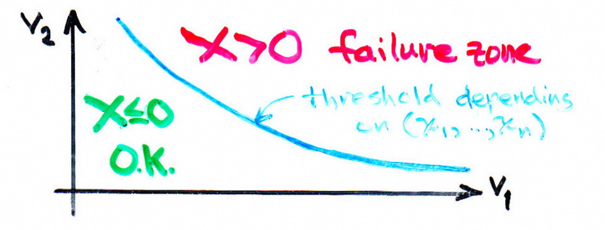
Replacing quantiles by superquantiles:

$\bar{c}(x) = Q_p[\underline{c}\,(x)]$ [taking $\mathcal{R}(X) = p$-superquantile of X]

then $\bar{c}(x) \leq b \iff \underline{c}\,(x) \leq b$ on average even in p-quantile-tail

An Engineering Perspective in Reliable Design

a "cost" triggering "failure": $X = c(x_1, \dots, x_n; V_1, \cdots, V_r)$



Probability of failure: $p_f(X) = \mathrm{prob}\{X > 0\}$

$$p_f(X) \leq 1 - p \quad \Longleftrightarrow \quad q_p(X) \leq 0$$

Troubles with this quantile concept:

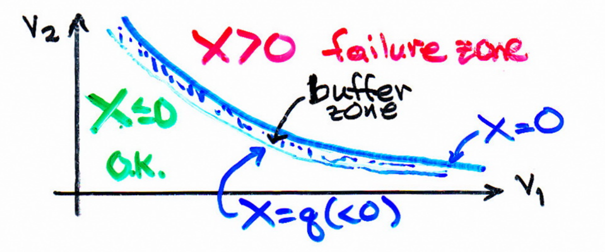
- How to compute or at least estimate?
- How to cope with dependence on $x_1, \dots, x_n$ in optimization?

both p_f and the threshold **shift** with changes in $x_1, \dots, x_n$

Alternative to consider: switch to a **superquantile** concept

Buffered Failure — Enhanced Safety

uncertain cost: $X = c(x_1, \ldots, x_n; V_1, \cdots, V_r)$



Buffered probability of failure: $P_f(X) = \mathrm{prob}\{X > q\}$

with q determined so as to make $E[X \mid X > q] = 0$!

$$P_f(X) \leq 1 - p \iff Q_p(X) \leq 0$$

Suggestion: adjust failure modeling to $P_f(X)$ in place of $p_f(X)$

safer by integrating tail information, and

easier also to work with in optimization! why?

Minimization Formula for VaR and CVaR

$$\mathrm{CVaR}_p(X) = \min_{C \in R} \left\{ C + \frac{1}{1-p} E\Big[\max\{0, X - C\} \Big] \right\} \quad p \in (0,1)$$
$$\mathrm{VaR}_p(X) = \mathrm{argmin} \;\; \text{(if unique, otherwise the lowest)}$$

Application to CVaR models: convert a problem in x like

minimize $\mathrm{CVaR}_{p_0}(\underline{c}_0(x))$ subject to [hard constraints and]
$$\mathrm{CVaR}_{p_i}(\underline{c}_i(x)) \leq 0, \;\; i = 1, \ldots, m$$

into a problem in x and auxiliary variables $C_0, C_1, \ldots, C_m$:

$$\text{minimize } C_0 + \frac{1}{1-p_0} E\Big[\max\{0, \underline{c}_0(x) - C_0\} \Big] \text{ while requiring}$$
$$C_i + \frac{1}{1-p_i} E\Big[\max\{0, \underline{c}_i(x) - C_i\} \Big] \leq 0, \;\; i = 1, \ldots, m$$

Important case: this converts to **linear programming** when
(1) each $\underline{c}_i(x) = c_i(x, V)$ depends **linearly** on x,
(2) the future state space Ω is modeled as **finite**

Axiomatization of Risk

Key axioms for risk measures: $\mathcal{R} : \mathcal{L}^p(\Omega, \mathcal{A}, P) \to (-\infty, \infty]$

(R1) $\mathcal{R}(C) = C$ for all constants C

(R2) convexity, (R3) lower semicontinuity

Supplementary properties of interest:

(R4) positive homogeneity: $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for $\lambda > 0$

(R5) monotonicity: $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $X \leq X'$

(R6) aversity: $\mathcal{R}(X) > EX$ for nonconstant X

Coherent measure of risk: $\mathcal{R}$ satisfying (R1), (R2), (R3), (R5)

Averse measure of risk: $\mathcal{R}$ satisfying (R1), (R2), (R3), (R6)

Artzner et al. (2000) introduced coherency without aversity

Preservation of convexity under coherency of $\mathcal{R}$

$\underline{c}(x) = c(x, V)$ convex in x $\implies$ $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$ convex in x

a further advantage of coherency: it promotes **dualizations**

Coherency or its Absence

- $\mathcal{R}(X) = q_p(X) = \mathrm{VaR}_p(X)$ fails (R2), (R3), and (R6)!
- $\mathcal{R}(X) = Q_p(X) = \mathrm{CVaR}_p(X)$ satisfies **all** axioms
- $\mathcal{R}(X) = \sup X$ satisfies **all** axioms
- $\mathcal{R}(X) = \mu(X) = EX$ fails (R6) (coherency without aversity)
- $\mathcal{R}(X) = \mu(X) + \lambda\sigma(X)$, $\lambda > 0$, fails (R5) (no monotonicity)

Mixtures of risk measures

$$\mathcal{R}(X) = \sum_{k=1}^{r} \lambda_k \mathcal{R}_k(X) \;\; \text{with} \;\; \lambda_k > 0, \;\; \sum_{k=1}^{r} \lambda_k = 1$$

- $\mathcal{R}_k$ coherent for $k = 1, \ldots, r \quad \Longrightarrow \quad \mathcal{R}$ coherent
- $\mathcal{R}_k$ averse for $k = 1, \ldots, r \quad \Longrightarrow \quad \mathcal{R}$ averse

a versatile example: enjoying both coherency and aversity

$$\mathcal{R}(X) = \sum_{k=1}^{r} \lambda_k Q_{p_k}(X) = \sum_{k=1}^{r} \lambda_k \, \mathrm{CVaR}_{p_k}(X)$$

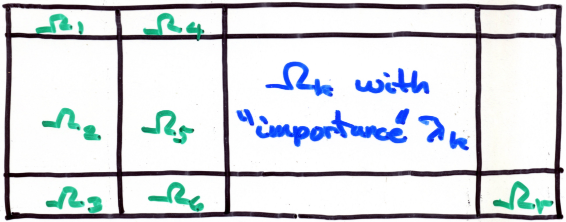
"continuous" mixtures work also $\longleftrightarrow$ dual utility risk profiles

Relaxations From Worst-Case Analysis

Interpolations with CVaR:

$\mathrm{CVaR}_p(X) \nearrow \sup X$ as $p \nearrow 1$, $\quad \mathrm{CVaR}_p(X) \searrow EX$ as $p \searrow 0$

Partitioning the future: Ω as a union of "likelihood regions" Ω_k



$\lambda_k > 0$ for $k = 1, \ldots, r, \quad \lambda_1 + \cdots + \lambda_r = 1$

$$\mathcal{R}(X) = \lambda_1 \Big[\sup_{\omega \in \Omega_1} X(\omega) \Big] \cdots + \lambda_r \Big[\sup_{\omega \in \Omega_r} X(\omega) \Big]$$ is **coherent**

Axiomatization of Deviation, a Partner to Risk

$\mathcal{D}(X)$ quantifies the **nonconstancy/uncertainty** in X

Key axioms for deviation measures: $\mathcal{D} : \mathcal{L}^1(\Omega, \mathcal{A}, P) \to [0, \infty]$

(D1) $\mathcal{D}(C) = 0$, but $\mathcal{D}(X) > 0$ for nonconstant X

(D2) convexity, (D3) lower semicontinuity

Supplementary properties of interest:

(D4) positive homogeneity: $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for $\lambda > 0$

(D5) upper range boundedness: $\mathcal{D}(X) \leq \sup X - EX$

Risk measures in relation to deviation measures

A **one-to-one** correspondence $\mathcal{D} \longleftrightarrow \mathcal{R}$ between deviation measures $\mathcal{D}$ and **averse** risk measures $\mathcal{R}$ is furnished by

$$\mathcal{R}(X) = EX + \mathcal{D}(X), \qquad \mathcal{D}(X) = \mathcal{R}(X - EX),$$

where moreover **coherency** is characterized by

$$\mathcal{R}(X) \text{ satisfies (R5)} \iff \mathcal{D}(X) \text{ satisfies (D5)}$$

$\mathcal{D}(X) = \lambda\sigma(X)$ fails (D5), so $\mathcal{R}(X) = EX + \lambda\sigma(X)$ isn't coherent!

Optimization and Estimation in Coping With Uncertainty

support for decision-making in a stochastic environment

Optimization:

minimize a “cost” expression under constraints on the decision
the constraints could involve bounds on other “costs”
the “costs” may have a background in statistical analysis

Estimation:

approximate some quantity from empirical/historical data
minimize an error expression to get regression coefficients
different interpretations of “error” yield different results

Interplay:

- optimization problems involving uncertainty depend on estimation methodology even in coming to a formulation
- estimation problems are optimization of a special sort
- new and deeper connections are now coming to light

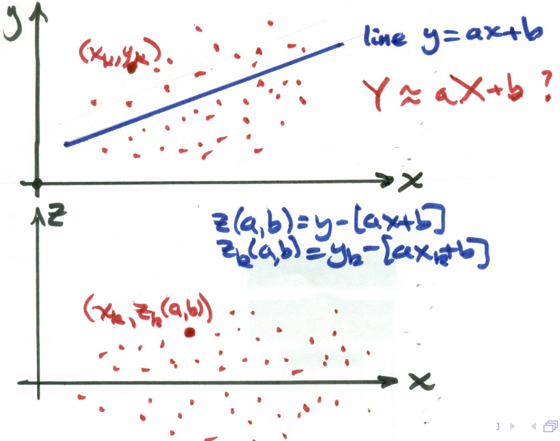
Databases: the Role of Estimation

a one-dimensional linear version for initial simplicity

Available information: a large(?) collection of pairs (x_k, y_k)

Perspective: empirical distribution in x, y space of r.v.'s X, Y

Approximation: $Y \approx aX + b$ error gap: $Z(a,b) = Y - [aX + b]$



Regression From a General Point of View

Y = random variable (scalar) to be understood in terms of
$X_1, \ldots, X_n$ = some "more basic" variables (e.g., "factors")

Approximation scheme: $Y \approx f(X_1, \ldots, X_n)$ for $f \in \mathcal{F}$

$\mathcal{F}$ = some specified class of functions $f : \boldsymbol{R}^n \to \boldsymbol{R}$

for instance $f(x_1, \ldots, x_n) = c_0 + c_1 x_1 + \cdots + c_n x_n$

corresponding error variable: $Z_f = Y - f(X_1, \ldots, X_n)$

Regression problem:

minimize $\mathcal{E}(Z_f)$ over all $f \in \mathcal{F}$ for some "error measure" $\mathcal{E}$

$\mathcal{E}$ quantifies the "**nonzeroness**" of the random variable Z_f

Important issue for connecting with optimization:

- $\underline{c}(x) = c(x, V)$ may only be supported by a database
- regression is essential then to get a "formula" for $\underline{c}(x)$
- for using $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$, shouldn't this be "tuned" to $\mathcal{R}$?

Example: an Application to Composition of Alloys

Alloy model: a mixture of various metals

amounts of chief ingredients: $x = (x_1, \ldots, x_n)$ "design"

amounts of other ingredients: $v = (v_1, \ldots, v_m)$ "contaminants"

a "characteristic" to be controlled: y ideally kept ≤ 0, say

due to uncertainty, a quantile constraint may be envisioned

Background information: $y = c(x, v)$? no available formula!

there is only a database in (x, v, y)-space, $\{(x^k, v^k, y^k)\}_{k=1}^N$

- view the database as an empirical distribution for random variables $X = (X_1, \ldots, X_n)$, $V = (V_1, \ldots, V_m)$, Y
- use regression of Y on $X_1, \ldots, X_N$ to get a function $y = \bar{c}(x)$
- then impose the constraint $\bar{c}(x) \leq 0$ on the design x

shouldn't the regression adapt then to the intended constraint?

The Effect of Regression in General

$\mathcal{E}$ = some measure of **error**: (E1) $\mathcal{E}(X) > 0$ when $X \not\equiv 0$
(E2) convexity, (E3) lower semicontinuity

for instance $\mathcal{E}(X) = E\{\varepsilon(X)\}$ or $\mathcal{E}(X) = \min_{w \in W} E\{\varepsilon(w, X)\}$

Error projection (with respect to constants C)

Let $\mathcal{D}(X) = \min_C \mathcal{E}(X - C)$, $\mathcal{S}(X) = \operatorname{argmin}_C \mathcal{E}(X - C)$.

Then $\mathcal{D}$ is a measure of deviation and $\mathcal{S}$ the associated "statistic"

$\mathcal{S}(X)$ is the constant C "nearest" to X with respect to $\mathcal{E}$

Regression problem "decomposition" (when $f \in \mathcal{F} \Rightarrow f + C \in \mathcal{F}$)

minimizing $\mathcal{E}(Z_f)$ over $f \in \mathcal{F}$ corresponds to minimizing $\mathcal{D}(Z_f)$
under the constraint $\mathcal{S}(Z_f) = 0$ where $Z_f = Y - f(X_1, \ldots, X_n)$

basic examples: least-squares regression, quantile regression

The Fundamental Quadrangle of Risk: A New Paradigm

an array of functionals to be applied to random "costs" X
serving to connect methodologies usually viewed separately

$$\begin{array}{ccccc} & \text{risk } \mathcal{R} & \longleftrightarrow & \mathcal{D} \text{ deviation} & \\ \text{optimization} & \uparrow\downarrow & \mathcal{S} & \downarrow\uparrow & \text{estimation} \\ & \text{regret } \mathcal{V} & \longleftrightarrow & \mathcal{E} \text{ error} & \end{array}$$

$\mathcal{R}(X)$ quantifies the "overall" cost in X
$\mathcal{D}(X)$ quantifies the nonconstancy in X
$\mathcal{E}(X)$ quantifies the nonzeroness in X
$\mathcal{V}(X)$ quantifies the net regret in outcomes $X > 0$ versus $X \leq 0$
$\mathcal{S}(X)$ is the "statistic" associated with X through $\mathcal{E}$

dualizations yield many insights and connect with relative entropy

Regret and Utility

$\mathcal{V}(X)$ quantifies (net) regret in outcomes $X > 0$ versus $X \leq 0$

Key axioms for regret measures: $\mathcal{V} : \mathcal{X} \to (-\infty, \infty]$

(V1) $\mathcal{V}(0) = 0$, (V2) convexity, (V3) closedness

Supplementary properties of interest:

(V4) positive homogeneity: $\mathcal{V}(\lambda X) = \lambda \mathcal{V}(X)$ for $\lambda > 0$

(V5) monotonicity: $\mathcal{V}(X) \leq \mathcal{V}(X')$ when $X \leq X'$

(V6) aversity: $\mathcal{V}(X) > EX$ for nonconstant X

Coherent measure of regret: $\mathcal{V}$ with (V1), (V2), (V3), (V5)

Averse measure of regret: $\mathcal{V}$ with (V1), (V2), (V3), (V6)

Connection with utility: $\mathcal{V} \longleftrightarrow \mathcal{U}$ expressed by

$$\mathcal{V}(X) = -\mathcal{U}(-X), \quad \mathcal{U}(Y) - \mathcal{V}(-Y)$$

- this is **relative** utility because of the $\mathcal{V}$ threshold: $\mathcal{U}(0) = 0$

Expectation case: $\mathcal{V}(X) = E[v(X)]$, $\mathcal{U}(Y) = E[u(Y)]$, where

$v(x) = -u(-x)$ for a utility $u : (-\infty, \infty) \to [-\infty, \infty)$

Risk Assessed From Regret

Trade-off formula

From a regret measure $\mathcal{V}$, a risk measure $\mathcal{R}$ can be derived by

$$\mathcal{R}(X) = \min_C \{ C + \mathcal{V}(X - C) \}$$

and in general this operation preserves aversity and coherency

Interpretation: write off a **certain** loss amount C and then only worry about the **uncertain residual** loss $X - C$

Ideal amount to write off: $\mathcal{S}(X) = \operatorname{argmin}_C \{ C + \mathcal{V}(X - C) \}$

Example: $\mathcal{R}(X) = Q_p(X)$ and $\mathcal{S}(X) = q_p(X)$ emerge from $\mathcal{V}(X) = \frac{1}{1-p} E[\max\{0, X\}]$ (classical penalty for loss)

Important challenge: inversion $\mathcal{R} \to \mathcal{V}$ more generally nonunique, but "natural" antecedents are now broadly known

Finalizing the Quadrangle

$$\begin{array}{ccccc} & \text{risk } \mathcal{R} & \longleftrightarrow & \mathcal{D} \text{ deviation} & \\ \text{optimization} & \uparrow\downarrow & \mathcal{S} & \downarrow\uparrow & \text{estimation} \\ & \text{regret } \mathcal{V} & \longleftrightarrow & \mathcal{E} \text{ error} & \end{array}$$

To consider now: $\mathcal{V} \longleftrightarrow \mathcal{E}$ as set forth through

$$\mathcal{E}(X) = \mathcal{V}(X) - EX, \qquad \mathcal{V}(X) = EX + \mathcal{E}(X)$$

expectation case: $\mathcal{E}(X) = E[\varepsilon(X)]$ for $\varepsilon : (-\infty, \infty] \to [0, \infty]$

Key consequence for $\mathcal{S}$ under $\mathcal{V} \longleftrightarrow \mathcal{E}$ and $\mathcal{R} \longleftrightarrow \mathcal{D}$

The derivations $\mathcal{E} \to \mathcal{D}$, $\mathcal{E} \to \mathcal{S}$, transform now exactly to

$$\mathcal{R}(X) = \min_C \{ C + \mathcal{V}(X - C) \}, \quad \mathcal{S}(X) = \operatorname*{argmin}_C \{ C + \mathcal{V}(X - C) \}$$

Final links: nonunique but "natural" inversions $\mathcal{D} \to \mathcal{E}$, $\mathcal{R} \to \mathcal{V}$

The Mean-Based Quadrangle

articulated with a scaling parameter $\lambda > 0$

$$\mathcal{S}(X) = EX = \mu(X)$$
$$= \text{mean}$$

$$\mathcal{E}(X) = \lambda ||X||_2$$
$$= L^2\text{-error, scaled}$$

$$\mathcal{D}(X) = \lambda\,\sigma(X)$$
$$= \text{standard deviation, scaled}$$

$$\mathcal{R}(X) = EX + \lambda\,\sigma(X)$$
$$= \text{standard-deviational tail risk}$$

$$\mathcal{V}(X) = EX + \lambda ||X||_2$$
$$= L^2\text{-regret}$$

properties: **aversity** with **convexity** but not coherency

The Quantile-Based Quadrangle

at any probability level $p \in (0,1)$

$$\mathcal{S}(X) = q_p(X) = \mathrm{VaR}_p(X)$$
$$= \text{quantile}$$

$$\mathcal{R}(X) = Q_p(X) = \mathrm{CVaR}_p(X)$$
$$= \text{superquantile}$$

$$\mathcal{D}(X) = Q_p(X - EX) = \mathrm{CVaR}_p(X - EX)$$
$$= \text{superquantile deviation}$$

$$\mathcal{E}(X) = E[\tfrac{p}{1-p}X_+ + X_-]$$
$$X_+ = \max\{0, X\},\ \ X_- = \max\{0, -X\}$$
$$= \text{"normalized" Koenker-Bassett error}$$

$$\mathcal{V}(X) = \tfrac{1}{1-p}E[X_+]$$
$$= \text{quantile-scaled absolute loss}$$

properties: **aversity** with **coherency**

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