Two-stage Stochastic Variational Inequalities

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5. X. Chen, H. Sun and H. Xu, Discrete approximation of two-stage stochastic and distributionally robust linear complementarity problems, (2017), revised version for Math. Program.
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1.1 Variational Inequalities (VI)

Given a nonempty closed-convex set $X \subseteq I\!R^n$ and a continuous function $F : I\!R^n \to I\!R^n$, the variational inequality problem is to find $x \in X$ such that

$$-F(x) \in \mathcal{N}_X(x)$$

$$\text{i.e.} \qquad (y-x)^T F(x) \geq 0, \quad \forall y \in X.$$

$\mathcal{N}_X(x)$ is the normal cone to the set X at x.
Complementarity problem as a special case: $X = I\!R^n_+$

$$-F(x) \in \mathcal{N}_{\boldsymbol{R}^n_+}(x), \qquad 0 \leq x \perp F(x) \geq 0$$

System of equations as a special case: $X = I\!R^n$

$$F(x) = 0$$

Stochastic variational inequalities

Single stage stochastic variational inequalities

A random variable ξ affects the function F and the set X.
$\xi \in \Xi \subseteq I\!R^L$, a set representing future states of knowledge.
Given $F : \Xi \times I\!R^n \to I\!R^n$ and $X_\xi \subset I\!R^n$, find $x \in X_\xi$ such that

$$-F(\xi, x) \in \mathcal{N}_{X_\xi}(x), \quad \text{i.e.,} \quad (y-x)^T F(\xi, x) \geq 0, \quad \forall y \in X_\xi.$$

This problem is well defined if ξ is known. “Wait-and-see”

Example

Consider $f : \Xi \times I\!R^n \to I\!R$ and $g : \Xi \times I\!R^n \to I\!R^m$

$$\text{minimize} \quad f(\xi, x) \qquad \text{subject to} \quad x \in X_\xi$$

where

$$X_\xi = \{x \in I\!R^n_+ \mid g(\xi, x) \geq 0\},$$

$-\nabla f(\xi, x) \in \mathcal{N}_{X_\xi}(x)$ —— First order optimality condition

Wait-and-see and Here-and-now

Wait-and-see solution Given $F : \Xi \times \mathbb{R}^n \to \mathbb{R}^n$ and $X_\xi \subset \mathbb{R}^n$, find $x_\xi \in X_\xi$ such that

$$-F(\xi, x_\xi) \in \mathcal{N}_{X_\xi}(x_\xi), \quad \text{i.e.,} \quad (y - x_\xi)^T F(\xi, x_\xi) \geq 0, \quad \forall y \in X_\xi.$$

Here-and-now solution One wants to make a decision x before knowing ξ. Let $X \equiv \mathbb{E}[X_\xi] = \{\mathbb{E}[x_\xi] \,|\, x_\xi \in X_\xi, \mathbb{E}[x_\xi] < \infty\}$.

- Expected Residual minimization (ERM) solution

$$\min_{x \in X} \mathbb{E}[\|r(\xi, x)\|^2],$$

  where $r(\xi, \cdot)$ is a residual function.

- Expected value (EV) solution

$$-\mathbb{E}[F(\xi, x)] \in \mathcal{N}_X(x)$$

Stochastic complementarity problems

A random variable ξ affects $F : \Xi \times I\!R^n \to I\!R^n$,

$$x \geq 0, \quad F(\xi, x) \geq 0, \quad x^T F(\xi, x) = 0, \quad \text{for} \quad \xi \in \Xi.$$

i.e. $\Phi(x, F(\xi, x)) = 0$, where $\Phi_i(x, F(\xi, x)) = \min(x_i, F_i(\xi, x)),\ 1 \leq i \leq n$

- Expected residual minimization (ERM) formulation
  Chen-Fukushima(2005)

$$\min_{x \in \boldsymbol{R}^n_+} I\!E[r(\xi, x)], \qquad r(\xi, x) = \|\Phi(x, F(\xi, x))\|^2$$

- Expected value (EV) formulation
  Gürkan-Özge-Robinson(1999), Ruszczynski-Shapiro(2003), Jiang-Xu(2008)

$$x \geq 0, \quad I\!E[F(\xi, x)] \geq 0, \quad x^T I\!E[F(\xi, x)] = 0$$

$$\Leftrightarrow \qquad \min_{x \in \boldsymbol{R}^n} r(x) := \|\Phi(x, I\!E[F(\xi, x)])\|^2$$

Two-stage stochastic variational inequalities

Given the (induced) probability space $(\Xi \subset \mathbb{R}^L, \mathcal{A}, P)$, find a pair $\big(x \in \mathbb{R}^{n_1}, u : \Xi \to \mathbb{R}^{n_2}\ \mathcal{A}\text{-measurable}\big)$, such that the following collection of variational inequalities is satisfied:

$$
\begin{aligned}
-\mathbb{E}[G\big(\boldsymbol{\xi}, x, u_{\boldsymbol{\xi}}\big)] &\in \mathcal{N}_D(x) \\
-F\big(\boldsymbol{\xi}, x, u_{\boldsymbol{\xi}}\big) &\in \mathcal{N}_{C_{\boldsymbol{\xi}}}(u_{\boldsymbol{\xi}}) \qquad \text{for a.e. } \xi \in \Xi.
\end{aligned}
$$

- $G : (\Xi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}) \to \mathbb{R}^{n_1}$ a vector-valued function, continuous with respect to (x, u) for all $\xi \in \Xi$, $\mathcal{A}$-measurable and integrable with respect to ξ.
- $\mathcal{N}_D(x)$ the normal cone to the nonempty closed-convex set $D \subset \mathbb{R}^{n_1}$ at $x \in \mathbb{R}^{n_1}$.
- $F : (\Xi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}) \to \mathbb{R}^{n_2}$ a vector-valued function, continuous with respect to (x, u) for all $\xi \in \Xi$ and $\mathcal{A}$-measurable with respect to ξ.
- $\mathcal{N}_{C_\xi}(v)$ the normal cone to the nonempty closed-convex set $C_\xi \subset \mathbb{R}^{n_2}$ at $v \in \mathbb{R}^{n_2}$, the random set $C_{\boldsymbol{\xi}}$ is $\mathcal{A}$-measurable.

Two-stage stochastic variational inequalities

The definition of the normal cone yields the following equivalent formulation:

find $\bar{x} \in D$ and $\bar{u} : \Xi \to I\!R^{n_2}$, $\mathcal{A}$-measurable, such that $\bar{u}_{\boldsymbol{\xi}} \in_{\mathrm{as}} C_{\boldsymbol{\xi}}$ and

$$\langle I\!E[G(\boldsymbol{\xi}, \bar{x}, \bar{u}_{\boldsymbol{\xi}})], x - \bar{x} \rangle \geq 0, \ \forall\, x \in D,$$

$$\langle F(\boldsymbol{\xi}, \bar{x}, \bar{u}_{\boldsymbol{\xi}}), v - \bar{u}_{\boldsymbol{\xi}} \rangle \geq 0, \ \ \forall v \in C_{\boldsymbol{\xi}}, \quad \text{for a.e. } \xi \in \Xi$$

Two-stage stochastic linear variational inequalities (SLVI)

$$\begin{aligned} 0 &\in Ax + I\!E[B(\xi)u(\xi)] + q_1 + \mathcal{N}_D(x), \\ 0 &\in N(\xi)x + M(\xi)u(\xi) + q_2(\xi) + \mathcal{N}_{C_\xi}(u(\xi)), \qquad \text{for a.e. } \xi \in \Xi. \end{aligned}$$

Two-stage stochastic linear complementarity problems (SLCP)

$D = I\!R^{n_1}_+, C_\xi = I\!R^{n_2}_+$

$$\begin{aligned} 0 \leq x \quad &\perp \quad Ax + I\!E[B(\xi)u(\xi)] + q_1 \geq 0, \\ 0 \leq u(\xi) \quad &\perp \quad N(\xi)x + M(\xi)u(\xi) + q_2(\xi) \geq 0, \qquad \text{for a.e. } \xi \in \Xi. \end{aligned}$$

Two-stage SLCP

X. Chen, H. Sun and H. Xu, Discrete approximation of two-stage stochastic and distributionally robust linear complementarity problems, (2017).

Two-stage stochastic linear complementarity problems (SLCP)

$$\begin{array}{rcl} 0 \le x & \perp & Ax + \mathbb{E}[B(\xi)u(\xi)] + q_1 \ge 0, \\ 0 \le u(\xi) & \perp & N(\xi)x + M(\xi)u(\xi) + q_2(\xi) \ge 0, \end{array} \qquad \text{for a.e. } \xi \in \Xi.$$

Assumption There exists a positive continuous function $\kappa(\xi)$ such that $\mathbb{E}[\kappa(\xi)] < +\infty$ and for almost every ξ,

$$(x^T, u^T)\begin{pmatrix} A & B(\xi) \\ N(\xi) & M(\xi) \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} \ge \kappa(\xi)(\|x\|^2 + \|u\|^2), \quad \forall x \in R^{n_1}, \; u \in R^{n_2}.$$

The two-stage SLCP has a unique solution $(x, u(\cdot)) \in R^{n_1} \times \mathcal{U}$.
$\mathcal{U}$ is the space of measurable functions defined on Ξ.
Convergence of the sample average approximation

Two-stage stochastic generalized equations

X. Chen, A. Shapiro and H. Sun, Convergence analysis of sample average approximation of two-stage stochastic generalized equations, (2017). without assuming relatively complete recourse

$$
\begin{aligned}
&0 \in \mathbb{E}[\Phi(x, u(\xi), \xi)] + \Gamma_1(x), \; x \in D, \\
&0 \in \Psi(x, u(\xi), \xi) + \Gamma_2(u(\xi), \xi), \quad \text{for a.e. } \xi \in \Xi.
\end{aligned}
$$

Here $D \subset \mathbb{R}^n$ is a nonempty closed convex set, $\Phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^L \to \mathbb{R}^{n_1}$, $\Psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^L \to \mathbb{R}^{n_2}$, $\Gamma_1 : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_1}$ and $\Gamma_2 : \mathbb{R}^{n_2} \rightrightarrows \mathbb{R}^{n_2}$ are multifunctions (point-to-set mappings).

If for almost all $\xi \in \Xi$, $\Theta(x, u(\xi), \xi) := \begin{pmatrix} \Phi(x, u(\xi), \xi) \\ \Psi(x, u(\xi), \xi) \end{pmatrix}$ is strongly monotone at $(x, u(\cdot))$,
then the two-stage SGE with $\Gamma_1(x) = \mathcal{N}_D(x)$ and $\Gamma_2(u(\xi), \xi) = \mathbb{R}^{n_2}_+$ has a unique solution $(x, u(\cdot)) \in \mathbb{R}^{n_1} \times \mathcal{U}$.

Algorithms for two-stage SVI

R.T. Rockafellar and J. Sun, Solving monotone stochastic variational inequalities and complementarity problems by progressive hedging, (2017). $\Xi = \{\xi^1, \ldots, \xi^\nu\}$.

Extend to non-monotone stochastic VI (joint work with D. Sun and J. Yang)

Example: two-stage stochastic linear VI

$$
\begin{aligned}
0 &\in Ax + \sum_{j=1}^{\nu} p_j B(\xi^j) u(\xi^j) + q_1 + \mathcal{N}_D(x), \\
0 &\in N(\xi^j)x + M(\xi^j)u(\xi^j) + q_2(\xi^j) + \mathcal{N}_{C_{\xi^j}}(u(\xi^j)), \quad \text{for } j = 1, \ldots, \nu,
\end{aligned}
$$

where $p_j > 0$ and $\sum_{j=1}^{\nu} p_j = 1$.

Let

$$C_j = C_{\xi^j}, \;\; B_j = B(\xi^j),\, N_j = N(\xi^j),\, M_j = M(\xi^j),\, q_{2j} = q(\xi^j)$$

$D \subset \mathbb{R}^{n_1}$ and $C_j \subset \mathbb{R}^{n_2}$ are boxes. Let $\Omega = D \times C_1 \times \cdots \times C_\nu$.

Example: two-stage linear SVI

$$0 \in Mz + q + \mathcal{N}_\Omega(z), \qquad (1)$$

$$M = \begin{pmatrix} A & p_1 B_1 & \dots & p_\nu B_\nu \\ N_1 & M_1 & & \\ \vdots & & \ddots & \\ N_\nu & & & M_\nu \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_{21} \\ \vdots \\ h_{2\nu} \end{pmatrix} \text{ and } z = \begin{pmatrix} x \\ u_1 \\ \vdots \\ u_\nu \end{pmatrix}.$$

When $\Omega = I\!R^{n_1+\nu n_2}$, (1) reduces to $Mz + q = 0$

When $\Omega = I\!R_+^{n_1+\nu n_2}$, (1) reduces to $0 \le z \perp Mz + q \ge 0$

PH Algorithm From x^k, u_j^k and w_j^k with $\sum_{j=1}^{\nu} p_j w_j^k = 0$, $j = 1, \dots, \nu$

Step 1 Determine $\hat{x}_j^k$, $\hat{u}_j^k$ for each j by solving

$$0 \in \begin{pmatrix} A & B_j \\ N_j & M_j \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} q_1 + w_j^k + r(x - x^k) \\ q_{2j} + r(u - u_j^k) \end{pmatrix} + \begin{pmatrix} \mathcal{N}_D(x) \\ \mathcal{N}_{C_j}(u) \end{pmatrix}$$

Step 2 Update by $x^{k+1} = \sum_{j=1}^{\nu} p_j \hat{x}_j^k, \quad u_j^{k+1} = \hat{u}_j^k$

$$w_j^{k+1} = w_j^k + r(\hat{x}_j^k - x^{k+1}), \ j = 1, \cdots, \nu.$$

Progressive Hedging Algorithm

Give initial points $x^0 \in I\!R^{n_1}$, $u_j^0 \in I\!R^{n_2}$ and $w_j^0 \in I\!R^{n_1}$, $j = 1, \cdots, \nu$ such that $\sum_{j=1}^{\nu} p_j w_j^0 = 0$. Choose $r > 0$. Let $k = 0$.

Step 1. For $j = 1, \cdots, \nu$, solve the VI

$$\begin{array}{rcl} -Ax - B_j u - q_1 - w_j^k - r(x - x^k) & \in & \mathcal{N}_D(x), \\ -N_j x - M_j u - q_{2j} - r(u - u_j^k) & \in & \mathcal{N}_{C_j}(u), \end{array}$$

and obtain a solution $(\hat{x}_j^k, \hat{u}_j^k)$, $j = 1, \cdots, \nu$.

Step 2. Let $x^{k+1} = \sum_{j=1}^{\nu} p_j \hat{x}_j^k$.

$$u_j^{k+1} = \hat{u}_j^k, \quad w_j^{k+1} = w_j^k + r(\hat{x}_j^k - x^{k+1}),\ j = 1, \cdots, \nu.$$

PHA for monotone SVI is an application of Douglas-Rachford splitting method; convergence analysis for non-monotone SVI. (joint work with D. Sun and J. Yang)

Algorithms for two-stage SVI

X. Chen, T.K. Pong and R. J-B Wets, Two-stage stochastic variational inequalities: an ERM-solution procedure, Math. Program., 165(2017).

Using suitable residual functions, the two-stage stochastic VI can be formulated as the two-stage stochastic optimization problem

$$\begin{array}{ll} \min & \theta(x) + \lambda \mathbb{E}[r(\xi, u(\xi, x)) + Q(\xi, x)] \\ \text{s.t.} & x \in D \\ & u(\xi, x) = x + W y^*_\xi, \quad Q(\xi, x) = \frac{1}{2}(y^*_\xi)^T H y^*_\xi, \qquad \xi \in \Xi, \\ \text{where} & \\ & y^*_\xi = \text{argmin}\{ \frac{1}{2} y_\xi^T H y_\xi \,|\, x + W y_\xi \in C_\xi \}. \end{array} \tag{2}$$

$\lambda > 0$, $H \in \mathbb{R}^{n_2 \times n_2}$ is positive definite, $y_\xi \in \mathbb{R}^{n_2}$ is the recourse variable, $W \in \mathbb{R}^{n_1 \times n_2}$ is the recourse matrix and θ, r are residual functions.

Douglas-Rachford splitting method

Applications & Future research

1. Optimality conditions for a stochastic program
2. A Walras equilibrium problem
3. Prevailing network flow analysis (traffic, data transmission, high-speed rail, airline, power system)
4. Stochastic convex game

I. Distributionally robust two-stage variational inequalities

$$\begin{array}{rcl} 0 & \in & Ax + \mathbb{E}_P[B(\xi)u(\xi)] + q_1 + \mathcal{N}_D(x), \quad P \in \mathcal{P} \\ 0 & \in & N(\xi)x + M(\xi)u(\xi) + q_2(\xi) + \mathcal{N}_{C_\xi}(u(\xi)), \qquad \text{for a.e. } \xi \in \Xi. \end{array}$$

II. Stochastic dynamic variational inequalities

$$\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + \mathbb{E}_{P_t}[B(t,\xi)u(t,\xi)] + q_1(t) + \mathcal{N}_D(x(t)), \quad t \in [0,T] \\ 0 & \in & N(t,\xi)x(t) + M(t,\xi)u(t,\xi) + q_2(t,\xi) + \mathcal{N}_{C_\xi}(u(t,\xi)), \text{ for a.e. } \xi \in \Xi. \end{array}$$

III. Multistage stochastic variational inequalities

$$x(\xi) = (x_1, x_2(\xi_1), x_3(\xi_1,\xi_2), \ldots, x_N(\xi_1,\xi_2,\ldots,\xi_{N-1}))$$

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Thank You